

**SOME GROWTH ASPECTS OF SPECIAL TYPE OF  
DIFFERENTIAL POLYNOMIAL GENERATED BY  
ENTIRE AND MEROMORPHIC FUNCTIONS ON THE  
BASIS OF THEIR RELATIVE  $(p, q)$ -TH ORDERS**

TANMAY BISWAS

ABSTRACT. In this paper we establish some results depending on the comparative growth properties of composite entire and meromorphic functions using relative  $(p, q)$ -th order and relative  $(p, q)$ -th lower order where  $p, q$  are any two positive integers and that of a special type of differential polynomial generated by one of the factors.

## 1. Introduction, Definitions and Notations

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [8, 11, 15, 16]. We also use the standard notations and definitions of the theory of entire functions which are available in [17] and therefore we do not explain those in details. Let  $f$  be an entire function defined in the open complex plane  $\mathbb{C}$ . The maximum modulus function  $M_f(r)$  corresponding to  $f$  is defined on  $|z| = r$  as  $M_f(r) = \max_{|z|=r} |f(z)|$ . In this connection the following definition is relevant:

---

Received April 27, 2019. Revised October 9, 2019. Accepted October 16, 2019.

2010 Mathematics Subject Classification: 30D20, 30D30, 30D35.

Key words and phrases: Entire function, meromorphic function, index-pair,  $(p, q)$ -th order, relative  $(p, q)$ -th order, composition, growth, Property (A), special type of differential polynomial.

© The Kangwon-Kyungki Mathematical Society, 2019.

This is an Open Access article distributed under the terms of the Creative Commons Attribution Non-Commercial License (<http://creativecommons.org/licenses/by-nc/3.0/>) which permits unrestricted non-commercial use, distribution and reproduction in any medium, provided the original work is properly cited.

DEFINITION 1. [2] A non-constant entire function  $f$  is said have the Property (A) if for any  $\sigma > 1$  and for all sufficiently large  $r$ ,  $[M_f(r)]^2 \leq M_f(r^\sigma)$  to holds. For examples of functions with or without the Property (A), one may see [2].

When  $f$  is meromorphic, one may introduce another function  $T_f(r)$  known as Nevanlinna’s characteristic function of  $f$ , playing the same role as  $M_f(r)$ . Now we just recall the following properties of meromorphic functions which will be needed in the sequel:

Let  $n_0, n_1, n_2, \dots, n_k$  are non negative integers. For a transcendental meromorphic function  $f$ , we call the expression  $M[f] = f^{n_0} (f^{(1)})^{n_1} (f^{(2)})^{n_2} \dots (f^{(k)})^{n_k}$  to be a monomial generated by  $f$ . The numbers  $\gamma_M = n_0 + n_1 + n_2 + \dots + n_k$  and  $\Gamma_M = n_0 + 2n_1 + 3n_2 + \dots + (k + 1)n_k$  are called respectively the degree and weight of the monomial. If  $M_1[f], M_2[f], \dots, M_n[f]$  denote monomials in  $f$ , then

$$Q[f] = a_1M_1[f] + a_2M_2[f] + \dots + a_nM_n[f],$$

where  $a_i \neq 0 (i = 1, 2, \dots, n)$  is called a differential polynomial generated by  $f$  of degree  $\gamma_Q = \max\{\gamma_{M_j} : 1 \leq j \leq n\}$  and weight  $\Gamma_Q = \max\{\Gamma_{M_j} : 1 \leq j \leq n\}$ . Also we call the numbers  $\underline{\gamma}_Q = \min_{1 \leq j \leq s} \gamma_{M_j}$  and  $k$  (the order of the highest derivative of  $f$ ) the lower degree and the order of  $Q[f]$  respectively. If  $\underline{\gamma}_Q = \gamma_Q$ ,  $Q[f]$  is called a homogeneous differential polynomial.

However, the Nevanlinna’s Characteristic function of a meromorphic function  $f$  is characterize as

$$T_f(r) = N_f(r) + m_f(r),$$

wherever the function  $N_f(r, a)$  ( $\bar{N}_f(r, a)$ ) known as counting function of  $a$ -points (distinct  $a$ -points) of meromorphic  $f$  is defined as follows:

$$N_f(r, a) = \int_0^r \frac{n_f(t, a) - n_f(0, a)}{t} dt + n_f(0, a) \log r$$

$$\left( \bar{N}_f(r, a) = \int_0^r \frac{\bar{n}_f(t, a) - \bar{n}_f(0, a)}{t} dt + \bar{n}_f(0, a) \log r \right),$$

in addition we represent by  $n_f(r, a)$  ( $\bar{n}_f(r, a)$ ) the number of  $a$ -points (distinct  $a$ -points) of  $f$  in  $|z| \leq r$  and an  $\infty$ -point is a pole of  $f$ . In

many occasions  $N_f(r, \infty)$  and  $\overline{N}_f(r, \infty)$  are symbolized by  $N_f(r)$  and  $\overline{N}_f(r)$  respectively.

On the other hand, the function  $m_f(r, \infty)$  alternatively indicated by  $m_f(r)$  known as the proximity function of  $f$  is defined as:

$$m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \text{ where}$$

$$\log^+ x = \max(\log x, 0) \text{ for all } x \geq 0.$$

Also we may employ  $m\left(r, \frac{1}{f-a}\right)$  by  $m_f(r, a)$ .

If  $f$  is entire, then the Nevanlinna's Characteristic function  $T_f(r)$  of  $f$  is defined as

$$T_f(r) = m_f(r).$$

Moreover for any non-constant entire function  $f$ ,  $T_f(r)$  is strictly increasing and continuous functions of  $r$ . Also its inverse  $T_f^{-1} : (|T_f(0)|, \infty) \rightarrow (0, \infty)$  is exists where  $\lim_{s \rightarrow \infty} T_f^{-1}(s) = \infty$ .

Now let us consider that  $x \in [0, \infty)$  and  $k \in \mathbb{N}$  where  $\mathbb{N}$  be the set of all positive integers. We define  $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$  and  $\log^{[k]} x = \log(\log^{[k-1]} x)$ . Also we denote that  $\log^{[0]} x = x$ ,  $\log^{[-1]} x = \exp x$ ,  $\exp^{[0]} x = x$  and  $\exp^{[-1]} x = \log x$ . Further we assume that throughout the present paper  $a, p, q, m$  and  $n$  always denote positive integers. Now considering this, let us recall that Shen et al. [13] defined the  $(m, n)$ - $\varphi$  order and  $(m, n)$ - $\varphi$  lower order of entire functions  $f$  which are as follows:

DEFINITION 2. [13] Let  $\varphi : [0, +\infty) \rightarrow (0, +\infty)$  be a non-decreasing unbounded function and  $m \geq n$ . The  $(m, n)$ - $\varphi$  order  $\rho^{(m,n)}(f, \varphi)$  and  $(m, n)$ - $\varphi$  lower order  $\lambda^{(m,n)}(f, \varphi)$  of entire functions  $f$  are defined as:

$$\rho^{(m,n)}(f, \varphi) = \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[m]} M_f(r)}{\log^{[n]} \varphi(r)} \text{ and } \lambda^{(m,n)}(f, \varphi) = \underline{\lim}_{r \rightarrow \infty} \frac{\log^{[m]} M_f(r)}{\log^{[n]} \varphi(r)}.$$

If  $f$  is a meromorphic function, then

$$\rho^{(m,n)}(f, \varphi) = \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[m-1]} T_f(r)}{\log^{[n]} \varphi(r)} \text{ and } \lambda^{(m,n)}(f, \varphi) = \underline{\lim}_{r \rightarrow \infty} \frac{\log^{[m-1]} T_f(r)}{\log^{[n]} \varphi(r)}.$$

Further for any non-decreasing unbounded function  $\varphi : [0, +\infty) \rightarrow (0, +\infty)$ , if we assume  $\lim_{r \rightarrow +\infty} \frac{\log^{[q]} \varphi(ar)}{\log^{[q]} \varphi(r)} = 1$  for all  $\alpha > 0$ , then for any

entire function  $f$ , using the inequality  $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$  {cf. [8]}, one can easily verify that (see [13])

$$\rho^{(m,n)}(f, \varphi) = \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[m]} M_f(r)}{\log^{[n]} \varphi(r)} = \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[m-1]} T_f(r)}{\log^{[n]} \varphi(r)}$$

and

$$\lambda^{(m,n)}(f, \varphi) = \underline{\lim}_{r \rightarrow \infty} \frac{\log^{[m]} M_f(r)}{\log^{[n]} \varphi(r)} = \underline{\lim}_{r \rightarrow \infty} \frac{\log^{[m-1]} T_f(r)}{\log^{[n]} \varphi(r)}.$$

If we take  $m = p$ ,  $n = 1$  and  $\varphi(r) = \log^{[q-1]} r$ , then the above definition reduce to the following definition:

**DEFINITION 3.** The  $(p, q)$ -th order and  $(p, q)$ -th lower order of an entire function  $f$  are defined as:

$$\rho^{(p,q)}(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \underline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}.$$

If  $f$  is a meromorphic function, then

$$\rho^{(p,q)}(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \underline{\lim}_{r \rightarrow \infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r}.$$

Definition 3 avoids the restriction  $p \geq q$  of the original definition of  $(p, q)$ -th order (respectively  $(p, q)$ -th lower order) of entire functions introduced by Juneja et al. [9]. An entire or meromorphic function for which  $(p, q)$ -th order and  $(p, q)$ -th lower order are the same is said to be of regular  $(p, q)$  growth. Functions which are not of regular  $(p, q)$  growth are said to be of irregular  $(p, q)$  growth.

However the above definition is very useful for measuring the growth of entire and meromorphic functions. If  $p = l$  and  $q = 1$  then we write  $\rho^{(l,1)}(f) = \rho^{(l)}(f)$  and  $\lambda^{(l,1)}(f) = \lambda^{(l)}(f)$  where  $\rho^{(l)}(f)$  and  $\lambda^{(l)}(f)$  are respectively known as generalized order and generalized lower order of entire or meromorphic function  $f$ . For details about generalized order one may see [14]. Also for  $p = 2$  and  $q = 1$ , we respectively denote  $\rho^{(2,1)}(f)$  and  $\lambda^{(2,1)}(f)$  by  $\rho(f)$  and  $\lambda(f)$  which are classical growth indicators such as order and lower order of entire or meromorphic function  $f$ .

In this connection we just recall the following definition of index-pair where we will give a minor modification to the original definition (see e.g. [9]) :

DEFINITION 4. An entire function  $f$  is said to have index-pair  $(p, q)$  if  $b < \rho^{(p,q)}(f) < \infty$  and  $\rho^{(p-1,q-1)}(f)$  is not a nonzero finite number, where  $b = 1$  if  $p = q$  and  $b = 0$  for otherwise. Moreover if  $0 < \rho^{(p,q)}(f) < \infty$ , then

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for } n = 1, 2, \dots \end{cases} .$$

Similarly for  $0 < \lambda^{(p,q)}(f) < \infty$ , one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for } n = 1, 2, \dots \end{cases} .$$

Analogously one can easily verify that the Definition 4 of index-pair can also be applicable for a meromorphic function  $f$ .

In order to compare the growth of entire functions having the same  $(p, q)$ -th order, Juneja, Kapoor and Bajpai [10] also introduced the concepts of  $(p, q)$ -th type and  $(p, q)$ -th lower type of entire function. Next we recall the definitions of  $(p, q)$ -th type and  $(p, q)$ -th lower type of entire function where we will give a minor modification to the original definition (see e.g. [10]):

DEFINITION 5. The  $(p, q)$ -th type and the  $(p, q)$ -th lower type of entire function  $f$  having non-zero finite positive  $(p, q)$ -th order  $\rho^{(p,q)}(f)$  are defined as :

$$\sigma^{(p,q)}(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\rho^{(p,q)}(f)}} \text{ and } \underline{\sigma}^{(p,q)}(f) = \underline{\lim}_{r \rightarrow \infty} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\rho^{(p,q)}(f)}} .$$

Likewise, to compare the growth of entire functions having the same  $(p, q)$ -th lower order, one can also introduced the concept of  $(p, q)$ -th weak type in the following manner :

DEFINITION 6. The  $(p, q)$ -th weak type of entire function  $f$  having non-zero finite positive  $(p, q)$ -th lower order  $\lambda^{(p,q)}(f)$  is defined as :

$$\tau^{(p,q)}(f) = \underline{\lim}_{r \rightarrow \infty} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}(f)}} .$$

Similarly one may define the growth indicator  $\bar{\tau}^{(p,q)}(f)$  of an entire function  $f$  in the following way :

$$\bar{\tau}^{(p,q)}(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}(f)}}, \quad 0 < \lambda^{(p,q)}(f) < \infty.$$

L. Bernal [1,2] introduced the relative order between two entire functions to avoid comparing growth just with  $\exp z$ . In the case of relative order, Sánchez Ruiz et al. [12] gave the definitions of relative  $(p, q)$ -th order and relative  $(p, q)$ -th lower order of an entire function with respect to another entire function and Debnath et al. [6] introduced the definitions of relative  $(p, q)$ -th order and relative  $(p, q)$ -th lower order of a meromorphic function with respect to another entire function in the light of index-pair. In order to keep accordance with Definition 3 and Definition 4, we will give a minor modification to the original definition of relative  $(p, q)$ -th order and relative  $(p, q)$ -th lower order of entire and meromorphic function (see e.g. [6, 12]).

**DEFINITION 7.** Let  $f$  and  $g$  be any two entire functions with index-pairs  $(m, q)$  and  $(m, p)$  respectively. Then the relative  $(p, q)$ -th order and relative  $(p, q)$ -th lower order of  $f$  with respect to  $g$  are defined as

$$\rho_g^{(p,q)}(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} M_g^{-1}(M_f(r))}{\log^{[q]} r} \quad \text{and} \quad \lambda_g^{(p,q)}(f) = \underline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} M_g^{-1}(M_f(r))}{\log^{[q]} r}.$$

If  $f$  is a meromorphic and  $g$  is entire, then

$$\rho_g^{(p,q)}(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} r} \quad \text{and} \quad \lambda_g^{(p,q)}(f) = \underline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} r}.$$

Further an entire or meromorphic function  $f$ , for which relative  $(p, q)$ -th order and relative  $(p, q)$ -th lower order with respect to an entire function  $g$  are the same is called a function of regular relative  $(p, q)$  growth with respect to  $g$ . Otherwise,  $f$  is said to be irregular relative  $(p, q)$  growth with respect to  $g$ .

In this paper we prove our results for a special type of differential polynomials. Actually in the paper we establish some new results depending on the comparative growth properties of composite transcendental entire or meromorphic functions using relative  $(p, q)$ -th order and relative  $(p, q)$ -th lower order and that of some special type of differential polynomials by one of the factors.

### 2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

LEMMA 1. [3] *Let  $f$  be meromorphic and  $g$  be entire then for all sufficiently large values of  $r$ ,*

$$T_{f \circ g}(r) \leq \{1 + o(1)\} \frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)).$$

LEMMA 2. [3] *Suppose that  $f$  is a meromorphic function and  $g$  be an entire function and suppose that  $0 < \mu < \rho_g \leq \infty$ . Then for a sequence of values of  $r$  tending to infinity,*

$$T_{f \circ g}(r) \geq T_f(\exp(r^\mu)) .$$

LEMMA 3. [7] *Let  $f$  be an entire function which satisfies the Property (A),  $\beta > 0$ ,  $\delta > 1$  and  $\alpha > 2$ . Then*

$$\beta T_f(r) < T_f(\alpha r^\delta) .$$

LEMMA 4. [5] *Let  $f$  be a transcendental meromorphic function and  $F = f^\alpha Q[f]$  where  $Q[f]$  is a differential polynomial in  $f$ , then for any  $\alpha \geq 1$*

$$\begin{aligned} T_f(r) &= O\{T_F(r)\} \text{ as } r \rightarrow \infty \\ \text{and } T_F(r) &= O\{T_f(r)\} \text{ as } r \rightarrow \infty. \end{aligned}$$

LEMMA 5. *Let  $f$  be a transcendental meromorphic function and  $g$  be a transcendental entire function with  $0 < \lambda_g(m, p) \leq \rho_g(m, p) < \infty$  where  $m > 1$ . Also let  $F = f^\alpha Q[f]$  and  $G = g^\beta Q[g]$  where  $Q[f]$  and  $Q[g]$  are differential polynomials in  $f$  and  $g$  respectively. Then for any  $\alpha \geq 1$  and  $\beta \geq 1$*

$$\frac{\lambda^{(m,p)}(g)}{\rho^{(m,p)}(g)} \leq \liminf_{r \rightarrow \infty} \frac{\log^{[p]} T_G^{-1}(T_F(r))}{\log^{[p]} T_g^{-1} T_f(r)} \leq \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_G^{-1}(T_F(r))}{\log^{[p]} T_g^{-1} T_f(r)} \leq \frac{\rho^{(m,p)}(g)}{\lambda^{(m,p)}(g)} .$$

*Proof.* Let us consider that  $\alpha_1, \beta_1, \gamma$  and  $\eta$  are all constant greater than 1. Now we get from Lemma 4 for all sufficiently large positive numbers of  $r$  that

$$(1) \qquad T_F(r) < \alpha_1 \cdot T_f(r)$$

and

$$(2) \quad T_F(r) > \frac{1}{\beta_1} \cdot T_f(r).$$

Also from Lemma 4, we get for all sufficiently large positive numbers of  $r$  that

$$\begin{aligned} T_G(r) &> \frac{1}{\gamma} \cdot T_g(r) \\ \text{i.e., } r &> T_G^{-1} \left( \frac{1}{\gamma} \cdot T_g(r) \right) \\ (3) \quad \text{i.e., } T_g^{-1}(\gamma \cdot r) &> T_G^{-1}(r) \end{aligned}$$

and

$$\begin{aligned} T_G(r) &< \eta \cdot T_g(r) \\ \text{i.e., } r &< T_G^{-1}(\eta \cdot T_g(r)) \\ (4) \quad \text{i.e., } T_g^{-1} \left( \frac{r}{\eta} \right) &< T_G^{-1}(r). \end{aligned}$$

Now from (1) and (3) it follows for all sufficiently large positive numbers of  $r$  that

$$\begin{aligned} T_G^{-1}(T_F(r)) &< T_G^{-1}(\alpha_1 \cdot T_f(r)) \\ (5) \quad \text{i.e., } T_G^{-1}(T_F(r)) &< T_g^{-1}(\gamma \alpha_1 \cdot T_f(r)). \end{aligned}$$

Again from (2) and (4), it follows for all sufficiently large positive numbers of  $r$  that

$$\begin{aligned} T_G^{-1}(T_F(r)) &> T_G^{-1} \left( \frac{1}{\beta_1} \cdot T_f(r) \right) \\ (6) \quad \text{i.e., } T_G^{-1}(T_F(r)) &> T_g^{-1} \left( \frac{1}{\eta \beta_1} \cdot T_f(r) \right). \end{aligned}$$

Now from (5) and (6), we for all sufficiently large positive numbers of  $r$  that

$$(7) \quad \log^{[p]} T_G^{-1}(T_F(r)) < \log^{[p]} T_g^{-1}(\gamma \alpha_1 \cdot T_f(r))$$

and

$$(8) \quad \log^{[p]} T_G^{-1}(T_F(r)) > \log^{[p]} T_g^{-1} \left( \frac{1}{\eta \beta_1} \cdot T_f(r) \right).$$



Now for the definition of  $(m, p)$ -th order and  $(m, p)$ -th lower order of  $g$ , we get for all sufficiently large positive numbers of  $r$  that

$$T_g \left( \exp^{[p-1]} \left[ \log^{[m-2]} T_f(r) \right]^{\frac{1}{\rho^{(m,p)}(g)+\varepsilon}} \right) \leq T_f(r)$$

(9) *i.e.*,  $\log^{[p]} T_g^{-1}(T_f(r)) \geq \frac{1}{(\rho^{(m,p)}(g) + \varepsilon)} \log^{[m-1]} T_f(r)$

and

$$T_g \left( \exp^{[p-1]} \left( \log^{[m-2]} (\gamma\alpha_1 \cdot T_f(r)) \right)^{\frac{1}{\lambda^{(m,p)}(g)-\varepsilon}} \right) \geq (\gamma\alpha_1 \cdot T_f(r))$$

*i.e.*,  $\exp^{[p-1]} \left( \log^{[m-2]} (\gamma\alpha_1 \cdot T_f(r)) \right)^{\frac{1}{\lambda^{(m,p)}(g)-\varepsilon}} \geq T_g^{-1}(\gamma\alpha_1 \cdot T_f(r))$

(10) *i.e.*,  $\frac{1}{(\lambda^{(m,p)}(g) - \varepsilon)} \log^{[m-1]} T_f(r) + O(1) \geq \log^{[p]} T_g^{-1}(\gamma\alpha_1 \cdot T_f(r))$ .

Therefore from (7) and (10), it follows for all sufficiently large positive numbers of  $r$  that

(11)  $\log^{[p]} T_G^{-1}(T_F(r)) < \frac{1}{(\lambda^{(m,p)}(g) - \varepsilon)} \log^{[m-1]} T_f(r) + O(1)$ .

Therefore from (9) and (11), it follows for all sufficiently large positive numbers of  $r$  that

$$\frac{\log^{[p]} T_G^{-1}(T_F(r))}{\log^{[p]} T_g^{-1}(T_f(r))} < \left( \frac{\rho^{(m,p)}(g) + \varepsilon}{\lambda^{(m,p)}(g) - \varepsilon} \right) \cdot \frac{\log^{[m-1]} T_f(r) + O(1)}{\log^{[m-1]} T_f(r)}$$

(12) *i.e.*,  $\overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_G^{-1}(T_F(r))}{\log^{[p]} T_g^{-1}(T_f(r))} \leq \frac{\rho^{(m,p)}(g)}{\lambda^{(m,p)}(g)}$ .

Similarly, from (8) it can be shown for all sufficiently large positive numbers of  $r$  that

(13)  $\underline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_G^{-1}(T_F(r))}{\log^{[p]} T_g^{-1}(T_f(r))} \geq \frac{\lambda^{(m,p)}(g)}{\rho^{(m,p)}(g)}$ .

Therefore from (12) and (13), we obtain that

$$\frac{\lambda^{(m,p)}(g)}{\rho^{(m,p)}(g)} \leq \underline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_G^{-1}(T_F(r))}{\log^{[p]} T_g^{-1}(T_f(r))} \leq \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_G^{-1}(T_F(r))}{\log^{[p]} T_g^{-1}(T_f(r))} \leq \frac{\rho^{(m,p)}(g)}{\lambda^{(m,p)}(g)}$$

Thus the lemma follows from above. □

LEMMA 6. Let  $f$  be a transcendental meromorphic function and  $g$  be a transcendental entire function with regular  $(m, p)$  growth where  $m > 1$ . Also let  $F = f^\alpha Q[f]$  and  $G = g^\beta Q[g]$  where  $Q[f]$  and  $Q[g]$  are differential polynomials in  $f$  and  $g$  respectively. Then for any  $\alpha \geq 1$  and  $\beta \geq 1$ , the relative  $(p, q)$ -th order and relative  $(p, q)$ -th lower order of  $F$  with respect to  $G$  are same as those of  $f$  with respect to  $g$ .

*Proof.* If  $g$  is of regular  $(m, p)$  growth with  $m > 1$ , then from Lemma 5 we get that

$$(14) \quad \lim_{r \rightarrow \infty} \frac{\log^{[p]} T_G^{-1}(T_F(r))}{\log^{[p]} T_g^{-1}(T_f(r))} = 1.$$

Now in view of (14), we obtain that

$$\begin{aligned} \rho_G^{(p,q)}(F) &= \lim_{r \rightarrow \infty} \frac{\log^{[p]} T_G^{-1}(T_F(r))}{\log^{[q]} r} \\ &= \lim_{r \rightarrow \infty} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} r} \cdot \lim_{r \rightarrow \infty} \frac{\log^{[p]} T_G^{-1}(T_F(r))}{\log^{[p]} T_g^{-1} T_f(r)} \\ &= \rho_g^{(p,q)}(f) \cdot 1 = \rho_g^{(p,q)}(f). \end{aligned}$$

In a similar manner,  $\lambda_G^{(p,q)}(F) = \lambda_g^{(p,q)}(f)$ .

Thus the lemma follows.  $\square$

### 3. Main Results

In this section we present the main results of the paper.

THEOREM 1. Let  $f$  be a transcendental meromorphic function and  $h$  be a transcendental entire function with regular  $(a, p)$  growth where  $a > 1$ . Also let  $F = f^\alpha Q[f]$ ,  $H = h^\gamma Q[h]$  where  $Q[f]$  and  $Q[h]$  are differential polynomials in  $f$  and  $h$  respectively,  $\lambda_h^{(p,q)}(f) > 0$  and  $g$  be an entire function with finite  $(m, n)$ -th order. If  $h$  satisfies the Property (A), then for every positive constant  $A$  and each  $\eta \in (-\infty, \infty)$ ,

$$(i) \quad \lim_{r \rightarrow \infty} \frac{\left\{ \log^{[p]} T_h^{-1}(T_{f \circ g}(r)) \right\}^{1+\eta}}{\log^{[p]} T_H^{-1}(T_F(\exp^{[q]} r^A))} = 0 \text{ if } q \geq m$$

and

$$(ii) \lim_{r \rightarrow \infty} \frac{\left\{ \log^{[p+m-q-1]} T_h^{-1} (T_{f \circ g} (r)) \right\}^{1+\eta}}{\log^{[p]} T_H^{-1} (T_F (\exp^{[q]} r^A))} = 0 \text{ if } q < m,$$

where  $A > (1 + \eta)\rho^{(m,n)}(g)$ ,  $\alpha \geq 1$  and  $\gamma \geq 1$ .

*Proof.* Let us consider that  $\alpha > 2$  and  $\delta \rightarrow 1^+$  in Lemma 3. If  $1 + \eta \leq 0$ , then the theorem is obvious. We consider  $1 + \eta > 0$ . Let us we choose  $\varepsilon$  such that

$$(15) \quad 0 < \varepsilon < \min \left\{ \lambda_h^{(p,q)}(f), \frac{A}{1 + \eta} - \rho^{(m,n)}(g) \right\}.$$

Since  $T_h^{-1}(r)$  is an increasing function of  $r$ , it follows from Lemma 1, Lemma 3 and the inequality  $T_g(r) \leq \log M_g(r)$  {cf. [8]} for all sufficiently large positive numbers of  $r$  that

$$\begin{aligned} T_h^{-1}(T_{f \circ g}(r)) &\leq T_h^{-1}(\{1 + o(1)\} T_f(M_g(r))) \\ \text{i.e., } T_h^{-1}(T_{f \circ g}(r)) &\leq \beta (T_h^{-1} T_f(M_g(r)))^\delta \\ \text{i.e., } \log^{[p]} T_h^{-1}(T_{f \circ g}(r)) &\leq \log^{[p]} T_h^{-1}(T_f(M_g(r))) + O(1) \end{aligned}$$

$$(16) \quad \text{i.e., } \log^{[p]} T_h^{-1}(T_{f \circ g}(r)) \leq \left( \rho_h^{(p,q)}(f) + \varepsilon \right) \log^{[q]} M_g(r) + O(1).$$

Now the following cases may arise :

**Case I.** Let  $q \geq m$ . Then we have from (16) for all sufficiently large positive numbers of  $r$  that

$$(17) \quad \log^{[p]} T_h^{-1}(T_{f \circ g}(r)) \leq \left( \rho_h^{(p,q)}(f) + \varepsilon \right) \log^{[m-1]} M_g(r) + O(1).$$

Now from the definition of  $(m, n)$ -th order of  $g$  in terms of maximum modulus, we get for arbitrary positive  $\varepsilon$  and for all sufficiently large positive numbers of  $r$  that

$$\begin{aligned} \log^{[m]} M_g(r) &\leq (\rho^{(m,n)}(g) + \varepsilon) \log^{[n]} r \\ (18) \quad \text{i.e., } \log^{[m]} M_g(r) &\leq (\rho^{(m,n)}(g) + \varepsilon) \log r. \end{aligned}$$

Also for all sufficiently large positive numbers of  $r$  it follows from (18) that

$$(19) \quad \log^{[m-1]} M_g(r) \leq r^{(\rho^{(m,n)}(g) + \varepsilon)}.$$

So from (17) and (19) it follows for all sufficiently large positive numbers of  $r$  that

$$(20) \quad \log^{[p]} T_h^{-1} (T_{f \circ g} (r)) \leq \left( \rho_h^{(p,q)} (f) + \varepsilon \right) r^{(\rho^{(m,n)}(g)+\varepsilon)} + O(1).$$

**Case II.** Let  $q < m$ . Then we get from(16) for all sufficiently large positive numbers of  $r$  that

$$(21) \quad \log^{[p]} T_h^{-1} (T_{f \circ g} (r)) \leq \left( \rho_h^{(p,q)} (f) + \varepsilon \right) \exp^{[m-q]} \log^{[m]} M_g (r) + O(1).$$

Also we obtain from (18) for all sufficiently large positive numbers of  $r$  that

$$(22) \quad \begin{aligned} \exp^{[m-q]} \log^{[m]} M_g (r) &\leq \exp^{[m-q]} \log r^{(\rho^{(m,n)}(g)+\varepsilon)} \\ i.e., \exp^{[m-q]} \log^{[m]} M_g (r) &\leq \exp^{[m-q-1]} r^{(\rho^{(m,n)}(g)+\varepsilon)}. \end{aligned}$$

Now from (21) and (22) we obtain for all sufficiently large positive numbers of  $r$  that

$$(23) \quad \begin{aligned} \log^{[p]} T_h^{-1} (T_{f \circ g} (r)) &\leq \left( \rho_h^{(p,q)} (f) + \varepsilon \right) \exp^{[m-q-1]} r^{(\rho^{(m,n)}(g)+\varepsilon)} + O(1) \\ i.e., \log^{[p+m-q-1]} T_h^{-1} (T_{f \circ g} (r)) &\leq r^{(\rho^{(m,n)}(g)+\varepsilon)} + O(1). \end{aligned}$$

Again in view of Lemma 6, we get for all sufficiently large positive numbers of  $r$  that

$$(24) \quad \begin{aligned} \log^{[p]} T_H^{-1} (T_F (\exp^{[q]} r^A)) &\geq \left( \lambda_H^{(p,q)} (F) - \varepsilon \right) \log^{[q]} \exp^{[q]} (r^A) \\ i.e., \log^{[p]} T_H^{-1} (T_F (\exp^{[q]} r^A)) &\geq \left( \lambda_h^{(p,q)} (f) - \varepsilon \right) \log^{[q]} \exp^{[q]} (r^A) \\ e.e., \log^{[p]} T_H^{-1} (T_F (\exp^{[q]} r^A)) &\geq \left( \lambda_h^{(p,q)} (f) - \varepsilon \right) r^A. \end{aligned}$$

Now if  $q \geq m$ , we get from (20), (24) and in view of (15) for all sufficiently large positive numbers of  $r$  that

$$\begin{aligned} \frac{\left\{ \log^{[p]} T_h^{-1} (T_{f \circ g} (r)) \right\}^{1+\eta}}{\log^{[p]} T_H^{-1} (T_F (\exp^{[q]} r^A))} &\leq \frac{\left( \rho_h^{(p,q)} (f) + \varepsilon \right)^{1+\eta} r^{(\rho^{(m,n)}(g)+\varepsilon)(1+\eta)} + O(1)}{\left( \lambda_h^{(p,q)} (f) - \varepsilon \right) r^A} \\ i.e., \lim_{r \rightarrow \infty} \frac{\left\{ \log^{[p]} T_h^{-1} (T_{f \circ g} (r)) \right\}^{1+\eta}}{\log^{[p]} T_H^{-1} (T_F (\exp^{[q]} r^A))} &= 0, \end{aligned}$$

which proves the first part of the theorem.

Again when  $q < m$ , we obtain from (23), (24) and (15) for all sufficiently large positive numbers of  $r$  that

$$\frac{\left\{ \log^{[p+m-q-1]} T_h^{-1} (T_{f \circ g} (r)) \right\}^{1+\eta}}{\log^{[p]} T_H^{-1} (T_F (\exp^{[q]} r^A))} \leq \frac{r^{(\rho^{(m,n)}(g)+\varepsilon)(1+\eta)} \left( 1 + \frac{O(1)}{r^{(\rho^{(m,n)}(g)+\varepsilon)}} \right)^{1+\eta}}{\left( \lambda_h^{(p,q)} (f) - \varepsilon \right) r^A}$$

$$i.e., \lim_{r \rightarrow \infty} \frac{\left\{ \log^{[p+m-q-1]} T_h^{-1} (T_{f \circ g} (r)) \right\}^{1+\eta}}{\log^{[p]} T_H^{-1} (T_F (\exp^{[q]} r^A))} = 0.$$

This proves the second part of the theorem. □

REMARK 1. In Theorem 1 if we take the condition  $\rho_h^{(p,q)} (f) > 0$  instead of  $\lambda_h^{(p,q)} (f) > 0$ , the theorem remains true with “limit inferior” in place of “limit”.

In view of Theorem 1 the following theorem can be carried out:

THEOREM 2. Let  $g$  and  $h$  be any two transcendental entire functions where  $h$  is of regular  $(m, p)$ -growth where  $m > 1$ . Also let  $f$  be a meromorphic function,  $g$  is of finite  $(m, n)$ -th order,  $\lambda_h^{(p,n)} (g) > 0$  and  $\rho_h^{(p,q)} (f) < \infty$ . If  $h$  satisfies the Property (A),  $G = g^\beta Q [g]$ ,  $H = h^\gamma Q [h]$  where  $Q [g]$  and  $Q [h]$  are differential polynomials in  $g$  and  $h$  respectively, then for every positive constant  $A$  and each  $\eta \in (-\infty, \infty)$ ,

$$(i) \lim_{r \rightarrow \infty} \frac{\left\{ \log^{[p]} T_h^{-1} (T_{f \circ g} (r)) \right\}^{1+\eta}}{\log^{[p]} T_H^{-1} (T_G (\exp^{[n]} r^A))} = 0 \text{ if } q \geq m$$

and

$$(ii) \lim_{r \rightarrow \infty} \frac{\left\{ \log^{[p+m-q-1]} T_h^{-1} (T_{f \circ g} (r)) \right\}^{1+\eta}}{\log^{[p]} T_H^{-1} (T_G (\exp^{[n]} r^A))} = 0 \text{ if } q < m,$$

where  $A > (1 + \eta)\rho^{(m,n)}(g)$ ,  $\beta \geq 1$  and  $\gamma \geq 1$ .

The proof is omitted.

REMARK 2. In Theorem 2, if we take the condition  $\rho_h^{(p,n)} (g) > 0$  instead of  $\lambda_h^{(p,n)} (g) > 0$ , the theorem remains true with “limit replaced by limit inferior”.

**THEOREM 3.** *Let  $f$  be a transcendental meromorphic function and  $h$  be a transcendental entire function with regular  $(m, p)$  growth where  $m > 1$ . Also let  $F = f^\alpha Q[f]$ ,  $H = h^\gamma Q[h]$  where  $Q[f]$  and  $Q[h]$  are differential polynomials in  $f$  and  $h$  respectively,  $g$  be an entire function,  $\rho_h^{(p,q)}(f) < \infty$  and  $\lambda_h^{(p,q)}(f \circ g) = \infty$ . Then for every  $A (> 0)$ ,*

$$\lim_{r \rightarrow \infty} \frac{\log^{[p]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p]} T_H^{-1}(T_F(r^A))} = \infty,$$

where  $\alpha \geq 1$  and  $\gamma \geq 1$ .

*Proof.* If possible, let there exists a constant  $\beta$  such that for a sequence of positive numbers of  $r$  tending to infinity we have

$$(25) \quad \log^{[p]} T_h^{-1}(T_{f \circ g}(r)) \leq \beta \cdot \log^{[p]} T_H^{-1}(T_F(r^A)).$$

Again from the definition of  $\rho_H^{(p,q)}(F)$  and in view of Lemma 6, it follows for all sufficiently large positive numbers of  $r$  that

$$\log^{[p]} T_H^{-1}(T_F(r^A)) \leq \left( \rho_H^{(p,q)}(F) + \varepsilon \right) \log^{[q]} r + O(1).$$

$$(26) \quad \text{i.e., } \log^{[p]} T_h^{-1}(T_{f \circ g}(r)) \leq \left( \rho_h^{(p,q)}(f) + \varepsilon \right) \log^{[q]} r + O(1).$$

Now combining (25) and (26) we obtain for a sequence of positive numbers of  $r$  tending to infinity that

$$\begin{aligned} \log^{[p]} T_h^{-1}(T_{f \circ g}(r)) &\leq \beta \cdot \left( \rho_h^{(p,q)}(f) + \varepsilon \right) \log^{[q]} r + O(1) \\ \text{i.e., } \lambda_h^{(p,q)}(f \circ g) &\leq \beta \cdot \left( \rho_h^{(p,q)}(f) + \varepsilon \right), \end{aligned}$$

which contradicts the condition  $\lambda_h^{(p,q)}(f \circ g) = \infty$ . So for all sufficiently large positive numbers of  $r$  we get that

$$\log^{[p]} T_h^{-1}(T_{f \circ g}(r)) > \beta \cdot \log^{[p]} T_H^{-1} T_F(r^A),$$

from which the theorem follows.  $\square$

In the line of Theorem 3, one can easily prove the following theorem and therefore its proof is omitted.

**THEOREM 4.** *Let  $g$  and  $h$  be any two transcendental entire functions where  $h$  is of regular  $(m, p)$  growth where  $m > 1$ . Also let  $f$  be a meromorphic function,  $G = g^\beta Q[g]$ ,  $H = h^\gamma Q[h]$  where  $Q[g]$  and  $Q[h]$*

are differential polynomials in  $g$  and  $h$  respectively,  $\rho_h^{(p,q)}(g) < \infty$  and  $\lambda_h^{(p,q)}(f \circ g) = \infty$ . Then for every  $A (> 0)$ ,

$$\lim_{r \rightarrow \infty} \frac{\log^{[p]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p]} T_H^{-1}(T_G(r^A))} = \infty$$

where  $\beta \geq 1$  and  $\gamma \geq 1$ .

REMARK 3. Theorem 3 is also valid with “limit superior” instead of “limit” if  $\lambda_h^{(p,q)}(f \circ g) = \infty$  is replaced by  $\rho_h^{(p,q)}(f \circ g) = \infty$  and the other conditions remain the same.

REMARK 4. Theorem 4 is also valid with “limit superior” instead of “limit” if  $\lambda_h^{(p,q)}(f \circ g) = \infty$  is replaced by  $\rho_h^{(p,q)}(f \circ g) = \infty$  and the other conditions remain the same.

COROLLARY 1. Under the assumptions of Theorem 3 and Remark 3,

$$\lim_{r \rightarrow \infty} \frac{\log^{[p-1]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p-1]} T_H^{-1}(T_F(r^A))} = \infty \text{ and } \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p-1]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p-1]} T_H^{-1}(T_F(r^A))} = \infty$$

respectively.

*Proof.* By Theorem 3 we obtain for all sufficiently large values of  $r$  and for  $K > 1$ ,

$$\begin{aligned} \log^{[p]} T_h^{-1}(T_{f \circ g}(r)) &\geq K \cdot \log^{[p]} T_H^{-1}(T_F(r^A)) \\ \text{i.e., } \log^{[p-1]} T_h^{-1}(T_{f \circ g}(r)) &\geq \left\{ \log^{[p-1]} T_H^{-1}(T_F(r^A)) \right\}^K, \end{aligned}$$

from which the first part of the corollary follows.

Similarly using Remark 3, we obtain the second part of the corollary. □

COROLLARY 2. Under the assumptions of Theorem 4 and Remark 4,

$$\lim_{r \rightarrow \infty} \frac{\log^{[p-1]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p-1]} T_H^{-1}(T_G(r^A))} = \infty \text{ and } \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p-1]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p-1]} T_H^{-1}(T_G(r^A))} = \infty$$

respectively.

In the line of Corollary 1, one can easily verify Corollary 2 with the help of Theorem 4 and Remark 4 respectively and therefore its proof is omitted.

**THEOREM 5.** *Let  $f$  be a transcendental meromorphic function and  $h$  be a transcendental entire function with regular  $(a, p)$  growth where  $a > 1$ . Also let  $F = f^\alpha Q[f]$ ,  $H = h^\gamma Q[h]$  where  $Q[f]$  and  $Q[h]$  are differential polynomials in  $f$  and  $h$  respectively,  $g$  be an entire function such that  $\rho^{(m,n)}(g) < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ . If  $h$  satisfies the Property (A), then*

$$(i) \lim_{r \rightarrow \infty} \frac{\log^{[p]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p-1]} T_H^{-1}(T_F(\exp^{[q-1]} r))} = 0 \text{ if } q \geq m$$

and

$$(ii) \lim_{r \rightarrow \infty} \frac{\log^{[p+m-q-1]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p-1]} T_H^{-1}(T_F(\exp^{[q-1]} r))} = 0 \text{ if } q < m$$

where  $\alpha \geq 1$  and  $\gamma \geq 1$ .

*Proof.* As  $\rho^{(m,n)}(g) < \lambda_h^{(p,q)}(f)$ , we can choose  $\varepsilon (> 0)$  in such a way that

$$(27) \quad \rho^{(m,n)}(g) + \varepsilon < \lambda_h^{(p,q)}(f) - \varepsilon.$$

From the definition of relative  $(p, q)$ -th order and in view of Lemma 6, we obtain for all sufficiently large positive numbers of  $r$  that

$$\log^{[p]} T_H^{-1}(T_F(\exp^{[q-1]} r)) \geq (\lambda_H^{(p,q)}(F) - \varepsilon) \log^{[q]} \exp^{[q-1]} r$$

$$i.e., \log^{[p]} T_H^{-1}(T_F(\exp^{[q-1]} r)) \geq (\lambda_h^{(p,q)}(f) - \varepsilon) \log^{[q]} \exp^{[q-1]} r$$

$$(28) \quad i.e., \log^{[p-1]} T_H^{-1}(T_F(\exp^{[q-1]} r)) \geq r^{(\lambda_h^{(p,q)}(f) - \varepsilon)}.$$

Now if  $q \geq m$ , combining (20), (28) and in view of (27) we have for all sufficiently large positive numbers of  $r$  that

$$\frac{\log^{[p]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p-1]} T_H^{-1}(T_F(\exp^{[q-1]} r))} \leq \frac{(\rho_h^{(p,q)}(f) + \varepsilon) r^{(\rho^{(m,n)}(g) + \varepsilon)} + O(1)}{r^{(\lambda_h^{(p,q)}(f) - \varepsilon)}}$$

$$i.e., \lim_{r \rightarrow \infty} \frac{\log^{[p]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p-1]} T_H^{-1}(T_F(\exp^{[q-1]} r))} = 0.$$

This proves the first part of the theorem.



When  $q < m$ , combining (23) and (28) it follows for all sufficiently large positive numbers of  $r$  that

$$\frac{\log^{[p+m-q-1]} T_h^{-1} (T_{f \circ g} (r))}{\log^{[p-1]} T_H^{-1} (T_F (\exp^{[q-1]} r))} \leq \frac{r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)} + O(1)}{r^{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)}}.$$

Since  $\rho^{(m,n)}(g) < \lambda_h^{(p,q)}(f)$  and  $\varepsilon (> 0)$  is arbitrary, we get from above

$$\lim_{r \rightarrow \infty} \frac{\log^{[p+m-q-1]} T_h^{-1} (T_{f \circ g} (r))}{\log^{[p-1]} T_H^{-1} (T_F (\exp^{[q-1]} r))} = 0,$$

which is the second part of the theorem. □

**THEOREM 6.** *Let  $f$  be a transcendental meromorphic function and  $h$  be a transcendental entire function with regular  $(a, p)$  growth where  $a > 1$ . Also let  $F = f^\alpha Q[f]$ ,  $H = h^\gamma Q[h]$  where  $Q[f]$  and  $Q[h]$  are differential polynomials in  $f$  and  $h$  respectively,  $g$  be an entire function such that  $\lambda^{(m,n)}(g) < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ . If  $h$  satisfies the Property (A), then*

$$(i) \quad \lim_{r \rightarrow \infty} \frac{\log^{[p]} T_h^{-1} (T_{f \circ g} (r))}{\log^{[p-1]} T_H^{-1} (T_F (\exp^{[q-1]} r))} = 0 \text{ if } q \geq m$$

and

$$(ii) \quad \lim_{r \rightarrow \infty} \frac{\log^{[p+m-q-1]} T_h^{-1} (T_{f \circ g} (r))}{\log^{[p-1]} T_H^{-1} (T_F (\exp^{[q-1]} r))} = 0 \text{ if } q < m$$

where  $\alpha \geq 1$  and  $\gamma \geq 1$ .

Proof of Theorem 6 is omitted as it can be carried out in the line of Theorem 5.

**THEOREM 7.** *Let  $f$  be a transcendental meromorphic function and  $h$  be a transcendental entire function with regular  $(a, p)$  growth where  $a > 1$ . Also let  $F = f^\alpha Q[f]$ ,  $H = h^\gamma Q[h]$  where  $Q[f]$  and  $Q[h]$  are differential polynomials in  $f$  and  $h$  respectively,  $g$  be an entire function with finite  $(m, q)$ -th order with  $m > q$  and  $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ . If  $h$  satisfies the Property (A), then*

$$\lim_{r \rightarrow \infty} \frac{\log^{[p+m-q]} T_h^{-1} (T_{f \circ g} (r))}{\log^{[p]} T_H^{-1} (T_F (r))} \leq \frac{\rho^{(m,q)}(g)}{\lambda_h^{(p,q)}(f)}$$

where  $\alpha \geq 1$  and  $\gamma \geq 1$ .

*Proof.* Since  $q < m$ , we get from(21) and in view of Lemma 6, for all sufficiently large positive numbers of  $r$  that

$$\begin{aligned} & \log^{[p+m-q]} T_h^{-1} (T_{f \circ g} (r)) \leq \log^{[m]} M_g (r) + O(1) \\ \text{i.e., } & \frac{\log^{[p+m-q]} T_h^{-1} (T_{f \circ g} (r))}{\log^{[p]} T_H^{-1} (T_F (r))} \leq \frac{\log^{[m]} M_g (r) + O(1)}{\log^{[q]} r} \cdot \frac{\log^{[q]} r}{\log^{[p]} T_H^{-1} (T_F (r))} \\ \text{i.e., } & \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p+m-q]} T_h^{-1} (T_{f \circ g} (r))}{\log^{[p]} T_H^{-1} (T_F (r))} \leq \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[m]} M_g (r) + O(1)}{\log^{[q]} r} \cdot \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[q]} r}{\log^{[p]} T_H^{-1} (T_F (r))} \\ \text{i.e., } & \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p+m-q]} T_h^{-1} (T_{f \circ g} (r))}{\log^{[p]} T_H^{-1} (T_F (r))} \leq \rho^{(m,q)} (g) \cdot \frac{1}{\lambda_H^{(p,q)} (F)} = \frac{\rho^{(m,q)} (g)}{\lambda_h^{(p,q)} (f)}. \end{aligned}$$

This proves the theorem. □

In the line of Theorem 7 we may state the following theorem without proof.

**THEOREM 8.** *Let  $g$  and  $h$  be any two transcendental entire functions where  $h$  is of regular  $(m, p)$  growth. Also let  $f$  be a meromorphic function,  $G = g^\beta Q [g]$ ,  $H = h^\gamma Q [h]$  where  $Q [g]$  and  $Q [h]$  are differential polynomials in  $g$  and  $h$  respectively,  $\rho_h^{(p,q)} (f) < \infty$ ,  $\lambda_h^{(p,n)} (g) > 0$  and  $\rho^{(m,n)} (g) < \infty$  where  $m > q$ . If  $h$  satisfies the Property (A), then*

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p+m-q]} T_h^{-1} (T_{f \circ g} (r))}{\log^{[p]} T_H^{-1} (T_G (r))} \leq \frac{\rho^{(m,n)} (g)}{\lambda_h^{(p,n)} (g)}$$

where  $\beta \geq 1$  and  $\gamma \geq 1$ .

**THEOREM 9.** *Let  $f$  be a transcendental meromorphic function and  $h$  be a transcendental entire function with regular  $(a, p)$  growth where  $a > 1$ . Also let  $F = f^\alpha Q [f]$ ,  $H = h^\gamma Q [h]$  where  $Q [f]$  and  $Q [h]$  are differential polynomials in  $f$  and  $h$  respectively,  $g$  be an entire function with finite  $(m, n)$ -th lower order and  $0 < \lambda_h^{(p,q)} (f) \leq \rho_h^{(p,q)} (f) < \infty$ . If  $h$  satisfies the Property (A), then*

$$\begin{aligned} (i) \quad & \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_H^{-1} (T_F(\exp^{[q]} (r^A)))}{\log^{[p]} T_h^{-1} (T_{f \circ g} (\exp^{[n]} r))} = \infty \text{ if } q \geq m \text{ and } A > 1, \\ (ii) \quad & \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_H^{-1} (T_F(\exp^{[q]} (r^A)))}{\log^{[p]} T_h^{-1} (T_{f \circ g} (\exp^{[n-1]} r))} = \infty \text{ if } q \geq m \end{aligned}$$

or  $m \neq 1$ ,  $q = m - 1$  and  $\lambda^{(m,n)} (g) < A$

and

$$(iii) \quad \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_H^{-1} (T_F(\exp^{[q]} (r^A)))}{\log^{[p+m-q-1]} T_h^{-1} (T_{f \circ g}(\exp^{[n-1]} r))} = \infty \text{ if } m > q + 1 \text{ and}$$

$$A > \lambda^{(m,n)}(g)$$

where  $\alpha \geq 1$  and  $\gamma \geq 1$ .

*Proof.* From the definition of  $\lambda_{W(h)}^{(p,q)}(W(f))$  and in view of Lemma 6, we obtain for arbitrary positive  $\varepsilon (> 0)$  and for all sufficiently large positive numbers of  $r$  that

$$\log^{[p]} T_H^{-1} (T_F(\exp^{[q]} (r^A))) \geq \left( \lambda_H^{(p,q)}(F) - \varepsilon \right) r^A$$

(29) *i.e.*,  $\log^{[p]} T_H^{-1} (T_F(\exp^{[q]} (r^A))) \geq \left( \lambda_h^{(p,q)}(f) - \varepsilon \right) r^A$ .

Also from the definition of  $(m, n)$ -th lower order of  $g$ , we get for a sequence of positive numbers of  $r$  tending to infinity that

$$\log^{[m]} M_g (\exp^{[n-1]} r) \leq (\lambda^{(m,n)}(g) + \varepsilon) \log^{[n]}(\exp^{[n-1]} r)$$

*i.e.*,  $\log^{[m]} M_g (\exp^{[n-1]} r) \leq (\lambda^{(m,n)}(g) + \varepsilon) \log r$

(30) *i.e.*,  $\log^{[m]} M_g (\exp^{[n-1]} r) \leq \log r^{\lambda^{(m,n)}(g) + \varepsilon}$

(31) *i.e.*,  $\log^{[m-1]} M_g (\exp^{[n-1]} r) \leq r^{\lambda^{(m,n)}(g) + \varepsilon}$ .

**Case I.** Let  $q \geq m$ . Then it follows from (16) for a sequence of positive numbers of  $r$  tending to infinity that

$$\log^{[p]} T_h^{-1} (T_{f \circ g} (\exp^{[n]} r)) \leq \left( \rho_h^{(p,q)}(f) + \varepsilon \right) \log^{[q]} M_g(\exp^{[n]} r) + O(1)$$

*i.e.*,  $\log^{[p]} T_h^{-1} (T_{f \circ g} (\exp^{[n]} r)) \leq \left( \rho_h^{(p,q)}(f) + \varepsilon \right) \log^{[m]} M_g(\exp^{[n]} r) + O(1)$

(32)

*i.e.*,  $\log^{[p]} T_h^{-1} (T_{f \circ g} (\exp^{[n]} r)) \leq$

$$\left( \rho_h^{(p,q)}(f) + \varepsilon \right) (\lambda^{(m,n)}(g) + \varepsilon) r + O(1).$$

**Case II.** Also let  $q \geq m$  or  $m \neq 1, q = m - 1$ . Then also we obtain from (31) and (16) for a sequence of positive numbers of  $r$  tending to infinity that

*i.e.*,  $\log^{[p]} T_h^{-1} (T_{f \circ g} (\exp^{[n-1]} r)) \leq \left( \rho_h^{(p,q)}(f) + \varepsilon \right) \log^{[q]} M_g(\exp^{[n-1]} r) + O(1)$

*i.e.*,  $\log^{[p]} T_h^{-1} (T_{f \circ g} (\exp^{[n-1]} r)) \leq \left( \rho_h^{(p,q)}(f) + \varepsilon \right) \log^{[m-1]} M_g(\exp^{[n-1]} r) + O(1)$

$$(33) \text{ i.e., } \log^{[p]} T_h^{-1} \left( T_{f \circ g} \left( \exp^{[n-1]} r \right) \right) \leq \left( \rho_h^{(p,q)}(f) + \varepsilon \right) r^{\lambda^{(m,n)}(g) + \varepsilon} + O(1).$$

**Case III.** Again let  $m > q + 1$ . Then we get from (30) and (16) for a sequence of positive numbers of  $r$  tending to infinity that

$$\begin{aligned} & \log^{[p]} T_h^{-1} \left( T_{f \circ g} \left( \exp^{[n-1]} r \right) \right) \leq \left( \rho_h^{(p,q)}(f) + \varepsilon \right) \log^{[q]} M_g \left( \exp^{[n-1]} r \right) + O(1) \\ \text{i.e., } & \log^{[p]} T_h^{-1} \left( T_{f \circ g} \left( \exp^{[n-1]} r \right) \right) \leq \left( \rho_h^{(p,q)}(f) + \varepsilon \right) \log^{[q-m]} \log^{[m]} M_g \left( \exp^{[n-1]} r \right) + O(1) \\ \text{i.e., } & \log^{[p]} T_h^{-1} \left( T_{f \circ g} \left( \exp^{[n-1]} r \right) \right) \leq \left( \rho_h^{(p,q)}(f) + \varepsilon \right) \exp^{[m-q]} \log^{[m]} M_g \left( \exp^{[n-1]} r \right) + O(1) \\ \text{i.e., } & \log^{[p]} T_h^{-1} \left( T_{f \circ g} \left( \exp^{[n-1]} r \right) \right) \leq \left( \rho_h^{(p,q)}(f) + \varepsilon \right) \exp^{[m-q]} \log r^{\lambda^{(m,n)}(g) + \varepsilon} + O(1) \\ \text{i.e., } & \log^{[p]} T_h^{-1} \left( T_{f \circ g} \left( \exp^{[n-1]} r \right) \right) \leq \left( \rho_h^{(p,q)}(f) + \varepsilon \right) \exp^{[m-q-1]} r^{\lambda^{(m,n)}(g) + \varepsilon} + O(1) \\ (34) \quad & \text{i.e., } \log^{[p+m-q-1]} T_h^{-1} \left( T_{f \circ g} \left( \exp^{[n-1]} r \right) \right) \leq r^{\lambda^{(m,n)}(g) + \varepsilon} + O(1). \end{aligned}$$

Now if  $q \geq m$  and  $A > 1$ , we get from (29) and (32) of Case I for a sequence of positive numbers of  $r$  tending to infinity that

$$\frac{\log^{[p]} T_H^{-1} \left( T_F \left( \exp^{[q]} \left( r^A \right) \right) \right)}{\log^{[p]} T_h^{-1} \left( T_{f \circ g} \left( \exp^{[n]} r \right) \right)} \geq \frac{\left( \lambda_h^{(p,q)}(f) - \varepsilon \right) r^A}{\left( \rho_h^{(p,q)}(f) + \varepsilon \right) \left( \lambda^{(m,n)}(g) + \varepsilon \right) r + O(1)},$$

from which the first part of the theorem follows.

Again combining (29) and (33) of Case II we obtain for a sequence of positive numbers of  $r$  tending to infinity when  $q \geq m$  or  $m \neq 1, q = m - 1$

$$(35) \quad \frac{\log^{[p]} T_H^{-1} \left( T_F \left( \exp^{[q]} \left( r^A \right) \right) \right)}{\log^{[p]} T_h^{-1} \left( T_{f \circ g} \left( \exp^{[n-1]} r \right) \right)} \geq \frac{\left( \lambda_h^{(p,q)}(f) - \varepsilon \right) r^A}{\left( \rho_h^{(p,q)}(f) + \varepsilon \right) r^{\lambda^{(m,n)}(g) + \varepsilon} + O(1)}.$$

As  $\lambda^{(m,n)}(g) < A$  we can choose  $\varepsilon (> 0)$  in such a way that

$$(36) \quad \lambda^{(m,n)}(g) + \varepsilon < A.$$

Thus from (35) and (36) we get that

$$\lim_{r \rightarrow \infty} \frac{\log^{[p]} T_H^{-1} \left( T_F \left( \exp^{[q]} \left( r^A \right) \right) \right)}{\log^{[p]} T_h^{-1} \left( T_{f \circ g} \left( \exp^{[n-1]} r \right) \right)} = \infty.$$

This establishes the second part of the theorem .

When  $m > q + 1$  and  $A > \lambda^{(m,n)}(g)$ , it follows from (29) and (34) of Case III for a sequence of positive numbers of  $r$  tending to infinity that

$$(37) \quad \frac{\log^{[p]} T_H^{-1} (T_F(\exp^{[q]}(r^A)))}{\log^{[p+m-q-1]} T_h^{-1} (T_{f \circ g}(\exp^{[n-1]} r))} \geq \frac{(\lambda_h^{(p,q)}(f) - \varepsilon) r^A}{r^{(\lambda^{(m,n)}(g) + \varepsilon)} + O(1)}.$$

Now from (36) and (37) we obtain that

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_H^{-1} (T_F(\exp^{[q]}(r^A)))}{\log^{[p+m-q-1]} T_h^{-1} (T_{f \circ g}(\exp^{[n-1]} r))} = \infty.$$

This proves the third part of the theorem.

Thus the theorem follows. □

In the line of Theorem 9 we may state the following theorem without proof.

**THEOREM 10.** *Let  $g$  be a transcendental entire function and  $h$  be a transcendental entire function with regular  $(m, p)$  growth where  $m > 1$ . Also let  $f$  be a meromorphic function,  $G = g^\beta Q[g]$ ,  $H = h^\gamma Q[h]$  where  $Q[g]$  and  $Q[h]$  are differential polynomials in  $g$  and  $h$  respectively,  $\rho_h^{(p,q)}(f)$  is finite,  $\lambda_h^{(p,n)}(g) > 0$  and  $\lambda^{(m,n)}(g) < \infty$ . If  $h$  satisfies the Property (A), then*

$$(i) \quad \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_H^{-1} (T_G(\exp^{[n]}(r^A)))}{\log^{[p]} T_h^{-1} (T_{f \circ g}(\exp^{[n]} r))} = \infty \text{ if } q \geq m \text{ and } A > 1,$$

$$(ii) \quad \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_H^{-1} (T_G(\exp^{[n]}(r^A)))}{\log^{[p]} T_h^{-1} (T_{f \circ g}(\exp^{[n-1]} r))} = \infty \text{ if } q \geq m$$

or  $m \neq 1$ ,  $q = m - 1$  and  $\lambda^{(m,n)}(g) < A$

and

$$(iii) \quad \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_H^{-1} (T_G(\exp^{[n]}(r^A)))}{\log^{[p+m-q-1]} T_h^{-1} (T_{f \circ g}(\exp^{[n-1]} r))} = \infty \text{ if } m > q + 1 \text{ and}$$

$$A > \lambda^{(m,n)}(g),$$

where  $\beta \geq 1$  and  $\gamma \geq 1$ .

**THEOREM 11.** *Let  $f$  be a transcendental meromorphic function and  $h$  be a transcendental entire function with regular  $(a, p)$  growth where  $a > 1$ . Also let  $F = f^\alpha Q[f]$ ,  $H = h^\gamma Q[h]$  where  $Q[f]$  and  $Q[h]$  are differential polynomials in  $f$  and  $h$  respectively,  $g$  be an entire function*

with finite  $(m, n)$ -th order and  $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ . If  $h$  satisfies the Property (A), then

$$(i) \quad \lim_{r \rightarrow \infty} \frac{\log^{[p]} T_H^{-1} (T_F(\exp^{[q]}(r^A)))}{\log^{[p]} T_h^{-1} (T_{f \circ g}(\exp^{[n]} r))} = \infty \text{ if } q \geq m \text{ and } A > 1,$$

$$(ii) \quad \lim_{r \rightarrow \infty} \frac{\log^{[p]} T_H^{-1} (T_F(\exp^{[q]}(r^A)))}{\log^{[p]} T_h^{-1} (T_{f \circ g}(\exp^{[n-1]} r))} = \infty \text{ if } q \geq m$$

or  $m \neq 1, q = m - 1$  and  $\rho^{(m,n)}(g) < A$

and

$$(iii) \quad \lim_{r \rightarrow \infty} \frac{\log^{[p]} T_H^{-1} (T_F(\exp^{[q]}(r^A)))}{\log^{[p+m-q-1]} T_h^{-1} (T_{f \circ g}(\exp^{[n-1]} r))} = \infty \text{ if } m > q + 1 \text{ and } A > \rho^{(m,n)}(g),$$

where  $\alpha \geq 1$  and  $\gamma \geq 1$ .

**THEOREM 12.** Let  $g$  be a transcendental entire function and  $h$  be a transcendental entire function with regular  $(m, p)$  growth where  $m > 1$ . Also let  $f$  be a meromorphic function,  $G = g^\beta Q[g], H = h^\gamma Q[h]$  where  $Q[g]$  and  $Q[h]$  are differential polynomials in  $g$  and  $h$  respectively,  $\rho_h^{(p,q)}(f)$  is finite,  $\lambda_h^{(p,n)}(g) > 0$  and  $\rho^{(m,n)}(g) < \infty$  where  $q, n$  are all positive integers with  $m \geq n$ . If  $h$  satisfies the Property (A), then

$$(i) \quad \lim_{r \rightarrow \infty} \frac{\log^{[p]} T_H^{-1} (T_G(\exp^{[n]}(r^A)))}{\log^{[p]} T_h^{-1} (T_{f \circ g}(\exp^{[n]} r))} = \infty \text{ if } q \geq m \text{ and } A > 1,$$

$$(ii) \quad \lim_{r \rightarrow \infty} \frac{\log^{[p]} T_H^{-1} (T_G(\exp^{[n]}(r^A)))}{\log^{[p]} T_h^{-1} (T_{f \circ g}(\exp^{[n-1]} r))} = \infty \text{ if } q \geq m$$

or  $m \neq 1, q = m - 1$  and  $\rho^{(m,n)}(g) < A$

and

$$(iii) \quad \lim_{r \rightarrow \infty} \frac{\log^{[p]} T_H^{-1} (T_G(\exp^{[n]}(r^A)))}{\log^{[p+m-q-1]} T_h^{-1} (T_{f \circ g}(\exp^{[n-1]} r))} = \infty \text{ if } m > q + 1 \text{ and } A > \rho^{(m,n)}(g),$$

where  $\beta \geq 1$  and  $\gamma \geq 1$ .

We omit the proof of Theorem 11 and Theorem 12 as those can be carried out in the line of Theorem 9 and Theorem 10 respectively.

**THEOREM 13.** *Let  $f$  be a transcendental meromorphic function and  $h$  be a transcendental entire function with regular  $(a, p)$  growth such that  $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$  where  $a > 1$ . Also let  $g$  be an entire function with non zero finite order,  $F = f^\alpha Q[f]$ ,  $H = h^\gamma Q[h]$  where  $Q[f]$  and  $Q[h]$  are differential polynomials in  $f$  and  $h$  respectively. Then for every positive constant  $A$  and every real number  $\alpha$ ,*

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_h^{-1} (T_{f \circ g}(\exp^{[n-1]} r))}{\left\{ \log^{[p]} T_H^{-1} (T_F(r^A)) \right\}^{1+\alpha}} = \infty,$$

where  $\alpha \geq 1$  and  $\gamma \geq 1$ .

*Proof.* If  $\alpha$  be such that  $1 + \alpha \leq 0$  then the theorem is trivial. So we suppose that  $1 + \alpha > 0$ . Now from the definition of  $\rho_H^{(p,q)}(F)$  and in view of Lemma 6, it follows for all sufficiently large positive numbers of  $r$  that

$$\log^{[p]} T_H^{-1} T_F(r^A) \leq \left( \rho_H^{(p,q)}(F) + \varepsilon \right) \log^{[q]} r + O(1)$$

$$i.e., \log^{[p]} T_H^{-1} T_F(r^A) \leq \left( \rho_h^{(p,q)}(f) + \varepsilon \right) \log^{[q]} r + O(1)$$

$$\log^{[p]} T_H^{-1} T_F(r^A) \leq \left( \rho_h^{(p,q)}(f) + \varepsilon \right) \log^{[q]} r \left( 1 + \frac{O(1)}{\left( \rho_h^{(p,q)}(f) + \varepsilon \right) \log^{[q]} r} \right)$$

$$(38) \quad i.e., \left\{ \log^{[p]} T_H^{-1} T_F(r^A) \right\}^{1+\alpha} \leq$$

$$\left( \rho_h^{(p,q)}(f) + \varepsilon \right)^{1+\alpha} \left( \log^{[q]} r \right)^{1+\alpha} \left( 1 + \frac{O(1)}{\left( \rho_h^{(p,q)}(f) + \varepsilon \right) \log^{[q]} r} \right)^{[1+\alpha]}.$$

Now from Lemma 2 we get for a sequence of positive numbers of  $r$  tending to infinity that

$$(39) \quad \log^{[p]} T_h^{-1} T_{f \circ g}(\exp^{[n-1]} r) \geq \left( \lambda_h^{(p,q)}(f) - \varepsilon \right) \log^{[q-1]} (\exp^{[n-1]} r)^\mu.$$

Now from (38) and (39) we have for a sequence of positive numbers of  $r$  tending to infinity that

$$\frac{\log^{[p]} T_h^{-1} T_{f \circ g}(\exp^{[n-1]} r)}{\left\{ \log^{[p]} T_H^{-1} T_F(r^A) \right\}^{1+\alpha}} \geq \frac{\left( \lambda_h^{(p,q)}(f) - \varepsilon \right) \log^{[q-1]} (\exp^{[n-1]} r)^\mu}{\left( \rho_h^{(p,q)}(f) + \varepsilon \right)^{1+\alpha} \left( \log^{[q]} r \right)^{1+\alpha} \left( 1 + \frac{O(1)}{\left( \rho_h^{(p,q)}(f) + \varepsilon \right) \log^{[q]} r} \right)^{[1+\alpha]}}.$$

Since  $\frac{\log^{[q-1]}(\exp^{[n-1]} r)^\mu}{(\log^{[q]} r)^{1+\alpha}} \rightarrow \infty$  as  $r \rightarrow \infty$ , then the theorem follows from above. □

**THEOREM 14.** *Let  $f$  be a transcendental meromorphic function,  $g$  be an entire function and  $h$  be a transcendental entire function with regular  $(a, p)$ -growth such that  $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$  and  $\sigma^{(m,n)}(g) < \infty$  where  $a > 1$  and  $q = m - 1$ . Also let  $F = f^\alpha Q[f]$  and  $H = h^\gamma Q[h]$  where  $Q[f]$  and  $Q[h]$  are differential polynomials in  $f$  and  $h$  respectively. If  $h$  satisfies the Property (A), then*

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p]} T_H^{-1}\left(T_F\left(\exp^{[q]} \left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right)} \leq \frac{\sigma^{(m,n)}(g) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)}$$

where  $\alpha \geq 1$  and  $\gamma \geq 1$ .

*Proof.* Since  $q = m - 1$ , we get from (16) for all sufficiently large positive numbers of  $r$  that

$$\begin{aligned} \text{i.e., } \log^{[p]} T_h^{-1}(T_{f \circ g}(r)) &\leq \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[m-1]} M_g(r) + O(1) \\ (40) \text{ i.e., } \log^{[p]} T_h^{-1}(T_{f \circ g}(r)) &\leq \\ &\left(\rho_h^{(p,q)}(f) + \varepsilon\right) \left(\sigma^{(m,n)}(g) + \varepsilon\right) \left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)} + O(1) . \end{aligned}$$

Now from the definition of  $\lambda_H^{(p,q)}(F)$  and in view of Lemma 6, we obtain for all sufficiently large positive numbers of  $r$  that

$$\begin{aligned} \log^{[p]} T_H^{-1}\left(T_F\left(\exp^{[q]} \left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right) &\geq \left(\lambda_H^{(p,q)}(F) - \varepsilon\right) \left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)} \\ (41) \text{ i.e., } \log^{[p]} T_H^{-1}\left(T_F\left(\exp^{[q]} \left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right) &\geq \\ &\geq \left(\lambda_h^{(p,q)}(f) - \varepsilon\right) \left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)} . \end{aligned}$$

Therefore from (40) and (41), it follows for all sufficiently large positive numbers of  $r$  that

$$\frac{\log^{[p]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p]} T_H^{-1}\left(T_F\left(\exp^{[q]} \left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right)} \leq$$



$$\begin{aligned}
 & \frac{(\rho_h^{(p,q)}(f) + \varepsilon) (\sigma^{(m,n)}(g) + \varepsilon) (\log^{[n-1]} r)^{\rho^{(m,n)}(g)} + O(1)}{(\lambda_h^{(p,q)}(f) - \varepsilon) (\log^{[n-1]} r)^{\rho^{(m,n)}(g)}} \\
 \text{i.e., } \overline{\lim}_{r \rightarrow \infty} & \frac{\log^{[p]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p]} T_H^{-1} \left( T_F \left( \exp^{[q]} (\log^{[n-1]} r)^{\rho^{(m,n)}(g)} \right) \right)} \leq \frac{\sigma^{(m,n)}(g) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)}.
 \end{aligned}$$

Thus the theorem is established. □

REMARK 5. In Theorem 14, if we replace “ $\sigma^{(m,n)}(g)$ ” with “ $\overline{\sigma}^{(m,n)}(g)$ ”, then Theorem 14 remains valid with “limit inferior” replaced by “limit superior”.

Now we state the following theorem without its proof as it can easily be carried out in the line of Theorem 14.

THEOREM 15. *Let  $f$  be meromorphic,  $g, h$  be any two transcendental entire functions where  $h$  is of regular  $(m, p)$  growth such that  $\lambda_h^{(p,n)}(g) > 0$ ,  $\rho_h^{(p,q)}(f) < \infty$  and  $\sigma^{(m,n)}(g) < \infty$  where  $m > 1$  and  $q = m - 1$ . Also let  $G = g^\beta Q[g]$  and  $H = h^\gamma Q[h]$  where  $Q[g]$  and  $Q[h]$  are differential polynomials in  $g$  and  $h$  respectively. If  $h$  satisfies the Property (A), then*

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p]} T_H^{-1} \left( T_G \left( \exp^{[n]} (\log^{[n-1]} r)^{\rho^{(m,n)}(g)} \right) \right)} \leq \frac{\sigma^{(m,n)}(g) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)},$$

where  $\beta \geq 1$  and  $\gamma \geq 1$ .

REMARK 6. In Theorem 15, if we replace “ $\sigma^{(m,n)}(g)$ ” with “ $\overline{\sigma}^{(m,n)}(g)$ ”, then Theorem 15 remains valid with “limit inferior” replaced by “limit superior”.

REMARK 7. We remark that in Theorem 15, if we will replace the condition “ $\rho_h^{(p,q)}(f) < \infty$ ” by “ $\lambda_h^{(p,q)}(f) < \infty$ ”, then

$$\begin{aligned}
 (42) \quad \overline{\lim}_{r \rightarrow \infty} & \frac{\log^{[p]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p]} T_H^{-1} \left( T_G \left( \exp^{[n]} (\log^{[n-1]} r)^{\rho^{(m,n)}(g)} \right) \right)} \\
 & \frac{\sigma^{(m,n)}(g) \cdot \lambda_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)}.
 \end{aligned}$$

REMARK 8. In Remark 7, if we replace the conditions “ $\lambda_h^{(p,n)}(g) > 0$  and  $\lambda_h^{(p,q)}(f) < \infty$ ” with “ $\rho_h^{(p,n)}(g) > 0$  and  $\rho_h^{(p,q)}(f) < \infty$ ” respectively, then we need to go the same replacement in right part of (42).

Using the concept of the growth indicator  $\bar{\tau}^{(m,n)}(g)$  of an entire function  $g$ , we may state the subsequent two theorems without their proofs since those can be carried out in the line of Theorem 14 and Theorem 15 respectively.

THEOREM 16. Let  $f$  be a transcendental meromorphic function,  $g$  be an entire function and  $h$  be a transcendental entire function with regular  $(a, p)$  growth such that  $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$  and  $\bar{\tau}_g(m, n) < \infty$  where  $a > 1$  and  $q = m - 1$ . Also let  $F = f^\alpha Q[f]$  and  $H = h^\gamma Q[h]$  where  $Q[f]$  and  $Q[h]$  are differential polynomials in  $f$  and  $h$  respectively. If  $h$  satisfies the Property (A), then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p]} T_H^{-1}\left(T_F\left(\exp^{[q]}(\log^{[n-1]} r)^{\lambda^{(m,n)}(g)}\right)\right)} \leq \frac{\bar{\tau}^{(m,n)}(g) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)},$$

where  $\alpha \geq 1$  and  $\gamma \geq 1$ .

REMARK 9. We remark that in Theorem 16, if we will replace the condition “ $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$  and  $\bar{\tau}_g(m, n) < \infty$ ” by “ $0 < \lambda_h^{(p,q)}(f) < \infty$  or  $0 < \rho_h^{(p,q)}(f) < \infty$  and  $\sigma^{(m,n)}(g) < \infty$ ”, then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p]} T_H^{-1}\left(T_F\left(\exp^{[q]}(\log^{[n-1]} r)^{\rho^{(m,n)}(g)}\right)\right)} \leq \sigma^{(m,n)}(g),$$

THEOREM 17. Let  $f$  be meromorphic,  $g, h$  be any two transcendental entire functions where  $h$  is of regular  $(m, p)$  growth such that  $\lambda_h^{(p,n)}(g) > 0$ ,  $\rho_h^{(p,q)}(f) < \infty$  and  $\bar{\tau}^{(m,n)}(g) < \infty$  where  $m > 1$  and  $q = m - 1$ . Also let  $G = g^\beta Q[g]$  and  $H = h^\gamma Q[h]$  where  $Q[g]$  and  $Q[h]$  are differential polynomials in  $g$  and  $h$  respectively. If  $h$  satisfies the Property (A), then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log^{[p]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p]} T_H^{-1}\left(T_G\left(\exp^{[n]}(\log^{[n-1]} r)^{\lambda^{(m,n)}(g)}\right)\right)} \leq \frac{\bar{\tau}^{(m,n)}(g) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)},$$

where  $\beta \geq 1$  and  $\gamma \geq 1$ .

Further using the notion of  $(p, q)$ -th weak type we may also state the following two theorems without proof because it can be carried out in the line of Theorem 14 and Theorem 15 respectively.

**THEOREM 18.** *Let  $f$  be a transcendental meromorphic function,  $g$  be an entire function and  $h$  be a transcendental entire function with regular  $(a, p)$  growth such that  $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$  and  $\tau^{(m,n)}(g) < \infty$  where  $a > 1$  and  $q = m - 1$ . Also let  $F = f^\alpha Q[f]$  and  $H = h^\gamma Q[h]$  where  $Q[f]$  and  $Q[h]$  are differential polynomials in  $f$  and  $h$  respectively. If  $h$  satisfies the Property (A), then*

$$\liminf_{r \rightarrow \infty} \frac{\log^{[p]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p]} T_H^{-1}\left(T_F\left(\exp^{[q]}(\log^{[n-1]} r)^{\lambda^{(m,n)}(g)}\right)\right)} \leq \frac{\tau^{(m,n)}(g) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)},$$

where  $\alpha \geq 1$  and  $\gamma \geq 1$ .

**REMARK 10.** We remark that in Theorem 18, if we will replace the condition “ $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$  and  $\tau^{(m,n)}(g) < \infty$ ” by “ $0 < \lambda_h^{(p,q)}(f) < \infty$  or  $0 < \rho_h^{(p,q)}(f) < \infty$  and  $\bar{\tau}^{(m,n)}(g) < \infty$ ”, then

$$\liminf_{r \rightarrow \infty} \frac{\log^{[p]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p]} T_H^{-1}\left(T_F\left(\exp^{[q]}(\log^{[n-1]} r)^{\lambda^{(m,n)}(g)}\right)\right)} \leq \bar{\tau}^{(m,n)}(g),$$

**THEOREM 19.** *Let  $f$  be meromorphic,  $g, h$  be any two transcendental entire functions where  $h$  is of regular  $(m, p)$  growth such that  $\lambda_h^{(p,n)}(g) > 0$ ,  $\rho_h^{(p,q)}(f) < \infty$  and  $\tau^{(m,n)}(g) < \infty$  where  $m > 1$  and  $q = m - 1$ . Also let  $G = g^\beta Q[g]$  and  $H = h^\gamma Q[h]$  where  $Q[g]$  and  $Q[h]$  are differential polynomials in  $g$  and  $h$  respectively. If  $h$  satisfies the Property (A), then*

$$\liminf_{r \rightarrow \infty} \frac{\log^{[p]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p]} T_H^{-1}\left(T_G\left(\exp^{[n]}(\log^{[n-1]} r)^{\lambda^{(m,n)}(g)}\right)\right)} \leq \frac{\tau^{(m,n)}(g) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)},$$

where  $\beta \geq 1$  and  $\gamma \geq 1$ .

**REMARK 11.** We remark that in Theorem 19, if we will replace the condition “ $\rho_h^{(p,q)}(f) < \infty$  and  $\tau^{(m,n)}(g) < \infty$ ” by “ $\lambda_h^{(p,q)}(f) < \infty$  and

$\bar{\tau}^{(m,n)}(g) < \infty$  , then

$$(43) \quad \lim_{r \rightarrow \infty} \frac{\log^{[p]} T_h^{-1}(T_{f \circ g}(r))}{\log^{[p]} T_H^{-1} \left( T_G \left( \exp^{[n]} \left( \log^{[n-1]} r \right)^{\lambda^{(m,n)}(g)} \right) \right)} \leq \frac{\bar{\tau}^{(m,n)}(g) \cdot \lambda_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)}.$$

REMARK 12. In Remark 11, if we will replace the conditions “ $\lambda_h^{(p,n)}(g) > 0$  and  $\lambda_h^{(p,q)}(f) < \infty$ ” by “ $\rho_h^{(p,n)}(g) > 0$  and  $\rho_h^{(p,q)}(f) < \infty$ ” respectively, then is need to go the same replacement in right part of (43).

#### Acknowledgment

The author is extremely grateful to the anonymous learned referee for his keen reading, valuable suggestion and constructive comments for the improvement of the paper.

#### References

- [1] L. Bernal-González, *Crecimiento relativo de funciones enteras. Aportaciones al estudio de las funciones enteras con índice exponencial finito*, Doctoral Thesis, 1984, Universidad de Sevilla, Spain.
- [2] L. Bernal, *Orden relative de crecimiento de funciones enteras* , Collect. Math., **39** (1988), 209–229.
- [3] W. Bergweiler, *On the Nevanlinna characteristic of a composite function*, Complex Variables Theory Appl. **10** (2-3) (1988), 225–236.
- [4] W. Bergweiler, *On the growth rate of composite meromorphic functions*, Complex Variables Theory Appl. **14** (1-4) (1990), 187–196.
- [5] S. S. Bhooshnurmath and K. S. L. N. Prasad, *The value distribution of some differential polynomials*, Bull. Cal. Math. Soc **101** (1) (2009), 55–62.
- [6] L. Debnath, S. K. Datta, T. Biswas and A. Kar, *Growth of meromorphic functions depending on (p,q)-th relative order*, Facta Univ. Ser. Math. Inform. **31** (3) (2016), 691–705.
- [7] S. K. Datta, T. Biswas and C. Biswas, *Measure of growth ratios of composite entire and meromorphic functions with a focus on relative order*, Int. J. Math. Sci. Eng. Appl. **8** (IV) (July, 2014), 207–218.
- [8] W.K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford (1964).
- [9] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, *On the (p,q)-order and lower (p,q)-order of an entire function*, J. Reine Angew. Math., **282** (1976), 53–67.
- [10] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, *On the (p,q)-type and lower (p,q)-type of an entire function*, J. Reine Angew. Math., **290** (1977), 180–190.

- [11] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, De Gruyter, Berlin, 1993.
- [12] L. M. S. Ruiz, S. K. Datta, T. Biswas and G. K. Mondal, *On the  $(p,q)$ -th relative order oriented growth properties of entire functions*, Abstract and Applied Analysis, Vol.2014, Article ID 826137, 8 pages, <http://dx.doi.org/10.1155/2014/826137>.
- [13] X. Shen, J. Tu and H. Y. Xu, *Complex oscillation of a second-order linear differential equation with entire coefficients of  $[p, q] - \varphi$  order*, Adv. Difference Equ. 2014,2014: 200, 14 pages, <http://www.advancesindifferenceequations.com/content/2014/1/200>.
- [14] D. Sato, *On the rate of growth of entire functions of fast growth*, Bull. Amer. Math. Soc. **69** (1963), 411–414.
- [15] C. C. Yang and H. X. Yi, *Uniqueness theory of meromorphic functions*, Mathematics and its Applications, 557. Kluwer Academic Publishers Group, Dordrecht, 2003.
- [16] L. Yang, *Value distribution theory*, Springer-Verlag, Berlin, 1993.
- [17] G. Valiron, *Lectures on the General Theory of Integral Functions*, Chelsea Publishing Company, (1949).

### Tanmay Biswas

Independent Researcher

Rajbari, Rabindrapalli, R. N. Tagore Road

P.O.-Krishnagar, Dist-Nadia, PIN- 741101, West Bengal, India..

*E-mail*: [tanmaybiswas\\_math@rediffmail.com](mailto:tanmaybiswas_math@rediffmail.com)