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MATRIX TRANSFORMATIONS AND COMPACT OPERATORS ON THE BINOMIAL SEQUENCE SPACES

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Abstract. In this work, we characterize some matrix classes concerning the Binomial sequence spaces $b_\infty^{r,s}$ and $b_p^{r,s}$, where $1 \leq p < \infty$. Moreover, by using the notion of Hausdorff measure of noncompactness, we characterize the class of compact matrix operators from $b_0^{r,s}$, $b_c^{r,s}$ and $b_\infty^{r,s}$ into c_0, c and ℓ_∞, respectively.

1. The Basic Notions And Notations

The family of all real(or complex) valued sequences is a vector space under point-wise addition and scalar multiplication. This space is denoted by w and every vector subspace of w is called a sequence space. The spaces of all bounded, null, convergent and absolutely p-summable sequences are symbolized with ℓ_∞, c_0, c and ℓ_p , respectively, where $p \in [1, \infty)$.

A sequence space X with a linear topology is called a K-space provided each of the maps $p_n : X \longrightarrow \mathbb{C}$ defined by $p_n(x) = x_n$ is continuous for all $n \in \mathbb{N}$. A Banach sequence space which has the property of K-space is called a BK-space [7]. A BK-space X that contains the set of all finite sequences is said to have AK property, if $x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)} \longrightarrow x$ $(as\ m \to \infty)$ for all $x = (x_k) \in X$, where $e^{(k)} = (0, 0, ..., 1, 0, 0, ...)$ with 1 in place k and 0 elsewhere [13].

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The classical sequence spaces ℓ_∞, c_0 and c are BK-spaces according to their *sup-norm* defined by $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$ and ℓ_p is a $BK-$space with respect to its ℓ_p-*norm* defined by

$$\|x\|_{\ell_p} = \Big(\sum_{k=0}^{\infty} |x_k|^p\Big)^{\frac{1}{p}}$$

where $1 \leq p < \infty$.

Let $x \in w$ and $A = (a_{nk})$ be an infinite matrix with $a_{nk} \in \mathbb{C}$, for each $n, k \in \mathbb{N}$. Then,

$$Ax = ((Ax)_n) = \left(\sum_{k=0}^{\infty} a_{nk}x_k\right)_{n=0}^{\infty} \tag{1}$$

is called the A-transform of x and the series $\sum_{k=0}^{\infty} a_{nk}x_k$ is assumed to be convergent for all $n \in \mathbb{N}$. Also, the sequence in the n-th row of A is denoted by $A_n = \{a_{nk}\}_{k=0}^{\infty}$ for all $n \in \mathbb{N}$. For simplicity of notation, we prefer throughout the paper that the summation without limits runs from 0 to ∞.

Let X be an arbitrary sequence space and $A = (a_{nk})$ be an infinite matrix. Then the set defined by

$$X_A = \Big\{x = (x_k) \in w : Ax \in X\Big\} \tag{2}$$

is called the domain of A and X_A is also a sequence space. The notation of $(X : Y)$ stands for the class of all infinite matrices A such that $X \subset Y_A$. The spaces of all bounded and convergent series are defined by means of the matrix domain of the summation matrix $S = (s_{nk})$ such that $bs = (\ell_\infty)_S$ and $cs = c_S$, respectively, where $S = (s_{nk})$ is defined by

$$s_{nk} = \begin{cases} 1 & , \quad 0 \leq k \leq n \\ 0 & , \quad k > n \end{cases}$$

for all $n, k \in \mathbb{N}$. For an infinite matrix $A = (a_{nk})$, we use the term of triangle, if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n, k \in \mathbb{N}$. A triangle matrix A has an inverse A^{-1} which is unique and also a triangle matrix.

Let X and Y be normed spaces. Then, the set of all bounded linear operators on X into Y is denoted by $B(X : Y)$. Also, the sets $S_X = \{x \in X : \|x\| = 1\}$ and $B_X = \{x \in X : \|x\| \leq 1\}$ are called the unit sphere and closed unit ball in X, respectively.

Let X be a BK-space containing the set of all finite sequences and $a \in w$. Then, we write $\|a\|_X^* = \sup_{x \in B_X} \left| \sum_{k=0}^{\infty} a_k x_k \right|$.

A linear operator $L : X \longrightarrow Y$ is said to be compact if and only if for each bounded sequence (x_n) in X the sequence $(L(x_n))$ has a subsequence such that the subsequence is convergent in Y. We denote the class of such operators by $C(X, Y)$ [13].

A functional $L : \ell_\infty \longrightarrow \mathbb{R}$ is called a Banach Limit, if the following conditions hold.

(i) $L(ax_n + by_n) = aL(x_n) + bL(y_n) \ \ a, b \in \mathbb{R}$

(ii) $L(x_n) \geq 0$ if $x_n \geq 0, \ \ n = 0, 1, 2, ...$

(iii) $L(P^j(x_n)) = L(x_n), \ \ P^j(x_n) = x_{n+j}, \ \ j = 1, 2, 3, ...$

(iv) $L(e) = 1$ where $e = (1, 1, ...)$

A bounded sequence $x = (x_n)$ is called almost convergent to the generalized limit λ if $L(x_n) = \lambda$ hold for every Banach Limit L [12]. Here λ is called $f - limit$ of (x_n) and denoted by $f - lim x_n = \lambda$.

The theory of matrix transformations is of great interest in the study of sequence spaces and was first motivated by Cesàro, Riesz Norlund, ... So, many authors have defined new sequence spaces by the aid of the matrix transformations. For example, $\tilde{c}$ and $\tilde{c_0}$ in [26], X_p and X_∞ in [22], $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$ in [11], e_0^r and e_c^r in [1], e_p^r and e_∞^r in [2] and [16], $e_0^r(\Delta), e_c^r(\Delta)$ and $e_\infty^r(\Delta)$ in [3], $e_0^r(\Delta^{(m)}), e_c^r(\Delta^{(m)})$ and $e_\infty^r(\Delta^{(m)})$ in [23]. Moreover, some authors have investigated compactness property by using the notion of Hausdorff measure of noncompactness in [17], [18], [19], [20] and [21].

In this work, we characterize some matrix classes concerning the Binomial sequence spaces $b_\infty^{r,s}$ and $b_p^{r,s}$, where $1 \leq p < \infty$. Moreover, by using the notion of Hausdorff measure of noncompactness, we characterize the class of compact matrix operators from $b_0^{r,s}$, $b_c^{r,s}$ and $b_\infty^{r,s}$ into c_0, c and ℓ_∞, respectively.

2. Some Matrix Classes Related To The Binomial Sequence Spaces

In this chapter, we remind some informations about the Binomial sequence spaces, almost convergent sequences space and almost convergent series space. Also, we charecterize some matrix classes concerning the Binomial sequence spaces $b_\infty^{r,s}$ and $b_p^{r,s}$, where $1 \leq p < \infty$.

The Binomial sequence spaces $b_0^{r,s}$, $b_c^{r,s}$, $b_\infty^{r,s}$ and $b_p^{r,s}$ have been defined in [5] and [6] as follows:

$$b_0^{r,s} = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k = 0 \right\},$$

$$b_c^{r,s} = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \ exists \right\},$$

$$b_\infty^{r,s} = \left\{ x = (x_k) \in w : \sup_{n\in\mathbb{N}} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right| < \infty \right\}$$

and

$$b_p^{r,s} = \left\{ x = (x_k) \in w : \sum_n \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right|^p < \infty \right\}$$

where $1 \le p < \infty$. Given a sequence $x = (x_k)$, the $B^{r,s}$-transform of $x = (x_k)$ is defined by

$$(B^{r,s}x)_k = y_k = \frac{1}{(s+r)^k} \sum_{j=0}^{k} \binom{k}{j} s^{k-j} r^j x_j \tag{3}$$

for all $k \in \mathbb{N}$, where the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ is defined by

$$b_{nk}^{r,s} = \begin{cases} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k & , \ 0 \le k \le n \\ 0 & , \ k > n \end{cases}$$

for all $k, n \in \mathbb{N}_0$.

Lorentz proved that $f - lim x_n = \lambda \Leftrightarrow \lim\limits_{i\to\infty} \sum\limits_{k=0}^{i} \frac{x_{n+k}}{i+1} = \lambda$ uniformly in n [12]. By using this fact, the space of all almost convergent sequences and almost convergent series are defined by

$$f = \left\{ x = (x_k) \in w : \exists \lambda \in \mathbb{C} \ni \lim_{i\to\infty} \sum_{k=0}^{i} \frac{x_{n+k}}{i+1} = \lambda \ uniformly \ in \ n \right\}$$

and

$$fs = \left\{ x = (x_k) \in w : \exists \lambda \in \mathbb{C} \ni \lim_{i\to\infty} \sum_{k=0}^{i} \sum_{j=0}^{n+k} \frac{x_j}{i+1} = \lambda \ uniformly \ in \ n \right\}$$

respectively.

Now, we give some lemmas which are needed in the proof of next theorems.

LEMMA 2.1 (see [4]). *Let X, Y be any two sequence spaces, A be an infinite matrix and B be a triangle matrix. Then, $A \in (X : Y_B)$ if and only if $BA \in (X : Y)$.*

LEMMA 2.2 (see [25]). *$A \in (\ell_1 : \ell_p)$ if and only if*

$$\sup_{k\in\mathbb{N}} \sum_n |a_{nk}|^p < \infty$$

where $1 \leq p < \infty$.

LEMMA 2.3 (see [6]). *Let $v_3^{r,s}$, $v_4^{r,s}$ and $v_6^{r,s}$ be defined as follows:*

$$v_3^{r,s} = \left\{ a = (a_k) \in w : \sum_{j=k}^{\infty} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \text{ exists for each } k \in \mathbb{N} \right\},$$

$$v_4^{r,s} = \left\{ a = (a_k) \in w : \sup_{n,k\in\mathbb{N}} \left| \sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \right| < \infty \right\}$$

and

$$v_6^{r,s} = \left\{ a = (a_k) \in w : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \right|^q < \infty \right\}, 1 < q < \infty.$$

Then, $\{b_1^{r,s}\}^\beta = v_3^{r,s} \cap v_4^{r,s}$ and $\{b_p^{r,s}\}^\beta = v_3^{r,s} \cap v_6^{r,s}$, where $1 < p < \infty$.

For simplicity of notation, we prefer to use the following equality.

$$t_{nk}^{r,s} = \sum_{j=k}^{\infty} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_{nj}$$

for all $n, k \in \mathbb{N}$.

THEOREM 2.4. *Let $A = (a_{nk})$ be an infinite matrix, then the followings hold:*

(i) $A \in (b_1^{r,s} : \ell_\infty)$ if and only if

$$\sup_{k,n\in\mathbb{N}} |t_{nk}^{r,s}| < \infty. \tag{4}$$

(ii) $A \in (b_p^{r,s} : \ell_\infty)$ *if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |t_{nk}^{r,s}|^q < \infty, \tag{5}$$

$$\{a_{nk}\}_{k\in\mathbb{N}} \in v_6^{r,s} \ (n \in \mathbb{N}) \tag{6}$$

where $1 < p < \infty$.

(iii) $A \in (b_\infty^{r,s} : \ell_\infty)$ *if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |t_{nk}^{r,s}| < \infty, \tag{7}$$

$$\lim_{m\to\infty} \sum_k \left| \sum_{j=k}^{m} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_{nj} \right| = \sum_k |t_{nk}^{r,s}| \ , \ (n \in \mathbb{N}) \tag{8}$$

Proof. We give the proof of theorem for only the case $1 < p < \infty$. The other parts of theorem can be proved by using a similar way.

Let us take any $x = (x_k) \in b_p^{r,s}$. We assume that the conditions (5) and (6) hold. Then $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{b_p^{r,s}\right\}^\beta$ for all $n \in \mathbb{N}$. So, A-transform of x exists.

By taking into account the relation (3), we can write the following equality:

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} \sum_{j=k}^{m} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_{nj} y_k \tag{9}$$

If we take limit (9) side by side as $m \to \infty$, we have

$$\sum_k a_{nk} x_k = \sum_k t_{nk}^{r,s} y_k \ (n \in \mathbb{N}) \tag{10}$$

By taking ℓ_∞-norm (10) side by side and applying Hölder's inequality respectively, we obtain that

$$\begin{aligned} \|Ax\|_\infty &= \sup_{n\in\mathbb{N}} \left| \sum_k a_{nk} x_k \right| \\ &\le \sup_{n\in\mathbb{N}} \left(\sum_k |t_{nk}^{r,s}|^q \right)^{\frac{1}{q}} \left(\sum_k |y_k|^p \right)^{\frac{1}{p}} < \infty \end{aligned}$$

This shows us that $A \in \ell_\infty$, that is $A \in (b_p^{r,s} : \ell_\infty)$.

Conversely, we assume that $A \in (b_p^{r,s} : \ell_\infty)$. This leads us to $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{b_p^{r,s}\right\}^\beta$ for all $n \in \mathbb{N}$. Then $\{t_{nk}^{r,s}\}_{n,k\in\mathbb{N}}$ exists and the condition (6) hods. Also, from the same reason, we deduce that the condition (10) holds and the sequences $a_n = (a_{nk})_{k\in\mathbb{N}}$ define the continuous linear functionals f_n on $b_p^{r,s}$ such that

$$f_n(x) = \sum_k a_{nk}x_k$$

for all $n \in \mathbb{N}$. Moreover, from Theorem 2.2 in [6], we know that $b_p^{r,s}$ and ℓ_p are norm isomorphic. If we combine this fact and the condition (10), we conclude that

$$\|f_n\| = \|(t_{nk}^{r,s})_{k\in\mathbb{N}}\|_q$$

This result says us that the functionals f_n are pointwise bounded. Therefore, by taking into account the Banach-Steinhaus theorem, one can says that the functionals f_n are uniformly bounded, that is there exists a constant $M > 0$ such that

$$\left(\sum_k |t_{nk}^{r,s}|^q\right)^{\frac{1}{q}} = \|f_n\| \leq M$$

for all $n \in \mathbb{N}$. As a consequence, the condition (5) hods. This step completes the proof. $\square$

THEOREM 2.5. *Let $A = (a_{nk})$ be an infinite matrix. Then, $A \in (b_1^{r,s} : \ell_p)$ if and only if*

$$\sup_{k\in\mathbb{N}} \sum_n |t_{nk}^{r,s}|^p < \infty \tag{11}$$

where $1 \leq p < \infty$.

Proof. For a given sequence $x = (x_k) \in b_1^{r,s}$, we suppose that the condition (11) holds. Then, $y = (y_k) \in \ell_1$ and $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b_1^{r,s}\}^\beta$ for all $n \in \mathbb{N}$. Therefore the A-transform of x exists. Then, it is clear that the series $\sum_k t_{nk}^{r,s} y_k$ are absolutely convergent for all fixed $n \in \mathbb{N}$ and every $y = (y_k) \in \ell_1$. If we apply the Minkowski inequality to (10), we obtain

$$\left(\sum_n |(Ax)_n|^p\right)^{\frac{1}{p}} \leq \sum_k |y_k| \left(\sum_n |t_{nk}^{r,s}|^p\right)^{\frac{1}{p}} < \infty.$$

Hence $Ax \in \ell_p$, that is $A \in (b_1^{r,s} : \ell_p)$.

On the contrary, assume that $A \in (b_1^{r,s} : \ell_p)$, where $1 \leq p < \infty$. This leads us to $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b_1^{r,s}\}^\beta$ for all $n \in \mathbb{N}$. Then the condition (10) holds. So, it is clear that $T^{r,s} = (t_{nk}^{r,s}) \in (\ell_1 : \ell_p)$. If we this fact and Lemma 2.2, we conclude that the condition (11) holds. This step completes the proof. $\square$

THEOREM 2.6. *Let $A = (a_{nk})$ be an infinite matrix. Then the following statements hold.*

(i) $A \in (b_1^{r,s} : f)$ if and only if (4) holds, and the limit

$$f - \lim t_{nk}^{r,s} = \lambda_k \ , \ \forall k \in \mathbb{N} \tag{12}$$

exists for each $k \in \mathbb{N}$,

(ii) $A \in (b_p^{r,s} : f)$ if and only if (5), (6) and (12) hold, where $1 < p < \infty$.

(iii) $A \in (b_\infty^{r,s} : f)$ if and only if (7), (8) and (12) hold, and

$$\lim_{m\to\infty} \sum_k \left| \frac{1}{m+1} \sum_{j=0}^{m} t_{n+j,k}^{r,s} - \lambda_k \right| = 0 \ uniformly\ in\ n.$$

Proof. The proof of theorem is given for only the case $1 < p < \infty$. The other parts of theorem can be proved by using a similar way.

For a given $x = (x_k) \in b_p^{r,s}$, we suppose that the conditions (5), (6) and (12) hold. Then A-transform of x exists. By considering the condition (12), we write that

$$\left| \frac{1}{m+1} \sum_{j=0}^{m} t_{n+j,k}^{r,s} \right|^q \longrightarrow |\lambda_k|^q$$

as $m \to \infty$, uniformly in n for all $k \in \mathbb{N}$. By using this fact and (5), we obtain that the inequality

$$\begin{aligned} \sum_{j=0}^{k} |\lambda_j|^q &= \lim_{m\to\infty} \sum_{j=0}^{k} \left| \frac{1}{m+1} \sum_{i=0}^{m} t_{n+i,j}^{r,s} \right|^q \ (uniformly\ in\ n) \\ &\leq \sup_{n,m\in\mathbb{N}} \sum_j \left| \frac{1}{m+1} \sum_{i=0}^{m} t_{n+i,j}^{r,s} \right|^q = M < \infty, \end{aligned}$$

holds for all $k \in \mathbb{N}$. Therefore, $(\lambda_k) \in \ell_q$. Furthermore, from Theorem 2.2 in [6], it is known that $b_p^{r,s} \cong \ell_p$. Thus, $y = (y_k) \in \ell_p$ whenever

$x = (x_k) \in b_p^{r,s}$. Then, by applying the Hölder's inequality, we obtain that

$$\sum_k |\lambda_k y_k| \leq \left(\sum_k |\lambda_k|^q\right)^{\frac{1}{q}} \left(\sum_k |y_k|^p\right)^{\frac{1}{p}} < \infty,$$

that is $(\lambda_k y_k) \in \ell_1$. Since, $y = (y_k) \in \ell_p$, for all $\epsilon > 0$ there exists a fixed $k_0 \in \mathbb{N}$ such that

$$\left(\sum_{k=k_0+1}^{\infty} |y_k|^p\right)^{\frac{1}{p}} < \frac{\epsilon}{4M^{\frac{1}{q}}}.$$

So by considering the the condition (12), there exists some $m_0 \in \mathbb{N}$ such that

$$\left|\sum_{k=0}^{k_0} \left[\frac{1}{m+1}\sum_{j=0}^{m} t_{n+j,k}^{r,s} - \lambda_k\right] y_k\right| < \frac{\epsilon}{2}$$

for all $m \geq m_0$, uniformly in n. By combining these two facts and applying the Hölder's inequality, we obtain that

$$\begin{aligned}
&\left|\frac{1}{m+1}\sum_{i=0}^{m}(Ax)_{n+i} - \sum_k \lambda_k y_k\right| \\
&= \left|\sum_k \left[\frac{1}{m+1}\sum_{j=0}^{m} t_{n+j,k}^{r,s} - \lambda_k\right] y_k\right| \\
&\leq \left|\sum_{k=0}^{k_0} \left[\frac{1}{m+1}\sum_{j=0}^{m} t_{n+j,k}^{r,s} - \lambda_k\right] y_k\right| \\
&+ \left|\sum_{k=k_0+1}^{\infty} \left[\frac{1}{m+1}\sum_{j=0}^{m} t_{n+j,k}^{r,s} - \lambda_k\right] y_k\right| \\
&< \frac{\epsilon}{2} + \left(\sum_{k=k_0+1}^{\infty} \left[\left|\frac{1}{m+1}\sum_{j=0}^{m} t_{n+j,k}^{r,s}\right| + |\lambda_k|\right]^q\right)^{\frac{1}{q}} \left(\sum_{k=k_0+1}^{\infty} |y_k|^p\right)^{\frac{1}{p}} \\
&< \frac{\epsilon}{2} + 2M^{\frac{1}{q}} \frac{\epsilon}{4M^1 q} = \epsilon
\end{aligned}$$

for all $m \geq m_0$, uniformly in n. This means that $Ax \in f$ and thereby $A \in (b_p^{r,s} : f)$.

On the contrary, suppose that $A \in (b_p^{r,s} : f)$. Then, by the aid of the inclusion $f \subset \ell_\infty$, we deduce that $A \in (b_p^{r,s} : \ell_\infty)$. If we consider the last result and Theorem 2.4, it is obvious that the conditions (5) and (6) hold.

Now, we consider the sequence $g^{(k)}(r,s) = \left\{g_n^{(k)}(r,s)\right\}_{n\in\mathbb{N}} \in b_p^{r,s}$ defined by

$$g_n^{(k)}(r,s) = \begin{cases} 0 & , \quad 0 \le n < k \\ \binom{n}{k}(-s)^{n-k}r^{-n}(s+r)^k & , \quad n \ge k \end{cases}$$

for all fixed $k \in \mathbb{N}$. Since $A \in (b_p^{r,s} : f)$, we have

$$Ag^{(k)}(r,s) = \left\{\sum_{j=k}^{\infty}\binom{j}{k}(-s)^{j-k}r^{-j}(r+s)^k a_{nj}\right\}_{n\in\mathbb{N}} \in f$$

for all $k \in \mathbb{N}$. As a consequence, the condition (12) holds. This step completes the proof. □

Let us substitute the ordinary limit instead of $f - limit$ in Theorem 2.6. Then, we can give the next corollary.

Corollary 2.7. *Given an infinite matrix $A = (a_{nk})$, the following statements hold.*

(i) $A \in (b_1^{r,s} : c)$ *if and only if* (4) *holds, and*

(13) $$\lim_{n\to\infty} t_{nk}^{r,s} = \lambda_k \ , \ \forall k \in \mathbb{N}$$

(ii) $A \in (b_p^{r,s} : c)$ *if and only if* (5), (6) *and* (13) *hold, where* $1 < p < \infty$.

(iii) $A \in (b_\infty^{r,s} : c)$ *if and only if* (7), (8) *and* (13) *hold, and*

$$\lim_{n\to\infty}\sum_k |t_{nk}^{r,s} - \lambda_k| = 0$$

Now, by using Lemma 2.1, Theorems 2.4, 2.5, 2.6 and Corollary 2.7, we can give the next corollaries.

Corollary 2.8. *Let us define a matrix $E^{u,v} = (e_{nk}^{u,v})$ via an infinite matrix $A = (a_{nk})$ as follows:*

$$e_{nk}^{u,v} = \frac{1}{(u+v)^n}\sum_{j=0}^{n}\binom{n}{j}v^{n-j}u^j a_{jk}$$

for all $n,k \in \mathbb{N}$, where $u,v \in \mathbb{R}$ and $uv > 0$. Then, the necessary and sufficient conditions in order that A belongs to any of the classes $(b_1^{r,s} : b_\infty^{u,v})$, $(b_p^{r,s} : b_\infty^{u,v})$, $(b_\infty^{r,s} : b_\infty^{u,v})$, $(b_1^{r,s} : b_p^{u,v})$ and $(b_p^{r,s} : b_c^{u,v})$ are *obtained by taking* $E^{u,v} = (e^{u,v}_{nk})$ *instead of* $A = (a_{nk})$ *in the required ones Theorems 2.4, 2.5 and Corollary 2.7.*

Corollary 2.9. *Let us define a matrix* $D = (d_{nk})$ *via an infinite matrix* $A = (a_{nk})$ *as follows:*

$$d_{nk} = \frac{1}{n+1}\sum_{j=0}^{n} a_{jk}$$

for all $n, k \in \mathbb{N}$. *Then, the necessary and sufficient conditions in order that* A *belongs to any of the classes* $(b_1^{r,s} : X_\infty)$, $(b_p^{r,s} : X_\infty)$, $(b_\infty^{r,s} : X_\infty)$, $(b_1^{r,s} : X_p)$ *and* $(b_p^{r,s} : \tilde{c})$ *are obtained by taking* $D = (d_{nk})$ *instead of* $A = (a_{nk})$ *in the required ones Theorems 2.4, 2.5 and Corollary 2.7, where* X_∞, X_p *and* $\tilde{c}$ *are defined in [22] and [26], respectively.*

Corollary 2.10. *Let us define two matrices* $H = (h_{nk})$ *and* $C = (c_{nk})$ *via an infinite matrix* $A = (a_{nk})$ *as follows:*

$$h_{nk} = a_{nk} - a_{n+1,k} \text{ and } c_{nk} = a_{nk} - a_{n-1,k}$$

for all $n, k \in \mathbb{N}$. *Then, the necessary and sufficient conditions in order that* A *belongs to any of the classes* $(b_1^{r,s} : \ell_\infty(\Delta))$, $(b_p^{r,s} : \ell_\infty(\Delta))$, $(b_\infty^{r,s} : \ell_\infty(\Delta))$, $(b_1^{r,s} : c(\Delta))$, $(b_p^{r,s} : c(\Delta))$, $(b_\infty^{r,s} : c(\Delta))$ *and* $(b_1^{r,s} : \ell_p(\Delta))$ *are obtained by taking* $H = (h_{nk})$ *or* $C = (c_{nk})$ *instead of* $A = (a_{nk})$ *in the required ones Theorems 2.4, 2.5 and Corollary 2.7, where* $\ell_\infty(\Delta)$ *and* $c(\Delta)$ *are defined in [11] and* $\ell_p(\Delta)$ *is defined in [4].*

Corollary 2.11. *Let us define a matrix* $D = (d_{nk})$ *via an infinite matrix* $A = (a_{nk})$ *as follows:*

$$d_{nk} = \sum_{j=0}^{n} a_{jk}$$

for all $n, k \in \mathbb{N}$. *Then, the necessary and sufficient conditions in order that* A *belongs to any of the classes* $(b_1^{r,s} : cs)$, $(b_p^{r,s} : cs)$, $(b_\infty^{r,s} : cs)$, $(b_1^{r,s} : bs)$, $(b_p^{r,s} : bs)$, $(b_\infty^{r,s} : bs)$, $(b_1^{r,s} : fs)$, $(b_p^{r,s} : fs)$ *and* $(b_\infty^{r,s} : fs)$ *are obtained by taking* $D = (d_{nk})$ *instead of* $A = (a_{nk})$ *in the required ones Theorems 2.4, 2.5 and Corollary 2.7.*

3. Compact Operators On The Binomial Sequence Spaces

In this chapter, we characterize the class of compact matrix operators from $b_0^{r,s}$, $b_c^{r,s}$ and $b_\infty^{r,s}$ into c_0, c and ℓ_∞, respectively by using the notion of Hausdorff measure of noncompactness.

We assume throughout the chapter that the matrices $D^{r,s} = (d_{nk}^{r,s})$, $H = (h_{nk})$, $U^{(a)} = (u_{nk}^{(a)})$, $V = (v_{nk})$ and $T^{r,s} = (t_{nk}^{r,s})$ are defined as follows:

$$d_{nk}^{r,s} = \begin{cases} \binom{n}{k}(-s)^{n-k}r^{-n}(s+r)^k & , \quad 0 \leq k \leq n \\ 0 & , \quad k > n \end{cases}$$

$H = (D^{r,s})^t$, the transpose of $D^{r,s} = (d_{nk}^{r,s})$,

$$u_{nk}^{(a)} = \begin{cases} \sum\limits_{j=n}^{\infty} a_j d_{jk}^{r,s} & , \quad 0 \leq k \leq n \\ 0 & , \quad k > n \end{cases},$$

$V = HA_n$ and

$$t_{nk}^{r,s} = \sum_{j=k}^{\infty} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_{nj}$$

for all $n, k \in \mathbb{N}$ and $a = (a_k) \in w$.

LEMMA 3.1 (see [14], Theorem 3.2).

(a) *Let X be a BK-space which has AK-property or $X = \ell_\infty$. Then, $a = (a_k) \in \{X_{B^{r,s}}\}^\beta$ if and only if $a = (a_k) \in (X^\beta)_H$ and $U^{(a)} \in (X : c_0)$.*
Also, if $a = (a_k) \in \{X_{B^{r,s}}\}^\beta$, then we write

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} (H_k a)(B_k^{r,s} x)$$

for all $x \in X_{B^{r,s}}$.

(b) *$a = (a_k) \in \{c_{B^{r,s}}\}^\beta$ if and only if $a = (a_k) \in (\ell_1)_H$ and $U^{(a)} \in (c : c)$*
Also, if $a = (a_k) \in \{c_{B^{r,s}}\}^\beta$, then we write

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} (H_k a)(B_k^{r,s} x) - \mu\lambda$$

for all $x \in c_{B^{r,s}}$, where $\mu = \lim\limits_{k\to\infty} B_k^{r,s} x$ and $\lambda = \lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} u_{nk}^{(a)}$.

LEMMA 3.2 (see [14], Theorem 3.4).

(a) *Let X be a BK-space which has AK-property or $X = \ell_\infty$ and Y be an arbitrary subset of w. Then $A \in (X_{B^{r,s}} : Y)$ if and only if $V \in (X : Y)$ and $U^{(A_n)} \in (X : c_0)$ for all $n \in \mathbb{N}$.*
Also, if $A \in (X_{B^{r,s}} : Y)$, then we write
$$Ax = V(B^{r,s}x)$$
for all $x \in X_{B^{r,s}}$.

(b) *Let Y be a linear subspace of w. Then, $A \in (c_{B^{r,s}} : Y)$ if and only if*
$$V \in (c_0 : Y),\ U^{(A_n)} \in (c : c)$$
for all $n \in \mathbb{N}$ and
$$Ve - (\lambda_n) \in Y$$
where $\lambda_n = \lim_{m\to\infty} \sum_{k=0}^{m} u_{nk}^{(A_n)}$ for all $n \in \mathbb{N}$.
Also, if $A \in (c_{B^{r,s}} : Y)$, then we write
$$Ax = V(B^{r,s}x) - \mu(\lambda_n)$$
for all $x \in c_{B^{r,s}}$, where $\mu = \lim_{k\to\infty} B_k^{r,s}x$.

Proposition 3.3 (see [8], Proposition 3.2).

(a) *If $X \in \{c_0,\ \ell_\infty\}$, then we write*
$$\|a\|^*_{X_{B^{r,s}}} = \|Ha\|_1 \tag{14}$$
for all $a = (a_k) \in \{X_{B^{r,s}}\}^\beta$.

(b) *We write*
$$\|a\|^*_{c_{B^{r,s}}} = \|Ha\|_1 + |\lambda| \tag{15}$$
for all $a = (a_k) \in \{c_{B^{r,s}}\}^\beta$, $\lambda = \lim_{n\to\infty} \sum_{k=0}^{n} u_{nk}^{(a)}$.

Definition 3.4 (see [15], Definition 2.10). Let (X, d) be a metric space and $Q \in \mathcal{M}_X$, where
$$\mathcal{M}_X = \{Q : Q \subset X \text{ and } Q \text{ bounded}\}$$
Then, the Hausdorff measure of noncompactness of Q is defined by
$$\chi(Q) = \inf\left\{\epsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, \delta_i),\ x_i \in X,\ \delta_i < \epsilon\ (i = 1, 2, ..., n)\ n \in \mathbb{N}\right\}$$

where $B(x_i, \delta_i) = \{y \in X : d(x_i, y) < \delta_i\}$. Here, the function χ is called the Hausdorff measure of noncompactness.

Definition 3.5 (see [15], Definition 2.24). Let χ_1 and χ_2 be Hausdorff measure of noncompactness defined on the Banach spaces X and Y, respectively. The operator $L : X \longrightarrow Y$ is called (χ_1, χ_2)-bounded if $L(Q) \in \mathcal{M}_Y$ and there exists a positive constant k such that $\chi_2(L(Q)) \leq k\chi_1(Q)$ for all $Q \in \mathcal{M}_X$. If an operator L is (χ_1, χ_2)-bounded, the (χ_1, χ_2)-measure of noncompactness of L is defined by

$$\|L\|_{(\chi_1,\chi_2)} = \inf\{k > 0 : \chi_2(L(Q)) \leq k\chi_1(Q) \ for \ every \ Q \in \mathcal{M}_X\}.$$

For brevity of notation , If $\chi_1 = \chi_2 = \chi$, then $\|L\|_{(\chi,\chi)} = \|L\|_\chi$.

Lemma 3.6. *Let X and Y be BK-spaces.*

(i) For all $A \in (X : Y)$, there exists a $L_A \in B(X, Y)$ such that $L_A(x) = Ax (x \in X)$, namely $(X : Y) \subset B(X, Y)$ (see [15], Theorem 1.23).

(ii) If X has the property AK, then every $L \in B(X, Y)$ is given by a matrix $A \in (X : Y)$ such that $Ax = L(x)$ $(x \in X)$, namely $B(X, Y) \subset (X : Y)$ (see [10], Theorem 1.9).

Lemma 3.7 (see [15], Theorem 1.23). *Let X be a BK-space and $Y \in \{\ell_\infty, c_0, c\}$. Then, if $A \in (X : Y)$, we have*

$$\|L_A\| = \|A\|_{(X,\infty)} = \sup_{n\in\mathbb{N}} \|A_n\|^*_X < \infty$$

Lemma 3.8 (see [15], Theorem 2.25 and Corollary 2.26). *Let X and Y be Banach spaces and $L \in B(X, Y)$. Then, we write*

$$\|L\|_\chi = \chi(L(B_X)) = \chi(L(S_X)) \tag{16}$$

$$\|L\|_\chi = 0 \ if \ and \ only \ if \ L \in C(X, Y) \tag{17}$$

$$\|L\|_\chi \leq \|L\| \tag{18}$$

Lemma 3.9 (Goldenštein, Gohberg, Markus [15], Theorem 2.23). *Let X be a Banach space which has a Schauder basis $\{e_1, e_2, ...\}$, $Q \in \mathcal{M}_X$ and $P_n : X \longrightarrow X$ be the projector onto linear span of $\{e_1, e_2, ...\}$. Then, the following inequality holds.*

$$\frac{1}{\sigma} \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I - P_n)(x)\| \right) \leq \chi(Q) \leq \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I - P_n)(x)\| \right)$$

where $\sigma = \limsup_{n\to\infty} \|I - P_n\|$.

LEMMA 3.10 (see [24], Theorem 2.8). *Let* $X \in \{\ell_p, c_0\}$, *where* $1 \leq p < \infty$ *and* $Q \in \mathcal{M}_X$. *For an operator* $P_n : X \longrightarrow X$ *defined by* $P_n(x) = x^{[n]}$ $(x \in X)$ *the following statement holds.*

$$\chi(Q) = \lim_{n\to\infty}\left(\sup_{x\in Q}\|(I-P_n)(x)\|\right)$$

THEOREM 3.11. *Let* $X \in \{b_0^{r,s}, b_\infty^{r,s}\}$. *Then the following statements hold.*

(i) If $A \in (X : c_0)$, *then*

$$\|L_A\|_\chi = \lim_{m\to\infty}\|T_{[m]}^{r,s}\|_{(X,\infty)} \tag{19}$$

(ii) If $A \in (X : \ell_\infty)$, *then*

$$0 \leq \|L_A\|_\chi \leq \lim_{m\to\infty}\|T_{[m]}^{r,s}\|_{(X,\infty)} \tag{20}$$

where $T_{[m]}^{r,s} = \sup_{n>m}\left(\sum\limits_{k=0}^{\infty}|t_{nk}^{r,s}|\right)$ *for all* $m \in \mathbb{N}$.

Proof. It is obvious that the limits in (19) and (20) exist.

(i) If we apply Lemmas 3.8 and 3.10, we obtain

$$\|L\|_\chi = \chi(L(B_X)) = \lim_{m\to\infty}\left(\sup_{x\in B_X}\|(I-P_m)(Ax)\|\right) \tag{21}$$

where the projector $P_m : c_0 \longrightarrow c_0$ is defined by $P_m(x) = x^{[m]}$ for $x = (x_k) \in c_0$ and $m \in \mathbb{N}_0$.

Let $A^{[m]} = (a_{nk}^m)_{n,k=0}^\infty$ be defined as follows:

$$a_{nk}^m = \begin{cases} 0 & , \quad 0 \leq n \leq m \\ a_{nk} & , \quad n > m \end{cases}$$

for all $n, k, m \in \mathbb{N}$. Then, since $A^{[m]} \in (X : c_0)$ and hence $A_n^{[m]} \in X^\beta$, by taking into account Lemmas 3.7 and 3.1 and (14) in Proposition 3.3, we obtain

$$\|A_n^{[m]}\|_X^* = \|HA_n^{[m]}\|_1 = \sum_{k=0}^\infty |H_kA_n^{[m]}| = \sum_{k=0}^\infty\sum_{j=k}^\infty d_{jk}^{r,s}a_{nj}^m.$$

By considering the definition $H = (h_{nk})$, we write

$$v_{nk} = H_kA_n = \sum_{j=k}^\infty a_{nj}\binom{j}{k}(-s)^{j-k}r^{-j}(r+s)^k$$

for all $k \in \mathbb{N}_0$. Therefore, we write

$$\|A_n^{[m]}\|_X^* = \sum_{k=0}^{\infty} |t_{nk}^{r,s}|$$

for $n > m$. As a consequence, we deduce that

(22)
$$\sup_{x \in B_X} \|(I-P_m)(Ax)\| = \|L_{A^{[m]}}\| = \sup_{n>m} \|A_n^{[m]}\|_X^* = \sup_{n>m} \left(\sum_{k=0}^{\infty} |t_{nk}^{r,s}| \right) = \|T_{[m]}^{r,s}\|_{(X,\infty)}$$

So (19) follows from (21) and (22).

(ii) Now, for $x = (x_k) \in \ell_\infty$ and $m \in \mathbb{N}_0$, we define the projector $P_m : \ell_\infty \longrightarrow \ell_\infty$ such that $P_m(x) = x^{[m]}$. We know that $L(B_X) \subset P_m(L(B_X)) + (I - P_m)(L(B_X))$. So if we combine this fact, elementary properties of the function χ(see [15], Theorem 2.12), the conditions (16) and (18) we obtain that

$$\begin{aligned} \chi(L(B_X)) &\leq \chi(P_m(L(B_X))) + \chi((I - P_m)(L(B_X))) \\ &= \chi((I - P_m)(L(B_X))) \\ &\leq \sup_{x \in B_X} \|(I - P_m)(Ax)\| \\ &= \|L_{A^{[m]}}\| \end{aligned}$$

Therefore, the condition (20) holds. This step completes the proof. □

If we combine Theorem 3.11 and the condition (17), we give next Corollary.

Corollary 3.12. *Let $X \in \{b_0^{r,s}, b_\infty^{r,s}\}$.*

(i) If $A \in (X : c_0)$, then L_A is compact if and only if

$$\lim_{m\to\infty} \left[\sup_{n>m} \left(\sum_{k=0}^{\infty} |t_{nk}^{r,s}| \right) \right] = 0 \tag{23}$$

(ii) If $A \in (X : \ell_\infty)$, then L_A is compact if the condition (23) holds.

Theorem 3.13. *The following statements hold.*

(i) If $A \in (b_c^{r,s} : c_0)$, then

$$\|L_A\|_\chi = \lim_{m\to\infty} \|T_{[m]}^{r,s}\|_{(X,\infty)} \tag{24}$$

(ii) If $A \in (b_c^{r,s} : \ell_\infty)$, then

$$0 \leq \|L_A\|_\chi \leq \lim_{m\to\infty} \|T_{[m]}^{r,s}\|_{(X,\infty)} \tag{25}$$

where $\|T_{[m]}^{r,s}\|_{(X,\infty)} = \sup_{n>m}\left(\sum_{k=0}^{\infty} |t_{nk}^{r,s}| + |\lambda_n|\right)$ *and* $\lambda_n = \lim_{m\to\infty} \sum_{k=0}^{m} u_{mk}^{(A_n)}$.

Proof. By using (15) instead of (14) in the proof of Theorem 3.11, the present theorem can be proved by using a similar way. □

If we combine Theorem 3.13 and the condition (17), we give next corollary.

Corollary 3.14.

(i) If $A \in (b_c^{r,s} : c_0)$, then L_A is compact if and only if

$$\lim_{m\to\infty}\left[\sup_{n>m}\left(\sum_{k=0}^{\infty} |t_{nk}^{r,s}| + |\lambda_n|\right)\right] = 0 \tag{26}$$

(ii) If $A \in (b_c^{r,s} : \ell_\infty)$, then L_A is compact if the condition (26) *holds.*

Here, we want to give two more results. For this, we need next proposition and the theorem.

Proposition 3.15 (see [9], Corollary 5.13). *Let $X \in \{\ell_\infty, c_0\}$. If $A \in (X_{B^{r,s}} : c)$, then the following inequality holds.*

$$\frac{1}{2}\lim_{m\to\infty}\left(\sup_{n\geq m}\|V_n - \xi\|_1\right) \leq \|L_A\|_\chi \leq \lim_{m\to\infty}\left(\sup_{n\geq m}\|V_n - \xi\|_1\right)$$

where the sequence $\xi = (\xi_k)$ is defined by $\xi_k = \lim_{n\to\infty} v_{nk}$ for all $k \in \mathbb{N}_0$.

Theorem 3.16 (see [9], Theorem 5.14). *Let $A \in (c_{B^{r,s}} : c)$. Then the following inequality holds.*

$$\frac{1}{2}\lim_{m\to\infty}\left[\sup_{n\geq m}\left(\sum_{k=0}^{\infty}|v_{nk} - \xi_k| + \left|\mu - \lambda_n - \sum_{k=0}^{\infty}\xi_k\right|\right)\right] \leq \|L_A\|_\chi$$
$$\leq \lim_{m\to\infty}\left[\sup_{n\geq m}\left(\sum_{k=0}^{\infty}|v_{nk} - \xi_k| + \left|\mu - \lambda_n - \sum_{k=0}^{\infty}\xi_k\right|\right)\right]$$

where $\mu = \lim_{n\to\infty}\left(\sum_{k=0}^{\infty} v_{nk} - \lambda_n\right)$.

THEOREM 3.17.

(i) Let $X \in \{b_0^{r,s}, b_\infty^{r,s}\}$. If $A \in (X : c)$, then the following inequality holds.

$$\frac{1}{2}\lim_{m\to\infty}\|T_{[m]}^{r,s} - \hat{\xi}\|_{(X,\infty)} = \lim_{m\to\infty}\left[\sup_{n\geq m}\left(\sum_{k=0}^{\infty}|t_{nk}^{r,s} - \hat{\xi}_k|\right)\right] \leq \|L_A\|_\chi$$
$$= \lim_{m\to\infty}\|T_{[m]}^{r,s} - \hat{\xi}\|_{(X,\infty)}$$

(ii) If $A \in (b_c^{r,s} : c)$, then the following inequality holds.

$$\frac{1}{2}\lim_{m\to\infty}\left[\sup_{n\geq m}\left(\sum_{k=0}^{\infty}|t_{nk}^{r,s} - \hat{\xi}_k| + \left|\hat{\mu} - \lambda_n - \sum_{k=0}^{\infty}\hat{\xi}_k\right|\right)\right] \leq \|L_A\|_\chi$$
$$\leq \lim_{m\to\infty}\left[\sup_{n\geq m}\left(\sum_{k=0}^{\infty}|t_{nk}^{r,s} - \hat{\xi}_k| + \left|\hat{\mu} - \lambda_n - \sum_{k=0}^{\infty}\hat{\xi}_k\right|\right)\right]$$

where the sequence $\hat{\xi} = (\hat{\xi}_k)$ is defined by $\hat{\xi}_k = \lim_{n\to\infty} t_{nk}^{r,s}$ for all $k \in \mathbb{N}$ and $\hat{\mu} = \lim_{n\to\infty}\left(\sum_{k=0}^{\infty} t_{nk}^{r,s} - \lambda_n\right)$.

Proof. If we consider Proposition 3.15 and Theorem 3.16, the proof of theorem is obvious. So we omit. □

If we combine Theorem 3.17 and the condition (17), we give next corollary.

COROLLARY 3.18.

(i) Let $X \in \{b_0^{r,s}, b_\infty^{r,s}\}$. If $A \in (X : c)$, then L_A is compact if and only if

$$\lim_{m\to\infty}\left[\sup_{n\geq m}\left(\sum_{k=0}^{\infty}|t_{nk}^{r,s} - \hat{\xi}_k|\right)\right] = 0$$

(ii) If $A \in (b_c^{r,s} : c)$, then L_A is compact if and only if

$$\lim_{m\to\infty}\left[\sup_{n\geq m}\left(\sum_{k=0}^{\infty}|t_{nk}^{r,s} - \hat{\xi}_k| + \left|\hat{\mu} - \lambda_n - \sum_{k=0}^{\infty}\hat{\xi}_k\right|\right)\right] = 0$$

4. Conclusion

By taking into account the definition of the Binomial matrix $B^{r,s} = (b^{r,s}_{nk})$, we can say that $B^{r,s} = (b^{r,s}_{nk})$ reduces in the case $r + s = 1$ to the $E^r = (e^r_{nk})$ which is called the method of Euler means of order r. Therefore, our results generalize the results obtained from the method of Euler means of order r. In addition, the Binomial matrix $B^{r,s} = (b^{r,s}_{nk})$ is not a special case of weighed mean matrices. Hence this work is filled up a gap in the existent literature.

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