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THE CHROMATIC POLYNOMIAL FOR CYCLE GRAPHS

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Abstract. Let $P(G, \lambda)$ denote the number of proper vertex colorings of G with λ colors. The chromatic polynomial $P(C_n, \lambda)$ for the cycle graph C_n is well-known as

$$P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$$

for all positive integers $n \geq 1$. Also its inductive proof is widely well-known by the *deletion-contraction recurrence*. In this paper, we give this inductive proof again and three other proofs of this formula of the chromatic polynomial for the cycle graph C_n.

1. Introduction

The number of proper colorings of a graph with finite colors was introduced only for planar graphs by George David Birkhoff [1] in 1912, in an attempt to prove the four color theorem, where the formula for this number was later called by the chromatic polynomial. In 1932, Hassler Whitney [3] generalized Birkhoff's formula from the planar graphs to general graphs. In 1968, Ronald Cedric Read [2] introduced the concept of chromatically equivalent graphs and asked which polynomials are the chromatic polynomials of some graph, that remains open.

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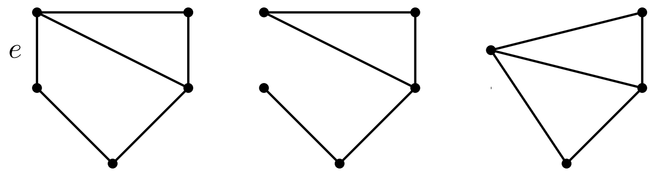


G $\quad$ $G - e$ $\quad$ G/e

FIGURE 1. G , $G - e$ and G/e

Chromatic polynomial. For a graph G, a *coloring* means almost always a *(proper) vertex coloring*, which is a labeling of vertices of G with colors such that no two adjacent vertices have the same colors. Let $P(G, \lambda)$ denote the number of (proper) vertex colorings of G with λ colors and $\chi(G)$ the least number λ satisfying $P(G, \lambda) > 0$, where $P(G, \lambda)$ and $\chi(G)$ are called a *chromatic polynomial* and *chromatic number* of G, respectively.

In fact, it is clear that the number of λ-colorings is a polynomial in λ from a deletion-contraction recurrence.

PROPOSITION 1 (Deletion-contraction recurrence). *For a given a graph G and an edge e in G, we have*

$$P(G, \lambda) = P(G - e, \lambda) - P(G/e, \lambda), \tag{1}$$

where $G - e$ is a graph obtained by deletion the edge e and G/e is a graph obtained by contraction the edge e.

EXAMPLE. The chromatic polynomials of graphs in Figure 1 are

$$\begin{aligned} P(G, \lambda) &= \lambda(\lambda - 1)^2(\lambda - 2), \\ P(G - e, \lambda) &= \lambda^2(\lambda - 1)(\lambda - 2), \textit{and} \\ P(G/e, \lambda) &= \lambda(\lambda - 1)(\lambda - 2). \end{aligned}$$

It is confirmed that (1) is true for the graph G and the edge e in Figure 1.

Cycle graph. A *cycle graph* C_n is a graph that consists of a single cycle of length n, which could be drown by a n-polygonal graph in a plane. The chromatic polynomial for cycle graph C_n is well-known as follows.

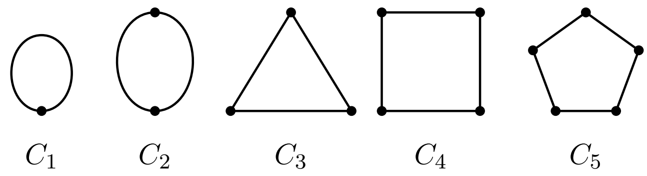


FIGURE 2. C_n $(1 \le n \le 5)$

THEOREM 2. *For a positive integer $n \ge 1$, the chromatic polynomial for cycle graph C_n is*

$$P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1) \tag{2}$$

EXAMPLE. For an integer $n \le 3$, it is easily checked that the chromatic polynomials of C_n are from (2) as follows.

$$\begin{aligned} P(C_1, \lambda) &= (\lambda - 1) + (-1)(\lambda - 1) = 0, \\ P(C_2, \lambda) &= (\lambda - 1)^2 + (-1)^2(\lambda - 1) = \lambda(\lambda - 1), \\ P(C_3, \lambda) &= (\lambda - 1)^3 + (-1)^3(\lambda - 1) = \lambda(\lambda - 1)(\lambda - 2). \end{aligned}$$

As shown in Figure 2, the cycle graph C_1 is a graph with one vertex and one loop and C_1 cannot be colored, that means $P(C_1, \lambda) = 0$. The cycle graph C_2 is a graph with two vertices, where two edges between two vertices, and C_2 can have colorings by assigning two vertices with different colors, that means $P(C_2, \lambda) = \lambda(\lambda - 1)$. The cycle graph C_3 is drawn by a triangle and C_3 can have colorings by assigning all three vertices with different colors, that means $P(C_3, \lambda) = \lambda(\lambda - 1)(\lambda - 2)$.

2. Four proofs of Theorem 2

In this section, we show the formula (2) in four different ways.

2.1. Inductive proof. This inductive proof is widely well-known. A *path graph P_n* is a connected graph in which $n-1$ edges connect n vertices of vertex degree at most 2, which could be drawn on a single straight line. The chromatic polynomial for path graph P_n is easily obtained by coloring all vertices $v_1, \ldots, v_n$ where v_i and v_{i+1} have different colors for $i = 1, \ldots, n-1$.

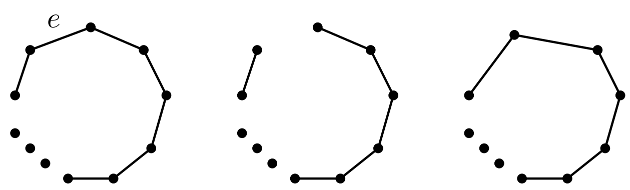


FIGURE 3. C_{n+1} , P_{n+1} and C_n

LEMMA 3. *For a positive integer $n \geq 1$, the chromatic polynomial for path graph P_n is*

$$P(P_n, \lambda) = \lambda(\lambda - 1)^{n-1}. \tag{3}$$

We use an induction on the number n of vertices by the deletion-contraction recurrence and the above lemma for path graph: It is already shown that (2) is true for $n \leq 3$ by the example in Section 1. Assume that (2) is true for a positive integer n. Using (1) and (3), we have

$$\begin{aligned} P(C_{n+1}, \lambda) &= P(C_{n+1} - e, \lambda) - P(C_{n+1}/e, \lambda) && \text{by (1)} \\ &= P(P_{n+1}, \lambda) - P(C_n, \lambda) \\ &= \lambda(\lambda - 1)^n - ((\lambda - 1)^n + (-1)^n(\lambda - 1)) && \text{by (3)} \\ &= (\lambda - 1)^{n+1} + (-1)^{n+1}(\lambda - 1). \end{aligned}$$

Thus, (2) is true for all positive integers $n \geq 1$.

2.2. Proof by inclusion-exclusion principle. The *inclusion-exclusion principle* is a technique of counting the size of the union of finite sets.

PROPOSITION 4 (Inclusion-exclusion principle). *Let $A_1, A_2, \ldots, A_n$ be subsets of a finite set U. Then number of elements excluding their union is as follows*

$$\begin{aligned} \left| \bigcap_{i=1}^{n} \overline{A_i} \right| &= \sum_{I \subset [n]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right| \\ &= |U| - \sum_{i=1}^{n} |A_i| + \sum_{i<j} |A_i \cap A_j| - \cdots + (-1)^n |A_1 \cap \cdots \cap A_n| \end{aligned}$$

where $\overline{A}$ is the complement of A in U.

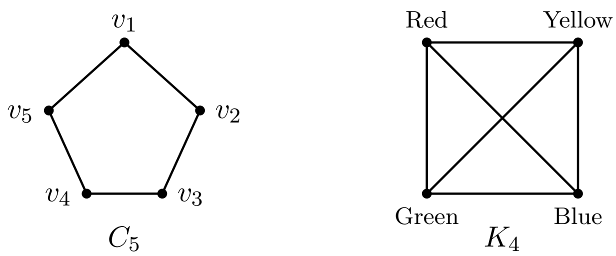


FIGURE 4. A cycle graph C_5 and a graph K_4 with names of colors

Considering every condition to assign different colors to two adjacent vertices, for each edge e, we define a finite sets of arbitrary (including improper) colorings to assign same color to two adjacent vertices by the edge e.

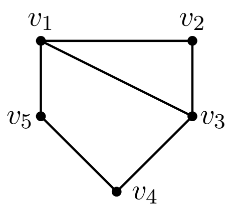
Let A_i be a set of colorings such that two vertices v_i and v_{i+1} are of same color, where v_{n+1} is regarded as v_1. Applying the inclusion-exclusion principle, we can write the following

$$\begin{aligned} P(C_n, \lambda) &= |U| - \sum_{i=1}^{n} |A_i| + \sum_{i<j} |A_i \cap A_j| + \cdots + (-1)^n |A_1 \cap \cdots \cap A_n| \\ &= \lambda^n - \binom{n}{1}\lambda^{n-1} + \binom{n}{2}\lambda^{n-2} + \cdots + (-1)^n \lambda \\ &= (\lambda - 1)^n - (-1)^n + (-1)^n \lambda \\ &= (\lambda - 1)^n + (-1)^n(\lambda - 1). \end{aligned}$$

Thus, (2) is true for all positive integers $n \geq 1$.

2.3. Algebric proof. Let us consider a case of $n = 5$ and $\lambda = 4$, that is, to assign the vertices of C_5 in four colors: red, blue, yellow, and green. Also let us consider a complete graph K_4 with vertex names red, blue, yellow, and green, see Figure 4.

When red-blue-red-yellow-green is assigned in order from the vertex v_1 to the vertex v_5 in C_5, it is corresponding to a closed walk of length 5 in K_4 which begins and ends at red, that is, it is red-blue-red-yellow-green-red in K_4. By generalizing it, we have a correspondence between



G

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

FIGURE 5. A graph G and its adjacency matrix A

λ-colorings of C_n and closed walks of length n in K_λ. By this correspondence, it is enough to count the number of closed walks of length n in K_λ, instead of the number of λ-colorings of C_n.

For a graph G with vertex set $\{v_1, \ldots, v_n\}$, the *adjacency matrix* of G is an $n \times n$ square matrix A such that its element A_{ij} is one when there is an edge between two vertices v_i and v_j, and zero when there is no edge between v_i and v_j.

The following related to an adjacency matrix is well-known.

PROPOSITION 5. *Let A be the adjacency matrix of the graph G on n vertices $v_1, \ldots, v_n$. Then the (i, j)th entry of the matrix A^n is the number of the walk of length n beginning at v_i and ending at v_j.*

By Proposition 5, we can calculate the number of closed walk of length n in the complete graph K_λ: Let A be an adjacency matrix of K_λ. Then A is a $\lambda \times \lambda$ matrix as follows

$$A = (a_{ij}) = \begin{pmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix},$$

where $a_{ij} = 0$ if $i = j$, and otherwise $a_{ij} = 1$. So the number of closed walks of length n in K_λ is enumerated by $tr(A^n)$, which equals the sum of all eigenvalues of A^n. Also let all eigenvalues of the matrix A be denoted by $u_1, \ldots, u_\lambda$, then all eigenvalues of the matrix A^n are $u_1^n, \ldots, u_\lambda^n$.

$$A = \begin{pmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} \lambda-1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix},$$

Since the matrix A have λ eigenvalues $u_1 = \lambda - 1$ and $u_2 = \cdots = u_\lambda = -1$, we have

$$tr(A^n) = \sum_{i=1}^{\lambda} u_i^n = (\lambda-1)^n + \underbrace{(-1)^n + \cdots + (-1)^n}_{\lambda - 1 \text{ times}}.$$

Thus, (2) is true for all positive integers $n \geq 1$.

2.4. Bijective proof. Let X_n denote the set of λ-colorings of C_n and $[\lambda-1]^n$ be the set of n-tuples of positive integers less than λ, where $[\lambda-1]$ means $\{1, \ldots, \lambda-1\}$. We consider a mapping φ from λ-colorings of C_n in X_n to n-tuples in $[\lambda-1]^n$.

A mapping φ from X_n to $[\lambda-1]^n$. The mapping $\varphi : X_n \to [\lambda-1]^n$ is defined as follows: Let ω be a λ-coloring of C_n in X_n, we write $\omega = (\omega_1, \ldots, \omega_n)$ where ω_i is the color of v_i in C_n and it is obvious that $\omega_i \neq \omega_{i+1}$ for $1 \leq i \leq \lambda$, where ω_{n+1} is regarded as ω_1. An entry ω_i is called a *cyclic descent* of C if $\omega_i > \omega_{i+1}$ for $1 \leq i \leq \lambda$. Then we define $\varphi(\omega) = \sigma = (\sigma_1, \ldots, \sigma_n)$ with

$$\sigma_i = \begin{cases} \omega_i - 1, & \text{if } \omega_i \text{ is a cyclic descent} \\ \omega_i, & \text{otherwise.} \end{cases}$$

Given a λ-coloring ω, if $\omega_i = \lambda$ then $\omega_{i+1} < \lambda$, so $\omega_i = \lambda$ should be a cyclic descent. Thus we have $\sigma_i < \lambda$ for all $1 \leq i \leq n$ and $\varphi(\omega)$ belongs to $[\lambda-1]^n$.

For example, in a case of $n = 9$ and $\lambda = 4$, $\omega = (1, 2, 1, 3, 2, 3, 1, 4, 2) \in X_9$ is given as an example of 4-colorings of C_9. Here $\omega_2 = 2$, $\omega_4 = 3$, $\omega_6 = 3$, $\omega_8 = 4$, and $\omega_9 = 2$ are cyclic descents of ω. So we have

$$\varphi(\omega) = \sigma = (1, 1, 1, 2, 2, 2, 1, 3, 1) \in [3]^9.$$

A mapping ψ as the inverse of φ. Let Z_n be the set of n-tuples $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ in $[\lambda-1]^n$ with

$$\sigma_1 = \sigma_2 = \cdots = \sigma_n$$

and it is obvious that the size of Z_n is $\lambda - 1$.

We would like to describe a mapping $\psi : ([\lambda-1]^n \setminus Z_n) \to X_n$ in order to satisfy $\varphi \circ \psi$ is the identity on $[\lambda-1]^n \setminus Z_n$ as follows: Given a $\sigma \in [\lambda-1]^n \setminus Z_n$, we define $\overline{\sigma} = (\overline{\sigma}_1, \ldots, \overline{\sigma}_n)$ with

$$\overline{\sigma}_i = \begin{cases} \sigma_i + 1, & \text{if } \sigma_i \text{ is a cyclic descent} \\ \sigma_i, & \text{otherwise.} \end{cases}$$

Since $\overline{\sigma}$ may have consecutive same entries, we define $\psi(\sigma) = \omega = (\omega_1, \ldots, \omega_n)$ from $\overline{\sigma}$ with $\omega_i = \overline{\sigma}_i + 1$ for any entry $\overline{\sigma}_i$ of $\overline{\sigma}$ with a finite positive even integer ℓ satisfying

$$\overline{\sigma}_i = \overline{\sigma}_{i+1} = \cdots = \overline{\sigma}_{i+\ell-1} \neq \overline{\sigma}_{i+\ell},$$

where $\overline{\sigma}_{n+k}$ is regarded as $\overline{\sigma}_k$ for $1 \le k \le n$, and $\omega_i = \overline{\sigma}_i$, otherwise. Thus ω has no consecutive same entries and $1 \le \omega_i \le \lambda$ for all $1 \le i \le n$, so $\psi(\sigma) = \omega$ belongs to X_n. Moreover, it is obvious that $\sigma_i \le \omega_i \le \sigma_i + 1$ for all $1 \le i \le n$ and if $\omega_i = \sigma_i + 1$ for some $1 \le i \le n$ then ω_i is a cyclic descent in ω. Hence $\varphi(\omega) = \sigma$ and $\sigma \in [\lambda-1]^n \setminus Z_n$ if and only if $\psi(\sigma) = \omega$.

In a previous example, $\sigma = (1,1,1,2,2,2,1,3,1)$ is denoted as an example of 9-tuples in $[3]^9$. Here $\sigma_6 = 2$, $\sigma_8 = 3$ are cyclic descents of σ and we obtain $\overline{\sigma} = (1,1,1,2,2,3,1,4,1)$. And then there exist only three entries $\overline{\sigma}_2$, $\overline{\sigma}_4$, and $\overline{\sigma}_9$ in $\overline{\sigma}$ satisfying the following

$$\begin{aligned} k = 2 :&\quad \overline{\sigma}_2 = \overline{\sigma}_3 \neq \overline{\sigma}_4 \quad (\ell = 2), \\ k = 4 :&\quad \overline{\sigma}_4 = \overline{\sigma}_5 \neq \overline{\sigma}_6 \quad (\ell = 2), \text{ and} \\ k = 9 :&\quad \overline{\sigma}_9 = \overline{\sigma}_1 = \overline{\sigma}_2 = \overline{\sigma}_3 \neq \overline{\sigma}_4 \quad (\ell = 4), \end{aligned}$$

so we get $\omega_2 = \overline{\sigma}_2 + 1 = 2$, $\omega_4 = \overline{\sigma}_4 + 1 = 3$, $\omega_9 = \overline{\sigma}_9 + 1 = 2$, and

$$\psi(\sigma) = \omega = (1,2,1,3,2,3,1,4,2) \in X_9.$$

Let Y_n be the set of λ-colorings ω in X_n with $\varphi(\omega) \in Z_n$. Since two mapping φ and ψ are bijections between $X_n \setminus Y_n$ and $[\lambda-1]^n \setminus Z_n$, the size of the set $X_n \setminus Y_n$ is same with the size of the $[\lambda-1]^n \setminus Z_n$, which is equal to $(\lambda-1)^n - (\lambda-1)$.

When n is even, for any $1 \leq i \leq \lambda - 1$, there exist only two n-tuples in X_n

$$\omega = (i+1, i, i+1, i, \ldots, i+1, i) \quad \text{and} \quad \omega = (i, i+1, i, i+1, \ldots, i, i+1)$$

satisfying $\varphi(\omega) = (i, i, \ldots, i) \in Z_n$. If n is even, the size of Y_n is equal to $2(\lambda - 1)$ and we obtain

$$\begin{aligned} P(C_n, \lambda) = |X_n| &= |X_n \setminus Y_n| + |Y_n| \\ &= [(\lambda-1)^n - (\lambda - 1)] + 2(\lambda - 1). \end{aligned} \tag{4}$$

When n is odd, there is no n-tuples satisfying $\varphi(\omega) \in Z_n$ and the set Y_n is empty. If n is odd, we obtain

$$\begin{aligned} P(C_n, \lambda) = |X_n| &= |X_n \setminus Y_n| + |Y_n| \\ &= [(\lambda-1)^n - (\lambda - 1)] + 0. \end{aligned} \tag{5}$$

Therefore, (2) yields from (4) and (5) for all positive integers $n \geq 1$.

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