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ORTHOGONALLY ADDITIVE AND ORTHOGONALLY QUADRATIC FUNCTIONAL EQUATION

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Abstract. Using the fixed point method, we prove the Ulam-Hyers stability of the orthogonally additive and orthogonally quadratic functional equation

$$\begin{aligned} & f\left(\frac{x}{2}+y\right)+f\left(\frac{x}{2}-y\right)+f\left(\frac{x}{2}+z\right)+f\left(\frac{x}{2}-z\right) \\ & = \frac{3}{2}f(x)-\frac{1}{2}f(-x)+f(y)+f(-y)+f(z)+f(-z) \end{aligned} \tag{0.1}$$

for all x, y, z with $x \perp y$, in orthogonality Banach spaces and in non-Archimedean orthogonality Banach spaces.

1. Introduction and preliminaries

In 1897, Hensel [19] introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [12, 27, 28, 35]).

A *valuation* is a function $|\cdot|$ from a field K into $[0, \infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r| \cdot |s|$ and the

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triangle inequality holds, i.e.,

$$|r+s| \le |r|+|s|, \qquad \forall r,s \in K.$$

A field K is called a *valued field* if K carries a valuation. Throughout this paper, we assume that the base field is a valued field, hence call it simply a field. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \le \max\{|r|,|s|\}, \qquad \forall r,s \in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Definition 1.1. ([34]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:
(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r|\|x\| \qquad (r \in K, x \in X)$;
(iii) the strong triangle inequality

$$\|x+y\| \le \max\{\|x\|,\|y\|\}, \qquad \forall x,y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space.*

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \le \varepsilon$$

for all $n, m \ge N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \le \varepsilon$$

for all $n \ge N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space.*

Assume that X is a real inner product space and $f : X \to \mathbb{R}$ is a solution of the orthogonal Cauchy functional equation $f(x + y) = f(x) + f(y), \langle x, y\rangle = 0$. By the Pythagorean theorem $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

G. Pinsker [40] characterized orthogonally additive functionals on an inner product space when the orthogonality is the ordinary one in such spaces. K. Sundaresan [50] generalized this result to arbitrary Banach spaces equipped with the Birkhoff-James orthogonality. The orthogonal Cauchy functional equation

$$f(x + y) = f(x) + f(y), \qquad x \perp y,$$

in which $\perp$ is an abstract orthogonality relation, was first investigated by S. Gudder and D. Strawther [18]. They defined $\perp$ by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation. In 1985, J. Rätz [47] introduced a new definition of orthogonality by using more restrictive axioms than of S. Gudder and D. Strawther. Moreover, he investigated the structure of orthogonally additive mappings. J. Rätz and Gy. Szabó [48] investigated the problem in a rather more general framework.

Let us recall the orthogonality in the sense of J. Rätz; cf. [47].

Suppose X is a real vector space with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(O_1) totality of $\perp$ for zero: $x \perp 0, 0 \perp x$ for all $x \in X$;

(O_2) independence: if $x, y \in X - \{0\}, x \perp y$, then x, y are linearly independent;

(O_3) homogeneity: if $x, y \in X, x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(O_4) the Thalesian property: if P is a 2-dimensional subspace of $X, x \in P$ and $\lambda \in \mathbb{R}_+$, which is the set of nonnegative real numbers, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space we mean an orthogonality space having a normed structure.

Some interesting examples are

(i) The trivial orthogonality on a vector space X defined by (O_1), and for non-zero elements $x, y \in X$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(X, \langle ., . \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.
(iii) The Birkhoff-James orthogonality on a normed space $(X, \|.\|)$ defined by $x \perp y$ if and only if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Clearly examples (i) and (ii) are symmetric but example (iii) is not. It is remarkable to note, however, that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric. There are several orthogonality notions on a real normed space such as Birkhoff-James, Boussouis, Singer, Carlsson, unitary-Boussouis, Roberts, Phythagorean, isosceles and Diminnie (see [1]–[3], [7, 14, 23, 24]).

The stability problem of functional equations originated from the following question of Ulam [52]: *Under what condition does there exist an additive mapping near an approximately additive mapping?* In 1941, Hyers [20] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1978, Th.M. Rassias [42] extended the theorem of Hyers by considering the unbounded Cauchy difference $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$, $(\varepsilon > 0, p \in [0, 1))$.

The first author treating the stability of the quadratic equation was F. Skof [49] by proving that if f is a mapping from a normed space X into a Banach space Y satisfying $\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varepsilon$ for some $\varepsilon > 0$, then there is a unique quadratic mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{\varepsilon}{2}$. P.W. Cholewa [8] extended the Skof's theorem by replacing X by an abelian group G. The Skof's result was later generalized by S. Czerwik [9] in the spirit of Ulam-Hyers-Rassias. The stability problem of functional equations has been extensively investigated by some mathematicians (see [10, 11, 21, 25, 39], [43]–[46]).

R. Ger and J. Sikorska [17] investigated the orthogonal stability of the Cauchy functional equation $f(x+y) = f(x)+f(y)$, namely, they showed that if f is a mapping from an orthogonality space X into a real Banach space Y and $\|f(x+y)-f(x)-f(y)\| \leq \varepsilon$ for all $x, y \in X$ with $x \perp y$ and some $\varepsilon > 0$, then there exists exactly one orthogonally additive mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{16}{3}\varepsilon$ for all $x \in X$.

The orthogonally quadratic equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \ x \perp y$$

was first investigated by F. Vajzović [53] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, H. Drljević [15], M. Fochi [16], M.S. Moslehian [31, 32] and Gy. Szabó [51] generalized this result. See also [33, 36].

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

THEOREM 1.3. [4, 13] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [22] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 26, 30, 37, 38, 41]).

This paper is organized as follows: In Section 2, we prove the Ulam-Hyers stability of the orthogonally additive and orthogonally quadratic functional equation (0.1) in orthogonality spaces for an odd mapping. In Section 3, we prove the Ulam-Hyers stability of the orthogonally additive and orthogonally quadratic functional equation (0.1) in orthogonality spaces for an even mapping. In Section 4, we prove the Ulam-Hyers stability of the orthogonally additive and orthogonally quadratic functional equation (0.1) in non-Archimedean orthogonality spaces for an odd mapping. In Section 5, we prove the Ulam-Hyers stability of the orthogonally additive and orthogonally quadratic functional equation (0.1) in non-Archimedean orthogonality spaces for an even mapping.

2. Ulam-Hyers stability of the orthogonally additive and orthogonally quadratic functional equation: an odd mapping case

Throughout this section, assume that $(X, \perp)$ is an orthogonality space and that $(Y, \|.\|_Y)$ is a real Banach space.

In this section, applying some ideas from [17, 21], we deal with the stability problem for the orthogonally additive and orthogonally quadratic functional equation

$$\begin{aligned} Df(x,y,z) := f\left(\frac{x}{2}+y\right) + f\left(\frac{x}{2}-y\right) + f\left(\frac{x}{2}+z\right) + f\left(\frac{x}{2}-z\right) \\ -\frac{3}{2}f(x) + \frac{1}{2}f(-x) - f(y) - f(-y) - f(z) - f(-z) = 0 \end{aligned}$$

for all $x, y, z \in X$ with $x \perp y$ in orthogonality spaces: an odd mapping case.

If f is an odd mapping with $Df(x,y,z) = 0$, then

$$f\left(\frac{x}{2}+y\right) + f\left(\frac{x}{2}-y\right) + f\left(\frac{x}{2}\right) + f\left(\frac{x}{2}\right) = 2f(x)$$

and so $f\left(\frac{x}{2}+y\right) + f\left(\frac{x}{2}-y\right) = f(x)$ for all $x, y \in X$ with $x \perp y$, and

$$f\left(\frac{x}{2}\right) + f\left(\frac{x}{2}\right) + f\left(\frac{x}{2}+z\right) + f\left(\frac{x}{2}-z\right) = 2f(x)$$

and so $f\left(\frac{x}{2}+z\right) + f\left(\frac{x}{2}-z\right) = f(x)$ for all $x, z \in X$. That is, f is orthogonally additive and additive.

Definition 2.1. A mapping $f : X \to Y$ is called an *orthogonally additive mapping* if

$$f\left(\frac{x}{2}+y\right) + f\left(\frac{x}{2}-y\right) + f\left(\frac{x}{2}+z\right) + f\left(\frac{x}{2}-z\right) = 2f(x)$$

for all $x, y, z \in X$ with $x \perp y$.

Theorem 2.2. *Let* $\varphi : X^3 \to [0, \infty)$ *be a function such that there exists* $0 < \alpha < 1$ *with*

$$\varphi(x,y,z) \le 2\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \tag{2.1}$$

for all $x, y, z \in X$ *with* $x \perp y$. *Let* $f : X \to Y$ *be an odd mapping satisfying*

$$\|Df(x,y,z)\|_Y \le \varphi(x,y,z) \tag{2.2}$$

for all $x, y, z \in X$ with $x \perp y$. Then there exists a unique orthogonally additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{\alpha}{2-2\alpha}\varphi(x,0,0) \tag{2.3}$$

for all $x \in X$.

Proof. Putting $y = z = 0$ in (2.2), we get

$$\left\|4f\left(\frac{x}{2}\right) - 2f(x)\right\|_Y \leq \varphi(x,0,0) \tag{2.4}$$

for all $x \in X$, since $x \perp 0$. So

$$\left\|f(x) - \frac{1}{2}f(2x)\right\|_Y \leq \frac{1}{4}\varphi(2x,0,0) \leq \frac{\alpha}{2}\varphi(x,0,0) \tag{2.5}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\|_Y \leq \mu\varphi(x,0,0),\ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [29]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\|_Y \leq \varphi(x,0,0)$$

for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\|_Y = \left\|\frac{1}{2}g(2x) - \frac{1}{2}h(2x)\right\|_Y \leq \alpha\varphi(x,0,0)$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq \alpha\varepsilon$. This means that

$$d(Jg, Jh) \leq \alpha d(g,h)$$

for all $g, h \in S$.

It follows from (2.5) that $d(f, Jf) \leq \frac{\alpha}{2}$.

By Theorem 1.3, there exists a mapping $L : X \to Y$ satisfying the following:

(1) L is a fixed point of J, i.e.,

$$L(2x) = 2L(x) \tag{2.6}$$

for all $x \in X$. The mapping L is a unique fixed point of J in the set

$$M = \{g \in S : d(h, g) < \infty\}.$$

This implies that L is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - L(x)\|_Y \leq \mu\varphi(x, 0, 0)$$

for all $x \in X$;

(2) $d(J^n f, L) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = L(x)$$

for all $x \in X$;

(3) $d(f, L) \leq \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality

$$d(f, L) \leq \frac{\alpha}{2 - 2\alpha}.$$

This implies that the inequality (2.3) holds.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|DL(x, y, z)\|_Y &= \lim_{n\to\infty} \frac{1}{2^n} \|Df(2^n x, 2^n y, 2^n z)\|_Y \\ &\leq \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y, 2^n z) \leq \lim_{n\to\infty} \frac{2^n \alpha^n}{2^n} \varphi(x, y, z) = 0 \end{aligned}$$

for all $x, y, z \in X$ with $x \perp y$. Since f is odd, L is odd. Hence

$$L\left(\frac{x}{2} + y\right) + L\left(\frac{x}{2} - y\right) + L\left(\frac{x}{2} + z\right) + L\left(\frac{x}{2} - z\right) = 2L(x)$$

for all $x, y, z \in X$ with $x \perp y$. So $L : X \to Y$ is an orthogonally additive mapping. Thus $L : X \to Y$ is a unique orthogonally additive mapping satisfying (2.3), as desired. $\square$

From now on, in corollaries, assume that $(X, \perp)$ is an orthogonality normed space.

Corollary 2.3. *Let θ be a positive real number and p a real number with $0 < p < 1$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x, y, z)\|_Y \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.7}$$

for all $x, y, z \in X$ with $x \perp y$. Then there exists a unique orthogonally additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{2^{p-1}\theta}{2 - 2^p}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$ with $x \perp y$. Then we can choose $\alpha = 2^{p-1}$ and we get the desired result. □

THEOREM 2.4. *Let $f : X \to Y$ be an odd mapping satisfying (2.2) for which there exists a function $\varphi : X^3 \to [0, \infty)$ such that*

$$\varphi(x, y, z) \leq \frac{\alpha}{2}\varphi(2x, 2y, 2z)$$

for all $x, y, z \in X$ with $x \perp y$. Then there exists a unique orthogonally additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{1}{2 - 2\alpha}\varphi(x, 0, 0) \tag{2.8}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (2.4) that $d(f, Jf) \leq \frac{1}{2}$. So

$$d(f, L) \leq \frac{1}{2 - 2\alpha}.$$

Thus we obtain the inequality (2.8).

The rest of the proof is similar to the proof of Theorem 2.2. □

COROLLARY 2.5. *Let θ be a positive real number and p a real number with $p > 1$. Let $f : X \to Y$ be an odd mapping satisfying (2.7). Then there exists a unique orthogonally additive mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \leq \frac{2^{p-1}\theta}{2^p - 2}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x,y,z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$ with $x \perp y$. Then we can choose $\alpha = 2^{1-p}$ and we get the desired result. $\square$

3. Ulam-Hyers stability of the orthogonally additive and orthogonally quadratic functional equation: an even mapping case

Throughout this section, assume that $(X, \perp)$ is an orthogonality space and that $(Y, \|.\|_Y)$ is a real Banach space.

In this section, applying some ideas from [17, 21], we deal with the stability problem for the orthogonally additive and orthogonally quadratic functional equation $Df(x,y,z) = 0$, given in the previous section, in orthogonality spaces: an even mapping case.

If f is an even mapping with $Df(x,y,z) = 0$, then

$$f\left(\frac{x}{2}+y\right)+f\left(\frac{x}{2}-y\right)+f\left(\frac{x}{2}\right)+f\left(\frac{x}{2}\right)=f(x)+2f(y)$$

and so $f\left(\frac{x}{2}+y\right)+f\left(\frac{x}{2}-y\right) = \frac{1}{2}f(x)+2f(y)$ for all $x, y \in X$ with $x \perp y$, and

$$f\left(\frac{x}{2}\right)+f\left(\frac{x}{2}\right)+f\left(\frac{x}{2}+z\right)+f\left(\frac{x}{2}-z\right)=f(x)+2f(z)$$

and so $f\left(\frac{x}{2}+z\right)+f\left(\frac{x}{2}-z\right) = \frac{1}{2}f(x)+2f(z)$ for all $x, z \in X$. That is, f is orthogonally quadratic and quadratic.

Definition 3.1. A mapping $f : X \to Y$ is called an *orthogonally quadratic mapping* if

$$f\left(\frac{x}{2}+y\right)+f\left(\frac{x}{2}-y\right)+f\left(\frac{x}{2}+z\right)+f\left(\frac{x}{2}-z\right)=f(x)+2f(y)+2f(z)$$

for all $x, y, z \in X$ with $x \perp y$.

Theorem 3.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists $0 < \alpha < 1$ with*

$$\varphi(x,y,z) \leq 4\alpha\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)$$

for all $x, y, z \in X$ with $x \perp y$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \leq \frac{\alpha}{1-\alpha}\varphi(x,0,0) \tag{3.1}$$

for all $x \in X$*.*

Proof. Putting $y = z = 0$ in (2.2), we get

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\|_Y \le \varphi(x, 0, 0) \tag{3.2}$$

for all $x \in X$, since $x \perp 0$. So

$$\left\| f(x) - \frac{1}{4} f(2x) \right\|_Y \le \frac{1}{4}\varphi(2x, 0, 0) \le \alpha \cdot \varphi(x, 0, 0) \tag{3.3}$$

for all $x \in X$.

By the same reasoning as in the proof of Theorem 2.2, one can obtain an orthogonally quadratic mapping $Q : X \to Y$ defined by

$$\lim_{n\to\infty} \frac{1}{4^n} f\left(2^n x\right) = Q(x)$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4} g\left(2x\right)$$

for all $x \in X$.

It follows from (3.3) that $d(f, Jf) \le \alpha$. So

$$d(f, Q) \le \frac{\alpha}{1 - \alpha}.$$

So we obtain the inequality (3.1). Thus $Q : X \to Y$ is a unique orthogonally quadratic mapping satisfying (3.1), as desired. □

Corollary 3.3. *Let* θ *be a positive real number and* p *a real number with* $0 < p < 2$*. Let* $f : X \to Y$ *be an even mapping satisfying* (2.7)*. Then there exists a unique orthogonally quadratic mapping* $Q : X \to Y$ *such that*

$$\|f(x) - Q(x)\|_Y \le \frac{2^p \theta}{4 - 2^p} \|x\|^p$$

for all $x \in X$*.*

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$ with $x \perp y$. Then we can choose $\alpha = 2^{p-2}$ and we get the desired result. □

Theorem 3.4. *Let $f : X \to Y$ be an even mapping satisfying* (2.2) *for which there exists a function $\varphi : X^3 \to [0, \infty)$ such that*

$$\varphi(x, y, z) \leq \frac{\alpha}{4}\varphi(2x, 2y, 2z)$$

for all $x, y, z \in X$ with $x \perp y$. Then there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \leq \frac{1}{1-\alpha}\varphi(x, 0, 0) \tag{3.4}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (3.2) that $d(f, Jf) \leq 1$. So we obtain the inequality (3.4).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.2. □

Corollary 3.5. *Let θ be a positive real number and p a real number with $p > 2$. Let $f : X \to Y$ be an even mapping satisfying* (2.7)*. Then there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\|_Y \leq \frac{2^p\theta}{2^p - 4}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$ with $x \perp y$. Then we can choose $\alpha = 2^{2-p}$ and we get the desired result. □

Let $f_o(x) = \frac{f(x)-f(-x)}{2}$ and $f_e(x) = \frac{f(x)+f(-x)}{2}$. Then f_o is an odd mapping and f_e is an even mapping such that $f = f_o + f_e$.

The above corollaries can be summarized as follows:

Theorem 3.6. *Assume that $(X, \perp)$ is an orthogonality normed space. Let θ be a positive real number and p a real number with $0 < p < 1$ (resp. $p > 2$). Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

(2.7). *Then there exist an orthogonally additive mapping* $L : X \to Y$ *and an orthogonally quadratic mapping* $Q : X \to Y$ *such that*

$$\|f(x) - L(x) - Q(x)\|_Y \leq \left(\frac{2^p}{2-2^p} + \frac{2^p}{4-2^p}\right)\theta\|x\|^p$$

$$(\textit{resp.}\quad \|f(x) - L(x) - Q(x)\|_Y \leq \left(\frac{2^p}{2^p-2} + \frac{2^p}{2^p-4}\right)\theta\|x\|^p)$$

for all $x \in X$.

4. Ulam-Hyers stability of the orthogonally additive and orthogonally quadratic functional equation in non-Archimedean orthogonality spaces: an odd mapping case

Throughout this section, assume that $(X, \perp)$ is a non-Archimedean orthogonality space and that $(Y, \|.\|_Y)$ is a real non-Archimedean Banach space. Assume that $|2| \neq 1$.

In this section, applying some ideas from [17, 21], we deal with the stability problem for the orthogonally additive and orthogonally quadratic functional equation $Df(x, y, x) = 0$, given in the second section, in non-Archimedean orthogonality spaces: an odd mapping case.

THEOREM 4.1. *Let* $\varphi : X^3 \to [0, \infty)$ *be a function such that there exists* $0 < \alpha < 1$ *with*

$$\varphi(x, y, z) \leq |2|\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x, y, z \in X$ *with* $x \perp y$. *Let* $f : X \to Y$ *be an odd mapping satisfying*

$$\|Df(x, y, z)\|_Y \leq \varphi(x, y, z) \tag{4.1}$$

for all $x, y, z \in X$ *with* $x \perp y$. *Then there exists a unique orthogonally additive mapping* $L : X \to Y$ *such that*

$$\|f(x) - L(x)\|_Y \leq \frac{\alpha}{|2| - |2|\alpha}\varphi(x, 0, 0) \tag{4.2}$$

for all $x \in X$.

Proof. Putting $y = z = 0$ in (4.1), we get

$$\left\|4f\left(\frac{x}{2}\right) - 2f(x)\right\|_Y \leq \varphi(x, 0, 0) \tag{4.3}$$

for all $x \in X$, since $x \perp 0$. So

$$\left\| f(x) - \frac{1}{2} f(2x) \right\|_Y \leq \frac{1}{|2|^2}\varphi(2x, 0, 0) \leq \frac{\alpha}{|2|}\varphi(x, 0, 0) \tag{4.4}$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g\left(2x\right)$$

for all $x \in X$.

It follows from (4.4) that $d(f, Jf) \leq \frac{\alpha}{|2|}$. Thus we obtain the inequality (4.2).

The rest of the proof is similar to the proof of Theorem 2.2. □

From now on, in corollaries, assume that $(X, \perp)$ is a non-Archimedean orthogonality normed space.

Corollary 4.2. *Let θ be a positive real number and p a real number with $p > 1$. Let $f : X \to Y$ be an odd mapping satisfying* (2.7). *Then there exists a unique orthogonally additive mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \leq \frac{|2|^{p-1}\theta}{|2| - |2|^p}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.1 by taking $\varphi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$ with $x \perp y$. Then we can choose $\alpha = |2|^{p-1}$ and we get the desired result. □

Theorem 4.3. *Let $f : X \to Y$ be an odd mapping satisfying* (4.1) *for which there exists a function $\varphi : X^3 \to [0, \infty)$ such that*

$$\varphi(x, y, z) \leq \frac{\alpha}{|2|}\varphi\left(2x, 2y, 2z\right)$$

for all $x, y, z \in X$ with $x \perp y$. Then there exists a unique orthogonally additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{1}{|2| - |2|\alpha}\varphi\left(x, 0, 0\right) \tag{4.5}$$

for all $x \in X$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (4.3) that $d(f, Jf) \le \frac{1}{|2|}$. So

$$d(f, L) \le \frac{1}{|2| - |2|\alpha}.$$

Thus we obtain the inequality (4.5).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 4.4. *Let θ be a positive real number and p a real number with $0 < p < 1$. Let $f : X \to Y$ be an odd mapping satisfying* (2.7). *Then there exists a unique orthogonally additive mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \le \frac{|2|^{p-1}\theta}{|2|^p - |2|}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.3 by taking $\varphi(x,y,z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$ with $x \perp y$. Then we can choose $\alpha = |2|^{1-p}$ and we get the desired result. □

5. Ulam-Hyers stability of the orthogonally additive and orthogonally quadratic functional equation in non-Archimedean orthogonality spaces: an even mapping case

Throughout this section, assume that $(X, \perp)$ is a non-Archimedean orthogonality space and that $(Y, \|.\|_Y)$ is a real non-Archimedean Banach space. Assume that $|2| \ne 1$.

In this section, applying some ideas from [17, 21], we deal with the stability problem for the orthogonally additive and orthogonally quadratic functional equation $Df(x,y,z) = 0$, given in the second section, in non-Archimedean orthogonality spaces: an even mapping case.

THEOREM 5.1. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists $0 < \alpha < 1$ with*

$$\varphi(x, y, z) \leq |4|\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x, y, z \in X$ with $x \perp y$. Let $f : X \to Y$ be an even mapping satisfying (4.1). *Then there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\|_Y \leq \frac{\alpha}{1-\alpha}\varphi(x, 0, 0) \tag{5.1}$$

for all $x \in X$.

Proof. Putting $y = z = 0$ in (4.1), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\|_Y \leq \varphi(x, 0, 0) \tag{5.2}$$

for all $x \in X$, since $x \perp 0$. So

$$\left\|f(x) - \frac{1}{4}f(2x)\right\|_Y \leq \frac{1}{|4|}\varphi(2x, 0, 0) \leq \alpha \cdot \varphi(x, 0, 0) \tag{5.3}$$

for all $x \in X$.

By the same reasoning as in the proof of Theorem 2.2, one can obtain an orthogonally quadratic mapping $Q : X \to Y$ defined by

$$\lim_{n\to\infty} \frac{1}{4^n} f\left(2^n x\right) = Q(x)$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g\left(2x\right)$$

for all $x \in X$.

It follows from (5.3) that $d(f, Jf) \leq \alpha$. So

$$d(f, Q) \leq \frac{\alpha}{1-\alpha}.$$

So we obtain the inequality (5.1). Thus $Q : X \to Y$ is a unique orthogonally quadratic mapping satisfying (5.1), as desired. $\square$

Corollary 5.2. *Let θ be a positive real number and p a real number with $p > 2$. Let $f : X \to Y$ be an even mapping satisfying* (2.7). *Then there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\|_Y \leq \frac{|2|^p \theta}{|2|^2 - |2|^p} \|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 5.1 by taking $\varphi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$ with $x \perp y$. Then we can choose $\alpha = |2|^{p-2}$ and we get the desired result. □

Theorem 5.3. *Let $f : X \to Y$ be an even mapping satisfying* (4.1) *and $f(0) = 0$ for which there exists a function $\varphi : X^3 \to [0, \infty)$ such that*

$$\varphi(x, y, z) \leq \frac{\alpha}{|4|} \varphi(2x, 2y, 2z)$$

for all $x, y, z \in X$ with $x \perp y$. Then there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \leq \frac{1}{1 - \alpha} \varphi(x, 0, 0) \tag{5.4}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (5.2) that $d(f, Jf) \leq 1$. So we obtain the inequality (5.4).

The rest of the proof is similar to the proofs of Theorems 2.2 and 5.1. □

Corollary 5.4. *Let θ be a positive real number and p a real number with $0 < p < 2$. Let $f : X \to Y$ be an even mapping satisfying* (2.7). *Then there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\|_Y \leq \frac{|2|^p \theta}{|2|^p - |2|^2} \|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 5.3 by taking $\varphi(x,y,z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x,y,z \in X$ with $x \perp y$. Then we can choose $\alpha = |2|^{2-p}$ and we get the desired result. $\square$

Let $f_o(x) = f(x) - f(-x)$ and $f_e(x) = f(x) + f(-x)$. Then f_o is an odd mapping and f_e is an even mapping such that $2f = f_o + f_e$.

The above corollaries can be summarized as follows:

THEOREM 5.5. *Assume that* $(X,\perp)$ *is a non-Archimedean orthogonality normed space. Let* θ *be a positive real number and* p *a real number with* $p > 2$ *(resp.* $0 < p < 1$*). Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and* (2.7)*. Then there exist an orthogonally additive mapping* $L : X \to Y$ *and an orthogonally quadratic mapping* $Q : X \to Y$ *such that*

$$\begin{aligned} &\|2f(x) - L(x) - Q(x)\|_Y \leq max\left(\tfrac{|2|^{p-1}}{|2|-|2|^p}, \tfrac{|2|^p}{|2|^2-|2|^p}\right) 2\theta\|x\|^p \\ (resp.\quad &\|2f(x) - L(x) - Q(x)\|_Y \leq max\left(\tfrac{|2|^{p-1}}{|2|^p-|2|}, \tfrac{|2|^p}{|2|^p-|2|^2}\right) 2\theta\|x\|^p) \end{aligned}$$

for all $x \in X$.

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