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SOME PROPERTIES OF BILINEAR MAPPINGS ON THE TENSOR PRODUCT OF C^*-ALGEBRAS

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C^*-algebras and $\mathcal{A} \otimes \mathcal{B}$ be their algebraic tensor product. For two bilinear maps on $\mathcal{A}$ and $\mathcal{B}$ with some specific conditions, we derive a bilinear map on $\mathcal{A} \otimes \mathcal{B}$ and study some characteristics. Considering two $\mathcal{A} \otimes \mathcal{B}$ bimodules, a centralizer is also obtained for $\mathcal{A} \otimes \mathcal{B}$ corresponding to the given bilinear maps on $\mathcal{A}$ and $\mathcal{B}$. A relationship between orthogonal complements of subspaces of $\mathcal{A}$ and $\mathcal{B}$ and their tensor product is also deduced with suitable example.

1. Introduction

The characterization of different types of mappings acting on different spaces is an interesting area of research in present times. In 1952, G. J. Wendel [36] first introduced the notion of centralizer in his work on group algebras. Helgason, in 1956 [17] introduced centralizer for Banach algebras. Centralizer for rings was introduced by B. E. Johnson [18] in 1964. Akemann et al. [1], investigated centralizers on C^*-algebras. In [14], Ghahramani studied about the centralizers and Jordan centralizers on Banach algebras considering bilinear maps satisfying a related condition. Recently a good number of prominent mathematicians have

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studied the behaviour of different maps like homomorphisms, derivations etc. when acting on special products (refer to [2], [31], [35],). Moreover, there are many extensions and generalizations of various existing results regarding the characterization of mappings in different directions with several applications (refer to ([7]- [12]), [15], [21]- [26], [28], [30]).

The theoretical study of tensor product of C^*-algebras was started in 1952 by T. Turumaru [34]. In 1969 A.Guichardiet [16] discussed about C^*-tensor norms and tensor product of C^*-algebras. In 1984, [20] Kaijser and Sinclair studied about the projective tensor product of C^*-algebras. In [3], Blecher investigated the geometrical properties of algebra norms on the tensor product of C^*-algebras. Many interesting results in this direction have been developed by different researchers (refer to [5], [6] etc.) time to time.

In this paper, we extend the works of Ghahramani [14] to the tensor product of C^*-algebras and obtain some specific properties of bilinear maps on such algebras. Using the bilinear map, we also give a characterization of centralizer in the tensor product.

2. Some basic definitions

Definition 2.1. [4] Let $\mathcal{A}$ and $\mathcal{B}$ be two normed spaces over the field $\mathbb{F}$ with dual spaces $\mathcal{A}^*$ and $\mathcal{B}^*$. For $a \in \mathcal{A}$ and $b \in \mathcal{B}$, let $a \otimes b$ be the element of $BL(\mathcal{A}^*, \mathcal{B}^*; \mathbb{F})$ defined by

$$a \otimes b(p, q) = p(a)q(b) \quad (p \in \mathcal{A}^*, \, q \in \mathcal{B}^*).$$

The algebraic tensor product of $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \otimes \mathcal{B}$ is defined as the linear span of $\{a \otimes b : a \in \mathcal{A}, \, b \in \mathcal{B}\}$ in $BL(\mathcal{A}^*, \mathcal{B}^*; \mathbb{F})$, where $BL(\mathcal{A}^*, \mathcal{B}^*; \mathbb{F})$ is the set of all bounded bilinear mappings from $\mathcal{A}^* \times \mathcal{B}^*$ to $\mathbb{F}$.

Definition 2.2. [3] Given normed spaces $\mathcal{A}$ and $\mathcal{B}$, the projective tensor norm (γ) on $\mathcal{A} \otimes \mathcal{B}$ is defined by

$$\gamma(u) = \inf\{\sum_{i=1}^{n} \|a_i\|\|b_i\| : u = \sum_{i=1}^{n} a_i \otimes b_i\}$$

where the infimum is taken over all (finite) representations of u. The completion of $\mathcal{A} \otimes \mathcal{B}$ with respect to γ is called the projective tensor product of $\mathcal{A}$ and $\mathcal{B}$ and it is denoted by $\mathcal{A} \otimes_\gamma \mathcal{B}$.

For example, Let μ, ν be positive σ-finite measures on measure spaces M, N respectively, and let $\mu \times \nu$ be the corresponding product measure on $M \times N$. Then there exists an isometric linear isomorphism of $L^1(\mu) \otimes_\gamma L^1(\nu)$ onto $L^1(\mu \times \nu)$. [4].

Definition 2.3. [4] A norm α on $\mathcal{A} \otimes \mathcal{B}$ is a cross norm if $\alpha(a \otimes b) = \|a\|\|b\|$, $\forall\, a \in \mathcal{A}$, $b \in \mathcal{B}$.

For example, projective tensor norm is a cross norms.

Lemma 2.4. *[4] Given $p \in \mathcal{A} \otimes \mathcal{B}$, there exist linearly independent sets $\{a_i\}$, $\{b_i\}$ such that $p = \sum_{i=1}^{n} a_i \otimes b_i$.*

Lemma 2.5. *[4] Let $\mathcal{A}$ and $\mathcal{B}$ be normed algebras over $\mathbb{F}$. There exists a unique product on $\mathcal{A} \otimes \mathcal{B}$ with respect to which $\mathcal{A} \otimes \mathcal{B}$ is an algebra and*

$$(a \otimes b)(c \otimes d) = ac \otimes bd \;\; (a,\, c\, \in \mathcal{A},\, b,\, d\, \in \mathcal{B}).$$

Definition 2.6. [4] In an algebra $\mathcal{A}$, for $x, x^* \in \mathcal{A}$, an involution is a map $x \to x^*$ such that $(x+y)^* = x^* + y^*$, $(x^*)^* = x$, $(xy)^* = y^*x^*$, $(\alpha x)^* = \bar{\alpha} x^*, \forall\, x, y \in \mathcal{A}$ and for all scalar α, where x^* is called the adjoint of x.

An algebra $\mathcal{A}$ with an involution * is called a *-algebra. The most common example of a *-algebra is the field of complex numbers $\mathbb{C}$ (over real) where * is complex conjugation.
If $\mathcal{A}$ and $\mathcal{B}$ are two *-algebras, then $\mathcal{A} \otimes \mathcal{B}$ is also a *-algebra where $(a \otimes b)^* = a^* \otimes b^*$.

Definition 2.7. [19] A norm on a $*$-algebra $\mathcal{A}$ that satisfies $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$ is called a C^*-norm and the algebra is called C^*-algebra.

An example of C^*-algebra is $B(\mathcal{H})$, the set of all bounded linear operator on a Hilbert space $\mathcal{H}$.

Definition 2.8. [14] Let $\mathcal{A}$ be a C^*-algebra and M be an $\mathcal{A}$-bimodule. A linear(additive) map $h : \mathcal{A} \to M$ is said to be a right (left) centralizer if

$$h(xy) = xh(y) \;\, (h(xy) = h(x)y) \;\forall\; x,\, y \in \mathcal{A}.$$

If h is both right and left centralizer then it is called a centralizer.

For example, let $h : \mathbb{R} \to \mathbb{R}$ be a mapping defined as $h(x) = \frac{x}{2}$, $x \in \mathbb{R}$. Clearly, h is a centralizer since $h(xy) = xh(y) = h(x)y = \frac{xy}{2}$, $\forall\; x, y \in \mathbb{R}$.

Definition 2.9. [14] For a C^*-algebra $\mathcal{A}$, with $\mathcal{A}$-bimodule M, a centralizer $h : \mathcal{A} \to M$ is called right (left) Jordan centralizer if

$$h(x^2) = xh(x)\ (h(x^2) = h(x)x),\ \text{for each } x \in \mathcal{A}.$$

h is said to be Jordan centralizer if

$$h(xy + yx) = xh(y) + h(y)x = yh(x) + h(x)y\ \forall x, y \in \mathcal{A}.$$

Every centralizer is a Jordan centralizer. But the converse is not true in general.

Example 2.10. Let $\mathcal{A}'$ be a C^*-algebra such that the square of each element in $\mathcal{A}'$ is zero but the product of some elements in $\mathcal{A}'$ is non-zero. Let $\mathcal{A} = \{a = \begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix} : p, q \in \mathcal{A}'\}$.

We define, $h : \mathcal{A} \to \mathcal{A}$ such that $h(a) = \begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix}$.

Then h is a Jordan centralizer. Also it can be easily verified that h is a right centralizer, but not a left centralizer.

3. Main Results

Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras with unit elements e_1 and e_2 respectively and $\mathcal{A} \otimes \mathcal{B}$ be their algebraic tensor product. Then for the unique product as given by Lemma 2.5, $\mathcal{A} \otimes \mathcal{B}$ is an algebra. Here we consider $\mathcal{A} \otimes \mathcal{B}$ with the projective tensor norm.

Let $\circ$ denote the Jordan product on $\mathcal{A} \otimes \mathcal{B}$ such that for $\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} a_i \otimes b_i,\ \sum_{j=1}^{m} v_j = \sum_{j=1}^{m} c_j \otimes d_j$ in $\mathcal{A} \otimes \mathcal{B}$,

$$(\sum_{i=1}^{n} u_i) \circ (\sum_{j=1}^{m} v_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} u_i v_j + \sum_{j=1}^{m}\sum_{i=1}^{n} v_j u_i.$$

Let the set of invertible elements of $\mathcal{A} \otimes \mathcal{B}$ be $Inv(\mathcal{A} \otimes \mathcal{B})$. Then $Inv(\mathcal{A} \otimes \mathcal{B})$ is an open subset of $\mathcal{A} \otimes \mathcal{B}$ and so, it is a disjoint union of open connected subsets, the components of $Inv(\mathcal{A} \otimes \mathcal{B})$ (refer to [14]). The component containing $e_1 \otimes e_2$ is called the principal component of $Inv(\mathcal{A} \otimes \mathcal{B})$ and it is denoted by $Inv_0(\mathcal{A} \otimes \mathcal{B})$.

In [14], Ghahramani used bilinear maps to study centralizers and Jordan centralizers on Banach algebras. From a given bilinear map on Banach algebra with some specific conditions, Ghahramani derived the following linear map.

THEOREM 3.1. [14] *Let $\mathcal{A}$ be a unital Banach algebra and X be a Banach space. Let $h : \mathcal{A} \times \mathcal{A} \to X$ be a continuous bilinear map such that*

$$a \in Inv_0(\mathcal{A}) \Rightarrow h(a, a^{-1}) = h(e_1, e_1). \tag{1}$$

$$\text{Then,} \quad h(a, a) = h(a^2, e_1) \text{ and } h(a, e_1) = h(e_1, a), \tag{2}$$

where $a \in \mathcal{A}$ and there exists a continuous linear map $P : \mathcal{A} \to X$ such that

$$h(a_1, a_2) + h(a_2, a_1) = P(a_1 \circ a_2), \ a_1, a_2 \in \mathcal{A}.$$

Here, we extend the above result for the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ with projective tensor norm, and considering Banach spaces X and Y with projective tensor product $X \otimes_\gamma Y$, (which is also a Banach space).

THEOREM 3.2. *Let $f_1 : \mathcal{A} \times \mathcal{A} \to X$ and $f_2 : \mathcal{B} \times \mathcal{B} \to Y$ be two continuous bilinear maps each satisfying the above property (1). Then there exists a continuous bilinear map*

$$f : \mathcal{A} \otimes \mathcal{B} \times \mathcal{A} \otimes \mathcal{B} \to X \otimes_\gamma Y$$

with the following properties:
for $\sum_{i=1}^n u_i = \sum_{i=1}^n a_i \otimes b_i$, $\sum_{j=1}^m v_j = \sum_{j=1}^m c_j \otimes d_j \in \mathcal{A} \otimes \mathcal{B}$,

$I) u_i \in Inv_0(\mathcal{A} \otimes \mathcal{B})$, $i = 1, 2, ..., n.$

$$\Rightarrow f(\sum_{i=1}^n u_i, \sum_{i=1}^n u_i^{-1}) = n f_1(e_1, e_1) \otimes f_2(e_2, e_2)$$
$$+ \frac{1}{2}[\sum_{i=1}^n \sum_{j\neq i, j=1}^n (f_1(a_i, a_j^{-1}) \otimes f_2(b_i, b_j^{-1}) + f_1(a_j^{-1}, a_i) \otimes f_2(b_j^{-1}, b_i))],$$

$$II) f(\sum_{i=1}^n u_i, \sum_{i=1}^n u_i) = f(\sum_{i=1}^n u_i^2, e_1 \otimes e_2)$$
$$+ \frac{1}{2}\sum_{i=1}^n \sum_{j\neq i, j=1}^n (f(a_i \otimes b_i, a_j \otimes b_j) + f(a_j \otimes b_j, a_i \otimes b_i)),$$

$$III) f(\sum_{i=1}^n u_i, e_1 \otimes e_2) = f(e_1 \otimes e_2, \sum_{i=1}^n u_i).$$

Moreover, there exists a continuous linear map

$$S : \mathcal{A} \otimes \mathcal{B} \to X \otimes_\gamma Y$$

such that

$$(3)\quad f(\sum_{i=1}^{n} u_i + \sum_{j=1}^{m} v_j, \sum_{i=1}^{n} u_i + \sum_{j=1}^{m} v_j) - \sum_{i=1}^{n} f(u_i, \sum_{j\neq i, j=1}^{n} u_j + \sum_{j=1}^{m} v_j)$$
$$+ \sum_{i=1}^{m} f(v_i, \sum_{j\neq i, j=1}^{n} u_j + \sum_{j=1}^{m} v_j)$$
$$= S(\sum_{i=1}^{n} u_i^2) + S(\sum_{j=1}^{m} v_j^2).$$

[Here, the representations $\sum_{i=1}^{n} u_i,\ \sum_{j=1}^{n} v_j \in \mathcal{A}\otimes\mathcal{B}$ follows by Lemma 2.4.]

Proof. Using f_1 and f_2, we define a map $f : \mathcal{A}\otimes\mathcal{B}\times\mathcal{A}\otimes\mathcal{B} \to X \otimes_\gamma Y$ by

$$f(\sum_{i=1}^{n} u_i, \sum_{j=1}^{m} v_j) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}[f_1(a_i, c_j)\otimes f_2(b_i, d_j) + f_1(c_j, a_i)\otimes f_2(d_j, b_i)].$$

First we show that f is a bilinear map.
For $\sum_{i=1}^{n} u_i,\ \sum_{j=1}^{m} v_j$ as already defined (without loss of generality let $m > n$), we take,
$x_l \otimes y_l = a_l \otimes b_l,\ l = 1, 2..., n$ and $x_{n+j} \otimes y_{n+j} = c_j \otimes d_j,\ j = 1, 2.., m$.
Let $\sum_{k=1}^{r} w_k = \sum_{k=1}^{r} p_k \otimes q_k \ \in \mathcal{A}\otimes\mathcal{B}$ and $\alpha\,, \beta$ be scalars. Now,

$$f(\sum_{i=1}^{n} u_i + \sum_{j=1}^{m} v_j, \sum_{k=1}^{r} w_k) = f(\sum_{l=1}^{n+m} x_l \otimes y_l, \sum_{k=1}^{r} w_k)$$
$$= \frac{1}{2}[\sum_{l=1}^{n+m}\sum_{k=1}^{r}(f_1(x_l, p_k)\otimes f_2(y_l, q_k) + f_1(p_k, x_l)\otimes f_2(q_k, y_l))]$$
$$= \frac{1}{2}[\sum_{l=1}^{n}\sum_{k=1}^{r}(f_1(x_l, p_k)\otimes f_2(y_l, q_k) + f_1(p_k, x_l)\otimes f_2(q_k, y_l))$$
$$+ \sum_{l=n+1}^{m}\sum_{k=1}^{r}(f_1(x_l, p_k)\otimes f_2(y_l, q_k) + f_1(p_k, x_l)\otimes f_2(q_k, y_l))]$$
$$= f(\sum_{i=1}^{n} u_i, \sum_{k=1}^{r} w_k) + f(\sum_{j=1}^{m} v_j, \sum_{k=1}^{r} w_k).$$

Similarly, we can show that

$$f(\sum_{i=1}^{n} u_i, \sum_{j=1}^{m} v_j + \sum_{k=1}^{r} w_k) = f(\sum_{i=1}^{n} u_i, \sum_{k=1}^{r} w_k) + f(\sum_{j=1}^{m} v_j, \sum_{k=1}^{r} w_k).$$

Also, using the bilinearity of the mapping f_1,

$$\begin{aligned} f(\sum_{i=1}^{n} \alpha u_i, \sum_{k=1}^{r} w_k) &= \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{r}[f_1(\alpha a_i, p_k) \otimes f_2(b_i, q_k) + f_1(p_k, \alpha a_i) \otimes f_2(q_k, b_i)] \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{r}[\alpha f_1(a_i, p_k) \otimes f_2(b_i, q_k) + \alpha f_1(p_k, a_i) \otimes f_2(q_k, b_i)] \\ &= \alpha f(\sum_{i=1}^{n} u_i, \sum_{k=1}^{r} w_k). \end{aligned}$$

Similarly, using the bilinearity of f_2, we can show that

$$f(\sum_{i=1}^{n} u_i, \sum_{k=1}^{r} \beta w_k) = \beta f(\sum_{i=1}^{n} u_i, \sum_{k=1}^{r} w_k).$$

Now, f_1 and f_2 are continuous (and hence bounded) mappings and the projective tensor norm on $\mathcal{A} \otimes \mathcal{B}$ is a cross norm. Hence, it follows that

$$\begin{aligned} \|f(\sum_{i=1}^{n} u_i, \sum_{j=1}^{m} v_j)\| &\leq \sum_{i=1}^{n}\sum_{j=1}^{m} \|f_1\|\|f_2\|\|a_i\|\|c_j\|\|b_i\|\|d_j\| \\ &= \|f_1\|\|f_2\| \sum_{i=1}^{n} \|a_i\|\|b_i\| \sum_{j=1}^{m} \|c_j\|\|d_j\|. \end{aligned}$$

Using the definition of projective tensor norm,

$$\|f(\sum_{i=1}^{n} u_i, \sum_{j=1}^{m} v_j)\| \leq \|f_1\|\|f_2\|\|\sum_{i=1}^{n} u_i\|\|\sum_{j=1}^{m} v_j\|,$$

$$i.e., \|f\| \leq \|f_1\|\|f_2\|,$$

showing that f is bounded and hence continuous.
(I) Let $u_i \in Inv_0(\mathcal{A} \otimes \mathcal{B}),\ \forall\ i = 1, 2, .., n.$

$$f(\sum_{i=1}^{n} u_i, \sum_{i=1}^{n} u_i^{-1}) = f(\sum_{i=1}^{n} u_i, \sum_{j=1}^{n} u_j^{-1}).$$

From the definition of f, the right hand expression equals to

$$\frac{1}{2}[\sum_{i=1}^{n}\sum_{j=1}^{n}(f_1(a_i, a_j^{-1}) \otimes f_2(b_i, b_j^{-1}) + f_1(a_j^{-1}, a_i) \otimes f_2(b_j^{-1}, b_i))]$$

$$= \sum_{i=1}^{n} f_1(e_1, e_1) \otimes f_2(e_2, e_2)$$

$$+ \frac{1}{2}[\sum_{i=1}^{n}\sum_{j\neq i, j=1}^{n}(f_1(a_i, a_j^{-1}) \otimes f_2(b_i, b_j^{-1}) + f_1(a_j^{-1}, a_i) \otimes f_2(b_j^{-1}, b_i))]$$

$$= nf_1(e_1, e_1) \otimes f_2(e_2, e_2)$$

$$+ \frac{1}{2}[\sum_{i=1}^{n}\sum_{j\neq i, j=1}^{n}(f_1(a_i, a_j^{-1}) \otimes f_2(b_i, b_j^{-1}) + f_1(a_j^{-1}, a_i) \otimes f_2(b_j^{-1}, b_i))].$$

(II) For $\sum_{i=1}^{n} u_i \in \mathcal{A} \otimes \mathcal{B}$,

$$f(\sum_{i=1}^{n} u_i, \sum_{i=1}^{n} u_i)$$

$$= \sum_{i=1}^{n} f_1(a_i, a_i) \otimes f_2(b_i, b_i)$$

$$+ \frac{1}{2}[\sum_{i=1}^{n}\sum_{j\neq i, j=1}^{n}(f_1(a_i, a_j) \otimes f_2(b_i, b_j) + f_1(a_j, a_i) \otimes f_2(b_j, b_i))].$$

By property (2) of the Theorem 3.1 of f_1 and f_2, the above expression equals

$$\sum_{i=1}^{n} f_1(a_i^2, e_1) \otimes f_2(b_i^2, e_2)$$

$$+ \frac{1}{2}[\sum_{i=1}^{n}\sum_{j\neq i, j=1}^{n}(f_1(a_i, a_j) \otimes f_2(b_i, b_j) + f_1(a_j, a_i) \otimes f_2(b_j, b_i))]$$

$$= f(\sum_{i=1}^{n} a_i^2 \otimes b_i^2, e_1 \otimes e_2)$$

$$+ \frac{1}{2}[\sum_{i=1}^{n}\sum_{j\neq i, j=1}^{n}(f(a_i \otimes b_i, a_j \otimes b_j) + f(a_j \otimes b_j, a_i \otimes b_i))]$$

$$= f(\sum_{i=1}^{n} u_i^2, e_1 \otimes e_2)$$

$$(4) \qquad + \frac{1}{2}[\sum_{i=1}^{n} \sum_{j \neq i, j=1}^{n} (f(a_i \otimes b_i, a_j \otimes b_j) + f(a_j \otimes b_j, a_i \otimes b_i))].$$

(III) For $\sum_{i=1}^{n} u_i \in \mathcal{A} \otimes \mathcal{B}$,

$$f(\sum_{i=1}^{n} u_i, e_1 \otimes e_2) = \frac{1}{2}[\sum_{i=1}^{n}(f_1(a_i, e_1) \otimes f_2(b_i, e_2) + f_1(e_1, a_i) \otimes f_2(e_2, b_i))]$$

$$= \frac{1}{2}[\sum_{i=1}^{n}(f_1(e_1, a_i) \otimes f_2(e_2, b_i) + f_1(e_1, a_i) \otimes f_2(e_2, b_i))],$$

(using the property (2) of f_1 and f_2.)

$$= \sum_{i=1}^{n} f_1(e_1, a_i) \otimes f_2(e_2, b_i)$$

$$= f(e_1 \otimes e_2, \sum_{i=1}^{n} u_i).$$

Next, with the help of f we define a map $S : \mathcal{A} \otimes \mathcal{B} \to X \otimes_\gamma Y$ by

$$S(\sum_{i=1}^{n} u_i) = f(\sum_{i=1}^{n} u_i, e_1 \otimes e_2).$$

Clearly, S is linear.

Also, S is bounded and hence continuous, since

$$\|S(\sum_{i=1}^{n} a_i \otimes b_i)\| = \| \sum_{i=1}^{n} f_1(a_i, e_1) \otimes f_2(b_i, e_2)\|$$

$$\leq \|f_1\|\|f_2\| \sum_{i=1}^{n} \|a_i\|\|b_i\|.$$

For the projective tensor norm on $\mathcal{A} \otimes \mathcal{B}$, we get,

$$\|S(\sum_{i=1}^{n} a_i \otimes b_i)\| \leq \|f_1\|.\|f_2\|\|\sum_{i=1}^{n} a_i \otimes b_i\|, i.e., \|S\| \leq \|f_1\|\|f_2\|.$$

Now, using (3) and taking $\sum_{i=1}^{n} u_i + \sum_{j=1}^{m} v_j = \sum_{k=1}^{m+n} w_k$, where

$w_k = u_k, k = 1, 2, .., n, w_{n+k} = v_k, k = 1, 2, .., m$ and $m > n$ we get,

$$f(\sum_{i=1}^{n} u_i + \sum_{j=1}^{m} v_j, \sum_{i=1}^{n} u_i + \sum_{j=1}^{m} v_j) = f(\sum_{k=1}^{m+n} w_k, \sum_{k=1}^{m+n} w_k).$$

Since f satisfies the property (II), so,

$$\begin{aligned}
&f(\sum_{k=1}^{m+n} w_k, \sum_{k=1}^{m+n} w_k)\\
&= f(\sum_{k=1}^{n+m} w_k^2, e_1 \otimes e_2) + \frac{1}{2}[\sum_{k=1}^{n+m} \sum_{\substack{l=1\\ l\neq k}}^{n+m} (f(w_k, w_l) + f(w_l, w_k))]\\
&= f(\sum_{k=1}^{n} w_k^2, e_1 \otimes e_2) + f(\sum_{k=1}^{m} w_{n+k}^2, e_1 \otimes e_2) + \frac{1}{2}[\sum_{k=1}^{n+m} \sum_{l=1, l\neq k}^{n} f(w_k, w_l)\\
&+ \sum_{k=1}^{n+m} \sum_{l=1, l\neq k}^{m} f(w_k, w_{n+l}) \sum_{l=1}^{n} \sum_{\substack{k=1\\ k\neq l}}^{n+m} f(w_l, w_k) + \sum_{l=1}^{m} \sum_{\substack{k=1\\ k\neq l}}^{n+m} f(w_{n+l}, w_k)]\\
&= f(\sum_{k=1}^{n} w_k^2, e_1 \otimes e_2) + f(\sum_{k=1}^{m} w_{n+k}^2, e_1 \otimes e_2)\frac{1}{2}[\sum_{k=1}^{n} \sum_{\substack{l=1\\ l\neq k}}^{n} f(w_k, w_l)\\
&+ \sum_{k=1}^{m} \sum_{\substack{l=1\\ l\neq k}}^{n} f(w_{n+k}, w_l) + \sum_{k=1}^{n} \sum_{\substack{l=1\\ l\neq k}}^{m} f(w_k, w_{n+l}) + \sum_{k=1}^{m} \sum_{\substack{l=1\\ l\neq k}}^{m} f(w_{n+k}, w_{n+l})\\
&+ \sum_{l=1}^{n} \sum_{\substack{k=1\\ k\neq l}}^{n} f(w_l, w_k) + \sum_{l=1}^{n} \sum_{\substack{k=1\\ k\neq l}}^{m} f(w_l, w_{n+k}) +\\
&\sum_{l=1}^{m} \sum_{\substack{k=1\\ k\neq l}}^{n} f(w_{n+l}, w_k) + \sum_{l=1}^{m} \sum_{\substack{k=1\\ k\neq l}}^{m} f(w_{n+l}, w_{n+k})]
\end{aligned}$$

After simplification, the above expression reduces to

$$f(\sum_{k=1}^{n} u_k^2, e_1 \otimes e_2) + f(\sum_{k=1}^{m} v_k^2, e_1 \otimes e_2) + \sum_{k\neq l,k,l=1}^{n} f(u_k, u_l)$$
$$+ \sum_{k=1}^{m} \sum_{k\neq l,l=1}^{n} f(v_k, u_l) + \sum_{k=1}^{n} \sum_{k\neq l,l=1}^{m} f(u_k, v_l)$$
$$+ \sum_{k\neq l,k,l=1}^{m} f(v_k, v_l),$$

which equals to

$$S(\sum_{k=1}^{n} u_k^2) + S(\sum_{k=1}^{m} v_k^2) + \sum_{k=1}^{n} f(u_k, \sum_{l\neq k,l=1}^{n} u_l + \sum_{l=1}^{m} v_l)$$
$$+ \sum_{k=1}^{m} f(v_k, \sum_{l\neq k,l=1}^{n} u_l + \sum_{l=1}^{m} v_l).$$

Then,

$$f(\sum_{i=1}^{n} u_i + \sum_{j=1}^{m} v_j, \sum_{i=1}^{n} u_i + \sum_{j=1}^{m} v_j) - \sum_{i=1}^{n} f(u_i, \sum_{j\neq i,j=1}^{n} u_j + \sum_{j=1}^{m} v_j)$$
$$+ \sum_{i=1}^{m} f(v_i, \sum_{j\neq i,j=1}^{n} u_j + \sum_{j=1}^{m} v_j)$$
$$= S(\sum_{i=1}^{n} u_i^2) + S(\sum_{j=1}^{m} v_j^2).$$

□

EXAMPLE 3.3. For the unital C^*-algebras l^1 (over $\mathbb{R}$) and $\mathbb{R}$, we define maps $f_1 : l^1 \times l^1 \to l^1$ by

$$f_1(\{a_1, a_2, ...\}, \{b_1, b_2, ...\}) = \{a_1b_1, a_2b_2, 0, 0...\}, \text{ for } \{a_n\}, \{b_n\} \in l^1$$

and $f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $f_2(x, y) = xy$, for $x, y \in \mathbb{R}$.
Clearly, f_1 and f_2 are continuous and bilinear maps.
Here $e_1 = \{1, 1, ...\}$ (the constant sequence) $\in l^1$ and $e_2 = 1 \in \mathbb{R}$.

Also, for $\{a_n\} \in Inv_0(l^1)$,

$$\begin{aligned} &f_1(\{a_1, a_2, ...\}, \{a_1^{-1}, a_2^{-1}, ...\}) \\ &= \{a_1 a_1^{-1}, a_2 a_2^{-1}, 0, 0...\} \\ &= \{1, 1, 0, 0...\} \\ &= f_1(e_1, e_1), \end{aligned}$$

and for $x \in \mathbb{R}$ invertible,

$$f_2(x, x^{-1}) = xx^{-1} = 1 = f_2(e_2, e_2).$$

Now, $l^1 \otimes \mathbb{R} \;\cong\; l^1(\mathbb{R})$(refer to [29]).
So, by the Theorem 3.2, there exists $f : (l^1 \otimes \mathbb{R}) \times (l^1 \otimes \mathbb{R}) \to l^1(\mathbb{R})$ such that

$$(5) \quad f(\sum_{i=1}^{n} a_i \otimes b_i, \sum_{j=1}^{m} c_j \otimes d_j) = \sum_{i=1}^{n}\sum_{j=1}^{m}\{p_{1i}q_{1j}b_i d_j, p_{2i}q_{2j}b_i d_j, 0, 0, ...\}$$

where $a_i = \{p_{ki}\}_k$, $c_j = \{q_{kj}\}_k \in l^1$ and $b_i, d_j \in \mathbb{R}, i = 1, 2, ..., n; j = i, 2, .., m$, and

$\sum_{i=1}^{n} a_i \otimes b_i = \sum_{i=1}^{n}\{p_{ki}b_i\}_k$ and $\sum_{j=1}^{m} c_j \otimes d_j = \sum_{j=1}^{m}\{q_{kj}d_j\}_k$.

Now, for the formula defined in Theorem 3.2,

$$\begin{aligned} &\frac{1}{2}[\sum_{i=1}^{n}\sum_{j=1}^{m}(f_1(a_i, c_j) \otimes f_2(b_i, d_j) + f_1(c_j, a_i) \otimes f_2(d_j, b_i)) \\ &= \frac{1}{2}[\sum_{i=1}^{n}\sum_{j=1}^{m}(\{p_{1i}q_{1j}, p_{2i}q_{2j}, 0, 0, ...\} \otimes b_i d_j + \{q_{1j}p_{1i}, q_{2j}p_{2i}, 0, 0, ...\} \otimes d_j b_i)] \\ &= \frac{1}{2}[\sum_{i=1}^{n}\sum_{j=1}^{m}(\{p_{1i}q_{1j}b_i d_j, p_{2i}q_{2j}b_i d_j, 0, 0, ...\} + \{q_{1j}p_{1i}d_j b_i, q_{2j}p_{2i}d_j b_i, 0, 0, ...\})] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\{p_{1i}q_{1j}b_i d_j, p_{2i}q_{2j}b_i d_j, 0, 0, ...\}. \end{aligned}$$

By (4) this is clearly equal to $f(\sum_{i=1}^{n} a_i \otimes b_i, \sum_{j=1}^{m} c_j \otimes d_j)$, which validates the Theorem 3.2.

Next, we consider two bilinear maps on the individual C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, and derive a centralizer on the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$. For this, let M and N be two $\mathcal{A} \otimes \mathcal{B}$-bimodules. Then by [32], $M \otimes N$ is also an $\mathcal{A} \otimes \mathcal{B}$-bimodule.

THEOREM 3.4. *Suppose* $\phi_1 : \mathcal{A} \times \mathcal{A} \to M$ *and* $\phi_2 : \mathcal{B} \times \mathcal{B} \to N$ *be two bilinear maps satisfying the properties :*

$$\begin{aligned} &i) x \in Inv_0(\mathcal{A}) \Rightarrow \phi_1(x, x^{-1}) = \phi_1(e_1, e_1) \text{ and} \\ &\quad y \in Inv_0(\mathcal{B}) \Rightarrow \phi_2(y, y^{-1}) = \phi_2(e_2, e_2) \\ &ii) \phi_1(e_1, x_1x_2) + \phi_1(x_1x_2, e_1) = (x_1 \otimes e_2)[\phi_1(e_1, x_2) + \phi_1(x_2, e_1)] \\ &\quad = [\phi_1(x_1, e_1) + \phi_1(e_1, x_1)](x_2 \otimes e_2) \ \forall \ x_1, \ x_2 \in \mathcal{A} \\ &iii) \phi_2(e_2, y_1y_2) + \phi_2(y_1y_2, e_2) = (e_1 \otimes y_1)[\phi_2(e_2, y_2) + \phi_2(y_2, e_2)] \\ &\quad = [\phi_2(y_1, e_2) + \phi_2(e_2, y_1)](e_2 \otimes y_2) \ \forall \ y_1, \ y_2 \ \in \mathcal{B} \end{aligned}$$

Then corresponding to ϕ_1 *and* ϕ_2, *there exists a Jordan centralizer from* $\mathcal{A} \otimes \mathcal{B}$ *to* $M \otimes N$.

Proof. Since ϕ_1 and ϕ_2 satisfy the property (i), so by Theorem 3.1 there exist two linear maps

$$h_1 : \mathcal{A} \to M \text{ and } h_2 : \mathcal{B} \to N \text{ given by}$$

$$\begin{aligned} &\phi_1(x_1, x_2) + \phi_1(x_2, x_1) = h_1(x_1 \circ x_2) \text{ and} \\ &\phi_2(y_1, y_2) + \phi_2(y_2, y_1) = h_2(y_1 \circ y_2), \ x_1, x_2 \in \mathcal{A}, \ y_1, y_2 \in \mathcal{B}. \end{aligned}$$

Replacing x_1 by e_1 and x_2 by x_1x_2, we have,

$$\begin{aligned} h_1(x_1x_2) &= \frac{1}{2}[\phi_1(e_1, x_1x_2) + \phi_1(x_1x_2, e_1)] \\ &= \frac{1}{2}[(x_1 \otimes e_2)[\phi_1(e_1, x_2) + \phi_1(x_2, e_1)]] \text{ by property (ii)} \\ &= (x_1 \otimes e_2)h_1(x_2). \end{aligned} \tag{6}$$

$$h_1(x_1x_2) = h_1(x_1)(x_2 \otimes e_2) \text{and} \tag{7}$$

$$h_2(y_1y_2) = (e_1 \otimes y_1)h_2(y_2) = h_2(y_1)(e_2 \otimes y_2). \tag{8}$$

Now, we define a map $h : \mathcal{A} \otimes \mathcal{B} \to M \otimes N$ by

$$h(\sum_{i=1}^{n} a_i \otimes b_i) = \sum_{i=1}^{n} h_1(a_i) \otimes h_2(b_i), \ \sum_{i=1}^{n} a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{B}.$$

Clearly, h is linear. Using Lemma 2.4 and by the definition of h, for

$$\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} a_i \otimes b_i,\ \sum_{j=1}^{m} v_j = \sum_{j=1}^{m} c_j \otimes d_j \in \mathcal{A} \otimes \mathcal{B},$$

$$h(\sum_{i=1}^{n} u_i \sum_{j=1}^{m} v_j + \sum_{j=1}^{m} v_j \sum_{i=1}^{n} u_i)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} (h_1(a_i c_j) \otimes h_2(b_i d_j) + h_1(c_j a_i) \otimes h_2(d_j b_i)).$$

Now, using the properties (5), (6), (7) of the mappings h_1 and h_2, and by the properties of module multiplication with respect to the tensor product [32], the above expression equals to

$$\sum_{i=1}^{n} \sum_{j=1}^{m} (a_i \otimes e_2) h_1(c_j) \otimes (e_1 \otimes b_i) h_2(d_j) + \sum_{i=1}^{n} \sum_{j=1}^{m} h_1(c_j)(a_i \otimes e_2) \otimes h_2(d_j)(e_1 \otimes b_i)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} ((a_i \otimes b_i)(h_1(c_j) \otimes h_2(d_j)) + (h_1(c_j) \otimes h_2(d_j))(a_i \otimes b_i))$$

$$= (\sum_{i=1}^{n} u_i) h(\sum_{j=1}^{m} v_j) + h(\sum_{j=1}^{m} v_j)(\sum_{i=1}^{n} u_i).$$

In a similar way, we can show that

$$h(\sum_{i=1}^{n} u_i \sum_{j=1}^{m} v_j + \sum_{j=1}^{m} v_j \sum_{i=1}^{n} u_i) = (\sum_{j=1}^{m} v_j) h(\sum_{i=1}^{n} u_i) + h(\sum_{i=1}^{n} u_i)(\sum_{j=1}^{m} v_j),$$

and thus h is a Jordan centralizer.□

Remark: The mapping h defined in the above Theorem 3.4 can also be shown to a centralizer, as

$$h(\sum_{i=1}^{n} u_i \sum_{j=1}^{m} v_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} (h_1(a_i c_j) \otimes h_2(b_i d_j))$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} ((a_i \otimes e_2) h_1(c_j) \otimes (e_1 \otimes b_i) h_2(d_j))$$

(by property (5) and (7) of h_1 and h_2)

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}(a_ie_1 \otimes e_2b_i)h(c_j \otimes d_j) \text{ (by Lemma 2.5)}$$
$$= (\sum_{i=1}^{n} u_i)h(\sum_{j=1}^{m} v_j).$$

In the same way, it follows that

$$h(\sum_{i=1}^{n} u_i \sum_{j=1}^{m} v_j) = h(\sum_{i=1}^{n} u_i)(\sum_{j=1}^{m} v_j).$$

□

EXAMPLE 3.5. For the C^*-algebras l^1 (over $\mathbb{R}$) and $\mathbb{R}$, we define $\phi_1 : l^1 \times l^1 \to l^1$ by

$$\phi_1(\{a_n\}, \{c_n\}) = \{a_1c_1, 0, 0...\}, \text{ for } \{a_n\}, \{c_n\} \in l^1$$

and $\phi_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\phi_2(b, d) = bd \text{ for } b, d \in \mathbb{R}.$$

Clearly, ϕ_1 and ϕ_2 are continuous bilinear maps satisfying property (i) of Theorem 3.4.

$$\begin{aligned}
&\phi_1(e_1, \{a_n\}\{c_n\}) + \phi_1(\{a_n\}\{c_n\}), e_1) \\
&= \phi_1(e_1, \{a_nc_n\}) + \phi_1(\{a_nc_n\}, e_1) \\
&= 2\{a_1c_1, 0, 0...\} \\
&= \{a_1, ...\}[\{c_1, 0, 0...\} + \{c_1, 0, 0...\}] \\
&= (\{a_n\} \otimes e_2)[\phi_1(e_1, \{c_n\}) + \phi_1(\{c_n\}, e_1)] \\
&= [\phi_1(\{a_n\}, e_1) + \phi_1(e_1, \{a_n\})](\{c_n\} \otimes e_2).
\end{aligned}$$

$$\text{Also, } \phi_2(1, bd) + \phi_2(bd, 1) = 2bd = (1 \otimes b)(\phi_2(1, d) + \phi_2(d, 1))$$
$$= \phi_2(bd, 1) + \phi_2(1, bd)(1 \otimes d). \tag{9}$$

So, properties (ii) and (iii) defined in Theorem 3.4 are also satisfied. Since ϕ_1 and ϕ_2 satisfy the property (1) of Theorem 3.1, so, there exist two linear maps

$$h_1 : l^1 \to l^1 \text{ and } h_2 : \mathbb{R} \to \mathbb{R} \text{ such that}$$

$$\phi_1(\{a_n\}, \{c_n\}) + \phi_1(\{c_n\}, \{a_n\}) = h_1(\{a_n\} \circ \{c_n\}), \text{ and}$$
$$\phi_2(b, d) + \phi_2(d, b) = h_2(b \circ d),$$

where $\{a_n\},\ \{c_n\} \in l^1$ and $b, d \in \mathbb{R}$.

Replacing $\{c_n\}$ by e_1 and d by 1 in (9) we get,

$$\begin{aligned} h_1(\{a_n\}) &= \frac{1}{2}[\phi_1(\{a_n\}, e_1) + \phi_1(e_1, \{a_n\})] \\ &= \frac{1}{2}(\{a_1, 0, 0, ...\} + \{a_1, 0, 0, ..\}) \\ &= \{a_1, 0, 0, ..\}, \end{aligned}$$

and $h_2(b) = \frac{1}{2}(\phi_2(b,1) + \phi_2(1,b)) = \frac{1}{2}(b+b) = b.$

By Theorem 3.4, the mapping

$h : l^1 \otimes \mathbb{R} \to l^1 \otimes \mathbb{R}$ is defined by

$$h(\sum_{l=1}^{n} x_l \otimes y_l) = \sum_{l=1}^{n} h_1(x_l) \otimes h_2(y_l),\ \sum_{l=1}^{n} x_l \otimes y_l \in l^1 \otimes \mathbb{R}.$$

Let $\{a_{k_i}\}_k, \{c_{k_j}\}_k \in l^1$ and $b_i, d_j \in \mathbb{R}, i = 1, 2, ..., n;\ j = 1, 2, .., m$.

Now,

$$h((\sum_{i=1}^{n}\{a_{k_i}\}_k \otimes b_i)(\sum_{j=1}^{m}\{c_{k_j}\}_k \otimes d_j))$$

$$= h(\sum_{i=1}^{n}\sum_{j=1}^{m}\{a_{k_i}\}_k\{c_{k_j}\}_k \otimes b_i d_j)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} h_1(\{a_{k_i}c_{k_j}\}_k) \otimes h_2(b_i d_j)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}\{a_{1_i}c_{1_j}b_i d_j, 0, 0...\}.$$

Again,

$$(\sum_{i=1}^{n}\{a_{k_i}\}_k \otimes b_i)h(\sum_{j=1}^{m}\{c_{k_j}\}_k \otimes d_j)$$

$$= \sum_{i=1}^{n}\{a_{k_i}b_i\}_k \sum_{j=1}^{m}\{c_{1_j}d_j, 0, 0, ...\}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}\{a_{1_i}b_i c_{1_j} d_j, 0, 0, ...\}$$

$$= h((\sum_{i=1}^{n}\{a_{k_i}\}_k \otimes b_i)(\sum_{j=1}^{m}\{c_{k_j}\}_k \otimes d_j).$$

Similarly,

$$h((\sum_{i=1}^{n}\{a_{k_i}\}_k \otimes b_i)(\sum_{j=1}^{m}\{c_{k_j}\}_k \otimes d_j) = h(\sum_{i=1}^{n}\{a_{k_i}\}_k \otimes b_i)(\sum_{j=1}^{m}\{c_{k_j}\}_k \otimes d_j),$$

which shows that h is a centralizer and hence a Jordan centralizer, which is guaranteed by Theorem 3.4.

COROLLARY 3.6. *Let ϕ_1 and ϕ_2 be as defined in Theorem 3.4.*

I) If the maps ϕ_1 and ϕ_2 are symmetric i.e., $\phi_1(a_1, a_2) = \phi_1(a_2, a_1)$ and $\phi_2(b_1, b_2) = \phi_2(b_2, b_1)$, for $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$, then $h(a \otimes b) = \phi_1(a, e_1) \otimes \phi_2(b, e_2)$ $a \in \mathcal{A}$, $b \in \mathcal{B}$.

II) If any of the maps ϕ_1 and ϕ_2 is skew-symmetric i.e., $\phi_1(a_1, a_2) = -\phi_1(a_2, a_1)$ or $\phi_2(b_1, b_2) = -\phi_2(b_2, b_1)$, then h is a zero mapping.

III) If the maps ϕ_1 and ϕ_2 are alternating i.e., $\phi_1(a, a) = 0$ and $\phi_2(b, b) = 0$ then, $h_1(a^2) = 0$ and $h_2(b^2) = 0$.

Proof. I) is obvious.
II) If ϕ_1 or ϕ_2 is skew-symmetric then

$$\phi_1(a, e_1) = -\phi_1(e_1, a) \text{ or } \phi_2(b, e_2) = -\phi_2(e_2, b).$$

Then $h_1(a) = 0$ or $h_2(b) = 0$, which implies that h is a zero mapping.
III) If $\phi_1(a, a) = 0$ then

$$h_1(a \circ a) = \phi_1(a, a) + \phi_1(a, a) = 0 \Rightarrow h_1(a^2) = 0.$$

Similarly, $h_2(b^2) = 0$ if, $\phi_2(b, b) = 0$. □

The following result gives some characteristics of the centralizer h on $\mathcal{A} \otimes \mathcal{B}$.

THEOREM 3.7. *The mapping h defined in Theorem 3.4 satisfies the following properties:*

$$(I)\, h(\sum_{i=1}^{n} a_i \otimes b_i) - (\sum_{i=1}^{n} a_i \otimes b_i)(h_1(e_1) \otimes h_2(e_2)) = 0, \textit{where}$$
$$\sum_{i=1}^{n} a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{B}.$$

$$(II)\, \textit{For} \sum_{i=1}^{n} a_i \otimes b_i, \sum_{i=1}^{n} c_i \otimes d_i \in \mathcal{A} \otimes \mathcal{B}, \textit{if}, a_i c_i \otimes b_i d_i = e_1 \otimes e_2,$$
$$i = 1, 2, 3, .., n, \textit{then}$$
$$\sum_{i=1}^{n} (a_i \otimes b_i)(h_1(c_i) \otimes h_2(d_i)) - n(h_1(e_1) \otimes h_2(e_2)) = 0.$$

$$(III)\, a_i \otimes b_i \in Inv(\mathcal{A} \otimes \mathcal{B}), i = 1, 2, ..n \Rightarrow \sum_{i=1}^{n} (a_i \otimes b_i) h(\sum_{i=1}^{n} (a_i \otimes b_i)^{-1})$$
$$= n(h_1(e_1) \otimes h_2(e_2)) + \sum_{i,j=1, i\neq j}^{n} h_1(a_i a_j^{-1}) \otimes h_2(b_i b_j^{-1}).$$

Proof. (I) Since h is a centralizer so, for any $\sum_{i=1}^{n} a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{B}$, we have,

$$\begin{aligned} h(\sum_{i=1}^{n} a_i \otimes b_i) &= h((\sum_{i=1}^{n} a_i \otimes b_i)(e_1 \otimes e_2)) \\ &= (\sum_{i=1}^{n} a_i \otimes b_i) h(e_1 \otimes e_2) \\ &= (\sum_{i=1}^{n} a_i \otimes b_i)(h_1(e_1) \otimes h_2(e_2)), \text{(using the definition of } h). \end{aligned}$$

(II) For $\sum_{i=1}^{n} a_i \otimes b_i$, $\sum_{i=1}^{n} c_i \otimes d_i \in \mathcal{A} \otimes \mathcal{B}$, if $a_ic_i \otimes b_id_i = e_1 \otimes e_2,\ i = 1, 2, 3, .., n$, then using (I),

$$h(\sum_{i=1}^{n} a_ic_i \otimes b_id_i) = \sum_{i=1}^{n}(a_ic_i \otimes b_id_i)(h_1(e_1) \otimes h_2(e_2))$$

$$i.e., h(\sum_{i=1}^{n} e_1 \otimes e_2) = \sum_{i=1}^{n}(a_i \otimes b_i)(c_i \otimes d_i)h(e_1 \otimes e_2).$$

Using definition of h and the centralizer property, it follows that

$$n(h_1(e_1) \otimes h_2(e_2)) = \sum_{i=1}^{n}(a_i \otimes b_i)h((c_i \otimes d_i)(e_1 \otimes e_2)),$$

$$i.e., n(h_1(e_1) \otimes h_2(e_2)) - \sum_{i=1}^{n}(a_i \otimes b_i)(h_1(c_i) \otimes h_2(d_i)) = 0.$$

(III) For $a_i \otimes b_i \in Inv(\mathcal{A} \otimes \mathcal{B}),\ i = 1, 2, .., n$,

$$\sum_{i=1}^{n}(a_i \otimes b_i)h(\sum_{i=1}^{n}(a_i \otimes b_i)^{-1})$$

$$= \sum_{i=1}^{n}(a_i \otimes b_i)(\sum_{i=1}^{n} h_1(a_i^{-1}) \otimes h_2(b_i^{-1}))$$

$$= \sum_{i=1}^{n}(a_i \otimes b_i)(h_1(a_i^{-1}) \otimes h_2(b_i^{-1})) + \sum_{i,j=1,i\neq j}^{n}(a_i \otimes b_i)(h_1(a_j^{-1}) \otimes h_2(b_j^{-1})).$$

By the definition of h, the above expression equals

$$n(h_1(e_1) \otimes h_2(e_2)) + \sum_{i,j=1,j\neq i}^{n}(a_i \otimes b_i)h(\sum_{j} a_j^{-1} \otimes b_j^{-1})$$

$$= n(h_1(e_1) \otimes h_2(e_2)) + (\sum_{i=1}^{n} a_i \otimes b_i)h(\sum_{\substack{j=1\\ j\neq i}}^{n} a_j^{-1} \otimes b_j^{-1})$$

$$= n(h_1(e_1) \otimes h_2(e_2)) + h(\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} a_ia_j^{-1} \otimes b_ib_j^{-1}) \text{ (since } h \text{ is a centralizer)}$$

$$= n(h_1(e_1) \otimes h_2(e_2)) + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} h_1(a_ia_j^{-1}) \otimes h_2(b_ib_j^{-1}).$$

□

Our next aim is to discuss the converse part of Theorem 3.4, i.e., from a centralizer on $\mathcal{A}\otimes\mathcal{B}$ we define a bilinear map for each of the individual C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. For this, we use the concept of Jordan product (which we denote by $\bullet$ in case of modules) on $\mathcal{A}\otimes\mathcal{B}$-bimodules M, N and $M\otimes N$. Here, also $\mathcal{A}\otimes\mathcal{B}$ is equipped with the projective tensor norm. For $\sum_{i=1}^{n} u_i \in \mathcal{A}\otimes\mathcal{B}$ and $\sum_{j=1}^{m} p_j \in M\otimes N$, where $u_i = a_i\otimes b_i,\ p_j = m_j\otimes n_j$, the Jordan product $\bullet$ on $M\otimes N$ is defined by

$$\sum_{i=1}^{n} u_i \bullet \sum_{j=1}^{m} p_j = \sum_{j=1}^{m} p_j \bullet \sum_{i=1}^{n} u_i = \sum_{i=1}^{n}\sum_{j=1}^{m} u_i p_j + \sum_{i=1}^{n}\sum_{j=1}^{m} p_j u_i.$$

THEOREM 3.8. *Corresponding to the centralizer $h : \mathcal{A}\otimes\mathcal{B} \to M\otimes N$, there exist two continuous bilinear maps $g_1 : \mathcal{A}\times\mathcal{A} \to M$ and $g_2 : \mathcal{B}\times\mathcal{B} \to N$ satisfying the property (1) of Theorem 3.1. Moreover, there exist a bilinear map $g : \mathcal{A}\otimes\mathcal{B}\times\mathcal{A}\otimes\mathcal{B} \to M\otimes N$ with $\|g\| \le \|h\|^2$, if h is also continuous.*

Proof. We define $g_1 : \mathcal{A}\times\mathcal{A} \to M$ and $g_2 : \mathcal{B}\times\mathcal{B} \to N$ by

$$g_1(a_1, a_2) = \frac{1}{2}(a_1\otimes e_2) \bullet h(a_2\otimes e_2) \text{ and}$$
$$g_2(b_1, b_2) = \frac{1}{2}(e_1\otimes b_1) \bullet h(e_1\otimes b_2),$$

where $a_1, a_2 \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$.
By the property of module multiplication and Jordan product, g_1 and g_2 are well defined. Also, clearly these are bilinear mappings. For $a \in Inv_0(\mathcal{A})$,

$$\begin{aligned} g_1(a, a^{-1}) &= \frac{1}{2}(a\otimes e_2) \bullet h(a^{-1}\otimes e_2) \\ &= \frac{1}{2}[(a\otimes e_2)h(a^{-1}\otimes e_2) + h(a^{-1}\otimes e_2)(a\otimes e_2)] \\ &= h(e_1\otimes e_2) \\ &= \frac{1}{2}(e_1\otimes e_2) \bullet h(e_1\otimes e_2) \\ &= g_1(e_1, e_1). \end{aligned}$$

Similarly we can show that
$g_2(b, b^{-1}) = g_2(e_2, e_2)$ for $b \in Inv_0(\mathcal{B})$. Hence, g_1 and g_2 satisfy the condition (1) of Theorem 3.1. Again,

$$\begin{aligned}
\|g_1(a_1, a_2)\| &= \|\frac{1}{2}(a_1 \otimes e_2) \bullet h(a_2 \otimes e_2)\| \\
&= \|\frac{1}{2}(a_1 \otimes e_2)h(a_2 \otimes e_2) + h(a_2 \otimes e_2)\frac{1}{2}(a_1 \otimes e_2)\| \\
&= \|\frac{1}{2}[h(a_1a_2 \otimes e_2) + h(a_2a_1 \otimes e_2)]\| \\
&\leq \frac{1}{2}(\|h(a_1a_2 \otimes e_2)\| + \|h(a_2a_1 \otimes e_2)\|) \\
&\leq \|h\|\|a_1\|\|a_2\| \\
\Rightarrow \|g_1\| &\leq \|h\|,
\end{aligned}$$

and similarly, $\|g_2\| \leq \|h\|$.
Thus, g_1 and g_2 are continuous.
So, by the Theorem 3.2, we have a continuous bilinear mapping

$g : \mathcal{A} \otimes \mathcal{B} \times \mathcal{A} \otimes \mathcal{B} \to M \otimes N$ such that

$$g(\sum_{i=1}^{n} u_i, \sum_{j=1}^{m} v_j) = \frac{1}{2}[\sum_{i=1}^{n}\sum_{j=1}^{m}(g_1(a_i, c_j) \otimes g_2(b_i, d_j) + g_1(c_j, a_i) \otimes g_2(d_j, b_i))].$$

Also, it follows that

$$\|g\| \leq \|g_1\|\|g_2\| \leq \|h\|\|h\| = \|h\|^2.$$

□

Now, we show an application of the Theorem 3.2 in deriving a relationship between orthogonal complements of subspaces of the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ and the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$. In [33], Shenoi studied some basic properties and characteristics of bilinear forms. We note that a bilinear mapping $f : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ is called a bilinear form on $\mathcal{A}$. A bilinear form f on a C^*-algebra $\mathcal{A}$ is called reflexive if for $a_1, a_2 \in \mathcal{A}$, $f(a_1, a_2) = 0$ implies $f(a_2, a_1) = 0$. An element $a \in \mathcal{A}$ is said to be orthogonal to $a' \in \mathcal{A}$ with respect to a bilinear form f if $f(a, a') = 0$.
For the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, let $f_1 : \mathcal{A} \times \mathcal{A} \to \mathbb{R}^+ \cup \{0\}$ and $f_2 : \mathcal{B} \times \mathcal{B} \to \mathbb{R}^+ \cup \{0\}$ be two bilinear forms and A_1 and B_1 be subspaces of

$\mathcal{A}$ and $\mathcal{B}$ respectively. Then the orthogonal complements of A_1 and B_1 with respect to f_1 and f_2 respectively are given by

$$(A_1^{\perp})_{f_1} = \{a \in \mathcal{A} : f_1(a, c) = 0 \,\forall\, c \in A_1\} \text{ and}$$
$$(B_1^{\perp})_{f_2} = \{b \in \mathcal{B} : f_2(b, d) = 0 \,\forall\, d \in B_1\}.$$

Now, using Theorem 3.2, corresponding to f_1 and f_2 there exists a bilinear form

$$f : \mathcal{A} \otimes \mathcal{B} \times \mathcal{A} \otimes \mathcal{B} \to \mathbb{R} \otimes_{\gamma} \mathbb{R}(\cong \mathbb{R})(\text{refer to [32]}).$$

The following result gives a relation between $((A_1^{\perp})_{f_1} \otimes \mathcal{B}) \cup (\mathcal{A} \otimes (B_1^{\perp})_{f_2})$ and $((A_1 \otimes B_1)^{\perp})_f$.

Theorem 3.9. *If f_1 and f_2 are reflexive then*

$$((A_1^{\perp})_{f_1} \otimes \mathcal{B}) \cup (\mathcal{A} \otimes (B_1^{\perp})_{f_2}) \subseteq ((A_1 \otimes B_1)^{\perp})_f.$$

Considering f_1 and f_2 as symmetric bilinear forms, the above inclusion can be replaced by equality.

Proof. Let $\sum_{i=1}^{n} a_i \otimes b_i \in ((A_1^{\perp})_{f_1} \otimes \mathcal{B}) \cup (\mathcal{A} \otimes (B_1^{\perp})_{f_2})$.

Then, $a_i \otimes b_i \in (A_1^{\perp})_{f_1} \otimes \mathcal{B}$ or $a_i \otimes b_i \in \mathcal{A} \otimes (B_1^{\perp})_{f_2}$, $\forall\, i = 1, 2, ..., n$,

i.e.,$a_i \in (A_1^{\perp})_{f_1}$ or $b_i \in (B_1^{\perp})_{f_2}$, $\forall\, i = 1, 2, .., n$,

i.e.,$f_1(a_i, c) = 0$ or $f_2(b_i, d) = 0$ $\forall\, c \in A_1$, $\forall\, d \in B_1$, $\forall\, i = 1, 2, .., n$.

Let $\sum_{j=1}^{m} c_j \otimes d_j \in A_1 \otimes B_1$ be arbitrary. Then $c_j \in A_1$ and $d_j \in B_1$, $j = 1, ..m$

So, $f_1(a_i, c_j) = 0$ or $f_2(b_i, d_j) = 0$, for all $i = 1, 2, .., n;\ j = 1, 2, .., m$.

If $f_1$and f_2 are reflexive, this implies

$$f_1(c_j, a_i) = 0 \text{ or } f_2(d_j, b_i) = 0, \text{ for all } i = 1, 2, .., n;\ j = 1, 2, .., m.$$

Hence,

$$f_1(a_i, c_j) \otimes f_2(b_i, d_j) + f_1(c_j, a_i) \otimes f_2(d_j, b_i) = 0, \forall i = 1, 2, .., n;\ j = 1, 2, .., m,$$

$$i.e., \sum_{i=1}^{n} \sum_{j=1}^{m} (f_1(a_i, c_j) \otimes f_2(b_i, d_j) + f_1(c_j, a_i) \otimes f_2(d_j, b_i)) = 0.$$

Using Theorem 3.2, by definition of f, this means that

$$f(\sum_{i=1}^{n} a_i \otimes b_i, \sum_{j=1}^{m} c_j \otimes d_j) = 0,\ i.e., \sum_{i=1}^{n} a_i \otimes b_i \in ((A_1 \otimes B_1)^{\perp})_f,$$

showing that $((A_1^{\perp})_{f_1} \otimes \mathcal{B}) \cup (\mathcal{A} \otimes (B_1^{\perp})_{f_2}) \subseteq ((A_1 \otimes B_1)^{\perp})_f$.

Conversely, let $\sum_{i=1}^{n} a_i \otimes b_i \in ((A_1 \otimes B_1)^{\perp})_f$. Then,

$$f(\sum_{i=1}^{n} a_i \otimes b_i, \sum_{j=1}^{m} c_j \otimes d_j) = 0, \ \forall \ \sum_{j=1}^{m} c_j \otimes d_j \in A_1 \otimes B_1,$$

$$i.e., \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} [f_1(a_i, c_j) \otimes f_2(b_i, d_j) + f_1(c_j, a_i) \otimes f_2(d_j, b_i)] = 0, \forall \, c_j \in A_1, d_j \in B_1$$

$$i.e., \sum_{i=1}^{n} \sum_{j=1}^{m} f_1(a_i, c_j) \otimes f_2(b_i, d_j) = 0, \forall \, c_j \in A_1, d_j \in B_1, \text{ since } f_1 \text{ and}$$

f_2 are symmetric.

By the definition of tensor product in $(\mathbb{R}^+ \cup \{0\}) \otimes (\mathbb{R}^+ \cup \{0\}) \cong \mathbb{R}^+ \cup \{0\}$,(refer to [32]), it follows that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f_1(a_i, c_j) f_2(b_i, d_j) = 0, \forall \, i = 1, 2, .., n; \forall \, j = 1, 2, .., m.$$

$$i.e., f_1(a_i, c_j) f_2(b_i, d_j) = 0, \forall \, i = 1, 2, .., n; j = 1, 2, .., m.$$

Thus, $a_i \in (A_1^{\perp})_{f_1}$ or $b_i \in (B_1^{\perp})_{f_2} \, \forall \, i = 1, 2, .., n; j = 1, 2, .., m$ and so,

$$\sum_{i=1}^{n} a_i \otimes b_i \in ((A_1^{\perp})_{f_1} \otimes \mathcal{B}) \cup (\mathcal{A} \otimes (B_1^{\perp})_{f_2}).$$

This shows that $((A_1^{\perp})_{f_1} \otimes \mathcal{B}) \cup (\mathcal{A} \otimes (B_1^{\perp})_{f_2}) = ((A_1 \otimes B_1)^{\perp})_f$. □

Example 3.10. For the C^*-algebra $\mathbb{R}$ and $\mathbb{R}^2$, we define the bilinear forms:

$f_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $f(x, y) = xy, \ x, y \in \mathbb{R}$ and
$f_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $f_2((a_1, a_2), (b_1, b_2)) = a_1 b_1 + a_2 b_2, \ \ ((a_1, a_2), (b_1, b_2) \in \mathbb{R}^2$.

Clearly, f_1 and f_2 are symmetric and satisfy the property (1) of Theorem 3.1.
So, by Theorem 3.2, there exists a bilinear form

$$f : \mathbb{R} \otimes \mathbb{R}^2 \times \mathbb{R} \otimes \mathbb{R}^2 \to \mathbb{R}$$

such that for $\alpha_i, \beta_j \in \mathbb{R}$, $(a_i, b_i), (x_i, y_i) \in \mathbb{R}^2$; $i = 1, 2, ..., n$; $j = 1, 2, .., m$,

$$\begin{aligned} & f(\sum_{i=1}^{n} \alpha_i \otimes (a_i, b_i), \sum_{j=1}^{m} \beta_j \otimes (x_j, y_j)) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} f_1(\alpha_i, \beta_j) \otimes f_2((a_i, b_i), (x_j, y_j)) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i \beta_j \otimes (a_i x_j + b_i y_j)) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j (a_i x_j + b_i y_j) \end{aligned}$$

We take $A_1 =< \frac{1}{2} >$ and $B_1 =< (1,1) >$ which are subspaces of $\mathbb{R}$ and $\mathbb{R}^2$ respectively. $(A_1^{\perp})_{f_1} = \{0\}$ and $(B_1^{\perp})_{f_2} =< (-1,1) >$ are their corresponding orthogonal complements.
Then, $(A_1^{\perp})_{f_1} \otimes \mathbb{R}^2 = \{0\}$ and $((A_1^{\perp})_{f_1} \otimes \mathbb{R}^2) \cup (\mathbb{R} \otimes B_1^{\perp})_{f_2}) =< (-1,1) >$.
Also, $((A_1 \otimes B_1)^{\perp})_f =< (-1,1) >$.
Thus Theorem 3.9 holds for A_1 and B_1 and the bilinear forms f_1, f_2.

4. Concluding Remarks

In this paper, we have extended the works of Ghahramani [14] to the algebraic tensor product of two C^*-algebras and derived different properties of bilinear maps and centralizers on such algebras. Using bilinear maps on the individual C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, we have obtained a centralizer on the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$, and conversely. In the last part, we apply the result to derive a relationship between orthogonal complements of subspaces of the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ and then algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ and a suitable example is provided.

Now a days, there is a wide application of bilinear maps in different areas of cryptography like encryption, signature and key agreement.(refer to [27], [13] etc.). The practical application of the results obtained in the paper in this direction is a scope for future study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] C. A. Akemann , G. K. Pedersen and J. Tomiyama, *Multipliers of C^*-algebras*, J. Funct. Anal. **13** (1973), 277–301.

[2] A. Alahamdi, H. Alhazmi, S. Ali and A.N. Khan, *A characterization of Jordan left *-centralizers in rings with involution*, Appl. Math. Inf. Sci. **11** (2) (2017), 441–447.

[3] D. P. Blecher, *Geometry of the tensor product of C^*-algebras*, Math. Proc. CambridgePhilos. Soc. (1988), 119–127.

[4] F. F. Bonsall and J. Duncan, *Complete normed Algebras*, Book, Springer- Verlag (1973).

[5] M. F. Bruckler , *Tensor products of C^*-algebras, Operator spaces and Hilbert C^*-modules*, Math. comm. **4** (1999), 257–268.

[6] T. K. Carne, *Tensor products and Banach algebras*, J. London Math. Soc. **17** (2) (1978), 480–488.

[7] D. Das and L. N. Mishra, *Some Fixed Point Results for $\mathcal{JHR}$ operator pairs in C^*-algebra Valued Modular b-Metric Spaces via C_* class functions with Applications*, Advanced Studies in Contemporary Mathematics, **29** (3) (2019), pp. 383-400.

[8] D. Das, N. Goswami, *Some fixed point theorems on the sum and product of operators in tensor product spaces*, IJPAM, **10D** (3), (2016), 651–663.

[9] D. Das, N. Goswami, V. N. Mishra, *Some results on fixed point theorems in Banach algebras*, International Journal of Analysis and Applications, **13** (1), (2017), 32–40.

[10] D. Das, N. Goswami, V. N. Mishra, *Some results on the projective cone normed tensor product spaces over Banach algebras*, Bol. Soc. Paran. Mat, **38** (1), (2020), 197–221.

[11] Deepmala, *A Study on Fixed Point Theorems for Nonlinear Contractions and its Applications*, Ph.D. Thesis, Pt. Ravishankar Shukla University, Raipur, Chhatisgarh, India., (2014), 492 010.

[12] Deepmala and H. K. Pathak, *A study on some problems on existence of solutions for nonlinear functional-integral equations*, Acta Mathematica Scientia, **33** B(5) (2013), 1305–1313.

[13] M. Gagne, *Applications of Bilinear maps in Cryptography. UWSpace*, M.sc thesis, University of Waterloo, Ontario, Canada,(2002).

[14] H. Ghahramani, *On centralizers of Banach algebras*, The Bull. of the Malaysian Math. Soc. Series 2, **38** (1) (2015), 155–164.

[15] N. Goswami, A. Sarma, *On a subset of invertible elements of C^*-algebras and some related fixed point theorems*, Journal of Tripura Math. Soc., **19** (2017), 21–37.

[16] A. Guichardet , *Tensor products of C^*-algebras*, Arhus Universitet, Matematisk Institute, (1969).

[17] S. Helgason , *Multipliers of Banach algebras*, Ann. of Math., **64** (1956), 240–254.

[18] B. E. Johnson, *An introduction to the theory of centralizers*, Proc. London Math. Soc., **14** (1964), 299–320.

[19] A. S. Kavruk, *Complete positivity in opeartor algebras*, an M.Sc. thesis, Bilkent University, (2006).

[20] S. Kaijser and A. M. Sinclair, *Projective tensor products of C^*-algebras*, Math.Scand **65** (1984), 161–187.

[21] X. Liu, M. Zhou, L. N. Mishra, V. N. Mishra, B. Damjanović, *Common fixed point theorem of six self-mappings in Menger spaces using (CLR_{ST}) property*, Open Mathematics, 2018; 16: 1423–1434.

[22] L. N. Mishra, M. Sen, *On the concept of existence and local attractivity of solutions for some quadratic Volterra integral equation of fractional order*, Applied Mathematics and Computation **285** (2016), 174–183. DOI: 10.1016/j.amc.2016.03.002

[23] L. N. Mishra, *On existence and behavior of solutions to some nonlinear integral equations with applications*, Ph.D. Thesis, National Institute of Technology, Silchar, Assam, India., (2017), 788 010.

[24] L. N. Mishra, K. Jyoti, A. Rani, Vandana, *Fixed point theorems with digital contractions image processing*, Nonlinear Sci. Lett. A, **9** (2) (2018), pp.104–115.

[25] L. N. Mishra, S. K. Tiwari, V. N. Mishra, I.A. Khan, *Unique Fixed Point Theorems for Generalized Contractive Mappings in Partial Metric Spaces*, Journal of Function Spaces, Volume 2015 (2015), Article ID 960827, 8 pages.

[26] L. N. Mishra, S. K. Tiwari, V. N. Mishra, *Fixed point theorems for generalized weakly S-contractive mappings in partial metric spaces*, Journal of Applied Analysis and Computation, **5** (4) (2015), pp. 600–612. doi:10.11948/2015047

[27] N. M. McCullagh, *Cryptographic Applications of Bilinear Maps*, Ph.D. Thesis, Dublin City University, Dublin, Ireland (2005).

[28] H. K. Pathak and Deepmala, *Common fixed point theorems for PD-operator pairs under Relaxed conditions with applications*, Journal of Computational and Applied Mathematics, **239** (2013), 103–113.

[29] A. R. Raymond, *Introduction to Tensor products of Banach Spaces*, Book Springer - Verlag London Limited, (2001).

[30] A. Sarma, N. Goswami, *Some results on positive elements in the tensor product of C^*-algebras*, Int. J. Math. Trends and Technology, **51** (3), (2017), 237–243.

[31] M. S. T. EI-Sayiad, N. M. Muthana, Z. S. Alkhamisi, *A note on Jordan left *-centralizers on prime and semiprime rings with involution*, J. Taibah Univ. Sci., (2018), 1080–1082.

[32] R. Schwartz, Notes on Tensor products, (2014), https://www.math.brown.edu/ res/M153/tensor.pdf

[33] G. N. Prasanth, *Introduction to bilinear forms*, Scholarly project, Govt. College of Chittur, (2014) *https://www.researchgate.net/publication/267982491.*

[34] T. Turumaru, *On the direct product of operator algebras I*, Tohoku Math. Journ., **4** (1952), 242–151.

[35] J. Vukman, *Centralizers on prime and semiprime rings*, Comment. Math. Univ. Carolinae **38** (2),(1997), 231–240.
[36] J. G. Wendel , *Left centralizers and isomorphism of group algebras*, Pacific J. Math. **2** (1952), 251–261.

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