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COMMUTATIVE SINGLE POWER CYCLIC HYPERGROUPS OF ORDER 4 AND PERIOD 2

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Abstract. In this paper we enumerate all commutative single power cyclic hypergroups of order 4 and period 2. Moreover, we prove some interesting properties regarding cyclic hypergroups.

1. Introduction and basic concepts

The composition of two elements in a group is an element, while the composition of two elements in a hypergroup is a set. If H is a non-empty set and $*$ is a mapping from $H \times H$ into the family of non-empty subsets of H, then $(H, *)$ is called a hypergroupoid. A hypergroup is a hypergroupoid $(H, *)$ with the following two conditions: (1) $(x*y)*z = x*(y*z)$, for every $x, y, z \in H$; (2) $x*H = H*x = H$, for every $x \in H$. Note that if A, B are non-empty subsets of H and $x \in H$, then we mean

$$A * B = \bigcup_{\substack{a\in A\\ b\in B}} a * b, \ \ x * A = \{x\} * A \text{ and } A * x = A * \{x\}.$$

If $(H, *)$ satisfies only in the first axiom, then it is called a semihypergroup. The concept of hyperstructures introduced by F. Marty in 1934 [15] and applied them to groups, algebraic functions and rational functions. Cyclic semihypergroups and cyclic hypergroups have been studied by Al Tahan and Davvaz [1–4], Corsini [7], Corsini and Leoreanu [8], Davvaz [9], De Salvo and and Freni [10], Freni [11], Konguetsof et

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al. [13], Leoreanu [14], Mirvakili et al. [16], Mousavi et al. [18], Vougiouklis [20,21], and many others. In [19], Novak et al. presented an overview and motivation of existing approaches to the cyclicity in algebraic hyperstructures. Moreover, they related these to EL-hyperstructures, a broad class of algebraic hyperstructures constructed from (pre)ordered (semi)groups. For more details about algebraic hyperstructures we refer the readers to [5–9, 12, 19, 21].

In [4], Al Tahan and Davvaz classified all commutative single power cyclic hypergroups of order three and period two.

Now, in this paper we classify all commutative single power cyclic hypergroups of order four and period two. The paper is organized as follows: After an introduction, in Section 2 we present some basic definitions concerning hyperstructures that are used throughout the paper. In Section 3 we present some new properties of cyclic hypergroups and in Section 4 find all commutative single power cyclic hypergroups of order four where each of its elements is a generator of period two. In Section 5 we show that some commutative single power cyclic hypergroups of order four and period two are isomorphic. Throughout this paper we denote the period of a cyclic hypergroup H by $per(H)$.

An element $e \in H$ is called an identity of $(H, *)$ if $x \in x * e \cap e * x$ for all $x \in H$ and it is called a scalar identity if $x * e = e * x = \{x\}$ for all $x \in H$ that is unique. An element $x \in H$ is called idempotent if $x*x = x$. The hypergroup $(H, *)$ is said to be commutative if $x*y = y*x$ for all $x, y \in H$. A non-empty subset K of H is called subhypergroup, if $(K, *)$ is hypergroup. A canonical hypergroup [17] is a non-empty set H endowed with a mapping $* : H \times H \longrightarrow \mathcal{P}^*(H)$, satisfying the following properties:

(1) $x * (y * z) = (x * y) * z$ for every $x, y, z \in H$,
(2) $x * y = y * x$ for every $x, y \in H$,
(3) there exists $e \in H$ such that $e * x = x * e = x$ for every $x \in H$,
(4) for every $x \in H$, there exists a unique element x' such that $e \in x*x'$,
(5) $z \in x * y$ implies that $y \in x' * z$ and $x \in z * y'$.

A hypergroup $(H, *)$ is called total hypergroup if $x * y = H$ for all $x, y \in H$. A hypergroup $(H, *)$ is cyclic if there exists $h \in H$ and $s \in N$ such that $H = h \cup h^2 \cup \cdots h^s \cup \cdots$. If $H = h \cup h^2 \cup \cdots h^s$ then H is cyclic hypergroup with finite period. Otherwise, H is called cyclic hypergroup with infinite period. A hypergroup $(H, *)$ is called single power cyclic hypergroup if there exists $h \in H$ and $s \in N$ such that

$H = h \cup h^2 \cup \cdots h^s \cup \cdots$ and $h \cup h^2 \cup \cdots h^{m-1} \subseteq h^m$ for every $m \in N$. Let $(H, *)$ and $(H', *')$ be two hypergroups. A function $f : (H, *) \longrightarrow (H', *')$ is said to be a homomorphism if $f(x_1 * x_2) \subseteq f(x_1) *' f(x_2)$ for all $x_1, x_2 \in H$. And it is called good homomorphism if $f(x_1 * x_2) = f(x_1) *' f(x_2)$. Two hypergroups are said to be isomorphic if there exists a bijective good homomorphism between them.

2. Properties of Cyclic Hypergroups

In this section we present some properties of cyclic hypergroups.

PROPOSITION 2.1. [4] *Let H be a cyclic hypergroup with generator $h \in H$. If there exists $k \in N$ satisfying $h^{k+1} = h^k$ then $per(H) \leq k$.*

PROPOSITION 2.2. [4] *Let H be a cyclic hypergroup with generator $h \in H$. If there exist $i \neq j \in N$ satisfying $h^i = h^j$ then $per(H) \leq max\{i, j\}$.*

PROPOSITION 2.3. [4] *If H is a finite cyclic hypergroup, then it has a finite period.*

In the following example we show that the converse of proposition 2.3 is not always true.

EXAMPLE 1. For every infinite total hypergroup H we have $per(H) = 2$. Thus $per(H)$ is finite but H is infinite.

PROPOSITION 2.4. [4] *Let H be a cyclic hypergroup. Then $per(H) = 1$ if and only if H is the trivial hypergroup.*

PROPOSITION 2.5. [4] *Let H be a cyclic hypergroup of order two. Then $per(H) = 2$.*

PROPOSITION 2.6. [4] *Let H be a cyclic hypergroup of order three. Then $per(H) = 2$ or $per(H) = 3$.*

PROPOSITION 2.7. [4] *If H is a single power cyclic hypergroup of order $n \geq 2$ then $2 \leq per(H) \leq n$.*

EXAMPLE 2. Let $H = \{a, b, c, d\}$ and define $(H, *)$ by the following table.

$*$	a	b	c	d
a	a,b,c,d	c	a,b,c,d	c
b	c	a,b,c,d	a,b,c,d	c
c	a,b,c,d	a,b,c,d	a,b,c,d	a,b,c,d
d	c	c	a,b,c	a,b,c,d

Then $(H,*)$ is a non commutative single power cyclic hypergroup of period two. It is clear from the table that $(H,*)$ is not commutative, satisfies the reproduction axiom and is generated by every member of H. Easy computations show that $(H,*)$ is associative.

3. Commutative Single Power Cyclic Hypergroups of Order Four and Period Two

In this section we find all commutative single power cyclic hypergroups of order four and period two where every element is a generator of period two.

Proposition 3.1. *Let $H=\{a,b,c,d\}$ and $*$ be a commutative hyperoperation on H. Then $(H,*)$ is associative if the all the following conditions are satisfied:*

(1) $a*(a*b)=(a*a)*b$,
(2) $a*(a*c)=(a*a)*c$,
(3) $a*(a*d)=(a*a)*d$,
(4) $a*(b*b)=(a*b)*b$,
(5) $a*(b*c)=(a*b)*c$,
(6) $a*(b*d)=(a*b)*d$,
(7) $a*(c*b)=(a*c)*b$,
(8) $a*(c*c)=(a*c)*c$,
(9) $a*(c*d)=(a*c)*d$,
(10) $a*(d*b)=(a*d)*b$,
(11) $a*(d*c)=(a*d)*c$,
(12) $a*(d*d)=(a*d)*d$,
(13) $b*(a*a)=(b*a)*a$,
(14) $b*(a*c)=(b*a)*c$,
(15) $b*(a*d)=(b*a)*d$,
(16) $b*(b*c)=(b*b)*c$,
(17) $b*(b*d)=(b*b)*d$,

(18) $b * (c * c) = (b * c) * c$,
(19) $b * (c * d) = (b * c) * d$,
(20) $b * (d * a) = (b * d) * a$,
(21) $b * (d * c) = (b * d) * c$,
(22) $b * (d * d) = (b * d) * d$,
(23) $c * (c * d) = (c * c) * d$,
(24) $c * (d * d) = (c * d) * d$.

Proof. Since $(H, *)$ is commutative, it follows that for every $x, y, z \in H$, we have $x * (x * x) = (x * x) * x$ and $z * (y * z) = (y * z) * z = (z * y) * z$. Having $(H, *)$ is commutative implies that $b*(c*a) = b*(a*c) = (a*c)*b$ and having $(a * c) * b = a * (c * b)$ implies that $b * (c * a) = b * (a * c) = (a * c) * b = a * (c * b) = a * (b * c) = (b * c) * a$ (using 7).

Having $(H, *)$ is commutative implies that $c * (a * a) = (a * a) * c = a * (a * c)$ by (2). We get now that $c * (a * a) = (a * a) * c = a * (a * c) = (a*c)*a = (c*a)*a$. In a similar manner, we prove that $c*(b*d) = (c*b)*d$ (using 19,21), $c * (d * a) = (c * d) * a$ (using 11), $c * (b * a) = (c * b) * a$ (using 5), $d * (c * b) = (d * c) * b$ (using 19). $\square$

Example 3. Let $H_4 = \{a, b, c, d\}$ and define $(H_4, *)$ by the following table:

$*$	a	b	c	d
a	a, b, c, d	a, b	a, b	c, d
b	a, b	a, b, c, d	c, d	a, b
c	a, b	c, d	a, b, c, d	c, d
d	c, d	a, b	c, d	a, b, c, d

It is clear $*$ is commutative, satisfy the reproduction axiom and that $H_4 = a^2 = b^2 = c^2 = d^2$. We show now that H_4 is associative by the using following table:

$a*(a*b)=H$	$(a*a)*b=H$
$a*(a*c)=H$	$(a*a)*c=H$
$a*(a*d)=H$	$(a*a)*d=H$
$a*(b*b)=H$	$(a*b)*b=H$
$a*(b*c)=H$	$(a*b)*c=H$
$a*(b*d)=H$	$(a*b)*d=H$
$a*(c*b)=H$	$(a*c)*b=H$
$a*(c*c)=H$	$(a*c)*c=H$
$a*(c*d)=H$	$(a*c)*d=H$
$a*(d*b)=H$	$(a*d)*b=H$
$a*(d*c)=H$	$(a*d)*c=H$
$a*(d*d)=H$	$(a*d)*d=H$
$b*(a*a)=H$	$(b*a)*a=H$
$b*(a*c)=H$	$(b*a)*c=H$
$b*(a*d)=H$	$(b*a)*d=H$
$b*(b*c)=H$	$(b*b)*c=H$
$b*(b*d)=H$	$(b*b)*d=H$
$b*(c*c)=H$	$(b*c)*c=H$
$b*(c*d)=H$	$(b*c)*d=H$
$b*(d*a)=H$	$(b*d)*a=H$
$b*(d*c)=H$	$(b*d)*c=H$
$b*(d*d)=H$	$(b*d)*d=H$
$c*(c*d)=H$	$(c*c)*d=H$
$c*(d*d)=H$	$(c*d)*d=H$

THEOREM 3.2. *There are 183398 commutative single power cyclic hypergroups of order four and period two.*

Proof. Indeed, by Algorithm 1 presented in Appendix and a computer program we can enumerate the commutative single power cyclic hypergroups of order 4 and period 2. □

EXAMPLE 4. In the following we present some of the commutative single power cyclic hypergroups of order 4 and period 2:

H_1	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	a,b
d	c,d	a,b	a,b	a,b,c,d

H_2	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	a,b,c
d	c,d	a,b	a,b,c	a,b,c,d

H_3	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	a,b,d
d	c,d	a,b	a,b,d	a,b,c,d

H_4	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	c,d
d	c,d	a,b	c,d	a,b,c,d

H_5	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	a,c,d
d	c,d	a,b	a,c,d	a,b,c,d

H_6	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	b,c,d
d	c,d	a,b	b,c,d	a,b,c,d

H_7	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	a,b,c,d
d	c,d	a,b	a,b,c,d	a,b,c,d

H_8	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,c
c	a,b	c,d	a,b,c,d	a,b
d	c,d	a,b,c	a,b	a,b,c,d

H_9	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,c
c	a,b	c,d	a,b,c,d	a,b,c
d	c,d	a,b,c	a,b,c	a,b,c,d

H_{10}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,c
c	a,b	c,d	a,b,c,d	a,b,d
d	c,d	a,b,c	a,b,d	a,b,c,d

H_{11}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,c
c	a,b	c,d	a,b,c,d	c,d
d	c,d	a,b,c	c,d	a,b,c,d

H_{12}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,c
c	a,b	c,d	a,b,c,d	a,c,d
d	c,d	a,b,c	a,c,d	a,b,c,d

H_{13}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,c
c	a,b	c,d	a,b,c,d	b,c,d
d	c,d	a,b,c	b,c,d	a,b,c,d

H_{14}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,c
c	a,b	c,d	a,b,c,d	a,b,c,d
d	c,d	a,b,c	a,b,c,d	a,b,c,d

H_{15}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,d
c	a,b	c,d	a,b,c,d	a,b
d	c,d	a,b,d	a,b	a,b,c,d

H_{16}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,d
c	a,b	c,d	a,b,c,d	a,b,c
d	c,d	a,b,d	a,b,c	a,b,c,d

H_{17}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,d
c	a,b	c,d	a,b,c,d	a,b,d
d	c,d	a,b,d	a,b,d	a,b,c,d

H_{18}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,d
c	a,b	c,d	a,b,c,d	c,d
d	c,d	a,b,d	c,d	a,b,c,d

H_{19}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,d
c	a,b	c,d	a,b,c,d	a,c,d
d	c,d	a,b,d	a,c,d	a,b,c,d

H_{20}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,d
c	a,b	c,d	a,b,c,d	b,c,d
d	c,d	a,b,d	b,c,d	a,b,c,d

H_{21}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,d
c	a,b	c,d	a,b,c,d	a,b,c,d
d	c,d	a,b,d	a,b,c,d	a,b,c,d

H_{22}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,c,d
c	a,b	c,d	a,b,c,d	a,b
d	c,d	a,b,c,d	a,b	a,b,c,d

H_{23}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,c,d
c	a,b	c,d	a,b,c,d	a,b,c
d	c,d	a,b,c,d	a,b,c	a,b,c,d

H_{24}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,c,d
c	a,b	c,d	a,b,c,d	a,b,d
d	c,d	a,b,c,d	a,b,d	a,b,c,d

H_{25}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,c,d
c	a,b	c,d	a,b,c,d	c,d
d	c,d	a,b,c,d	c,d	a,b,c,d

H_{26}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,c,d
c	a,b	c,d	a,b,c,d	a,c,d
d	c,d	a,b,c,d	a,c,d	a,b,c,d

H_{27}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,c,d
c	a,b	c,d	a,b,c,d	b,c,d
d	c,d	a,b,c,d	b,c,d	a,b,c,d

H_{28}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,c,d
c	a,b	c,d	a,b,c,d	a,b,c,d
d	c,d	a,b,c,d	a,b,c,d	a,b,c,d

H_{29}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	a,b
d	a,c,d	a,b	a,b	a,b,c,d

H_{30}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	a,b,c
d	a,c,d	a,b	a,b,c	a,b,c,d

H_{31}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	a,b,d
d	a,c,d	a,b	a,b,d	a,b,c,d

H_{32}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	c,d
d	a,c,d	a,b	c,d	a,b,c,d

H_{33}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	a,c,d
d	a,c,d	a,b	a,c,d	a,b,c,d

H_{34}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	b,c,d
d	a,c,d	a,b	b,c,d	a,b,c,d

H_{35}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	a,b,c,d
d	a,c,d	a,b	a,b,c,d	a,b,c,d

H_{36}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b,c
c	a,b	c,d	a,b,c,d	a,b
d	a,c,d	a,b,c	a,b	a,b,c,d

H_{37}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b,c
c	a,b	c,d	a,b,c,d	a,b,c
d	a,c,d	a,b,c	a,b,c	a,b,c,d

H_{38}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b,c
c	a,b	c,d	a,b,c,d	a,b,d
d	a,c,d	a,b,c	a,b,d	a,b,c,d

H_{39}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b,c
c	a,b	c,d	a,b,c,d	c,d
d	a,c,d	a,b,c	c,d	a,b,c,d

H_{40}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b,c
c	a,b	c,d	a,b,c,d	a,c,d
d	a,c,d	a,b,c	a,c,d	a,b,c,d

H_{41}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b,c
c	a,b	c,d	a,b,c,d	b,c,d
d	a,c,d	a,b,c	b,c,d	a,b,c,d

H_{42}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b,c
c	a,b	c,d	a,b,c,d	a,b,c,d
d	a,c,d	a,b,c	a,b,c,d	a,b,c,d

H_{43}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b,d
c	a,b	c,d	a,b,c,d	a,b
d	a,c,d	a,b,d	a,b	a,b,c,d

H_{44}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b,d
c	a,b	c,d	a,b,c,d	a,b,c
d	a,c,d	a,b,d	a,b,c	a,b,c,d

H_{45}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b,d
c	a,b	c,d	a,b,c,d	a,b,d
d	a,c,d	a,b,d	a,b,d	a,b,c,d

H_{46}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b,d
c	a,b	c,d	a,b,c,d	c,d
d	a,c,d	a,b,d	c,d	a,b,c,d

H_{47}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b,d
c	a,b	c,d	a,b,c,d	a,c,d
d	a,c,d	a,b,d	a,c,d	a,b,c,d

H_{48}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b,d
c	a,b	c,d	a,b,c,d	b,c,d
d	a,c,d	a,b,d	b,c,d	a,b,c,d

H_{49}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b,d
c	a,b	c,d	a,b,c,d	a,b,c,d
d	a,c,d	a,b,d	a,b,c,d	a,b,c,d

H_{50}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b,c,d
c	a,b	c,d	a,b,c,d	a,b
d	a,c,d	a,b,c,d	a,b	a,b,c,d

PROPOSITION 3.3. *Let $H = \{a, b, c, d\}$ and $(H, *)$ be a commutative single power cyclic hypergroup of period two such that all of its elements are generators of period two. Then, $y \neq x * y = y * x \neq x$, for any $x, y \in H$.*

Proof. (a) If $a*b = a$ then $a*(b*b) = a*H = H \neq (a*b)*b = a*b = a$ and if $a * b = b$ then $a * (a * b) = a * b = b \neq (a * a) * b = H * b = H$,

(b) If $a * c = c$ then $a * (a * c) = a * c = c \neq (a * a) * c = H * c = H$ and if $a * c = a$ then $a * (c * c) = a * H = H \neq (a * c) * c = a * c = a$,

(c) If $a * d = a$ then $a * (d * d) = a * H = H \neq (a * d) * d = a * d = a$ and if $a * d = d$ then $a * (a * d) = a * d = d \neq (a * a) * d = H * d = H$,

(d) If $b * c = b$ then $b * (c * c) = b * H = H \neq (b * c) * c = b * c = b$ and if $b * c = c$ then $b * (b * c) = b * c = c \neq (b * b) * c = H * c = H$,

(e) If $b * d = b$ then $b * (d * d) = b * H = H \neq (b * d) * d = b * d = b$ and if $b * d = d$ then $b * (b * d) = b * d = d \neq (b * b) * d = H * d = H$,

(f) If $c * d = c$ then $c * (d * d) = c * H = H \neq (c * d) * d = c * d = c$ and if $c * d = d$ then $d * (c * c) = d * H = H \neq (d * c) * c = d * c = d$. $\square$

PROPOSITION 3.4. *Let $H = \{a, b, c, d\}$ and $(H, *)$ be a commutative single power cyclic hypergroup of period two such that all of its elements are generators of period two. Then:*

(a) *If $a * b = c$ then $a * c = b * c = H$,*
(b) *If $a * b = d$ then $a * d = b * d = H$,*
(c) *If $a * c = b$ then $a * b = c * b = H$,*
(d) *If $a * c = d$ then $a * d = c * d = H$.*
(e) *If $a * d = b$ then $a * b = d * b = H$.*
(f) *If $a * d = c$ then $a * c = d * c = H$.*
(g) *If $b * c = d$ then $b * d = c * d = H$.*
(h) *If $b * c = a$ then $b * a = c * a = H$,*
(i) *If $b * d = c$ then $b * c = d * c = H$,*
(j) *If $b * d = a$ then $b * a = d * a = H$,*
(k) *If $c * d = a$ then $c * a = d * a = H$,*
(l) *If $c * d = b$ then $c * b = d * b = H$,*

Proof. (a) By hypothesis we have $a * b = c$. Now since $a * (a * b) = (a*a)*b$, we obtain $a*c = H$. On the other hand since $b*(b*a) = (b*b)*a$ we obtain $b * c = H$,

(b) By hypothesis we have $a * b = d$. Now since $a * (a * b) = (a * a) * b$, we obtain $a * d = H$. On the other hand since $a * (b * b) = (a * b) * b$ we obtain $b * d = H$,

(c) By hypothesis we have $a * c = b$. Now since $a * (c * c) = (a * c) * c$, we obtain $b * c = H$. On the other hand since $c * (a * a) = (c * a) * a$ we obtain $b * a = H$,

(d) By hypothesis we have $a * c = d$. Now since $a * (a * c) = (a * a) * c$, we obtain $a * d = H$. On the other hand since $c * (c * a) = (c * c) * a$ we obtain $d * c = H$.

(e) By hypothesis we have $a * d = b$. Now since $a * (a * d) = (a * a) * d$, we obtain $a * b = H$. On the other hand since $d * (d * a) = (d * d) * a$ we obtain $d * b = H$.

(f) By hypothesis we have $a * d = c$. Now since $a * (a * d) = (a * a) * d$, we obtain $a * c = H$. On the other hand since $a * (d * d) = (a * d) * d$ we obtain $c * d = H$.

(g) By hypothesis we have $b * c = d$. Now since $b * (b * c) = (b * b) * c$, we obtain $b * d = H$. On the other hand since $b * (c * c) = (b * c) * c$ we obtain $c * d = H$.

(h) By hypothesis we have $b * c = a$. Now since $b * (b * c) = (b * b) * c$, we obtain $b * a = H$. On the other hand since $b * (c * c) = (b * c) * c$ we obtain $a * c = H$.

(i) By hypothesis we have $b * d = c$. Now since $b * (d * d) = (b * d) * d$, we obtain $c * d = H$. On the other hand since $b * (b * d) = (b * b) * d$ we obtain $b * c = H$,

(j) By hypothesis we have $b * d = a$. Now since $b * (d * d) = (b * d) * d$, we obtain $a * d = H$. On the other hand since $d * (b * b) = (d * b) * b$ we obtain $a * b = H$,

(k) By hypothesis we have $c * d = a$. Now since $c * (d * d) = (c * d) * d$, we obtain $a * d = H$. On the other hand since $d * (c * c) = (d * c) * c$ we obtain $a * c = H$,

(l) By hypothesis we have $c * d = b$. Now since $c * (c * d) = (c * c) * d$, we obtain $c * b = H$. On the other hand since $c * (d * d) = (c * d) * d$ we obtain $b * d = H$, □

Proposition 3.5. *Let $H = \{a, b, c, d\}$ and $(H, *)$ be a commutative single power cyclic hypergroup of period two such that all of its elements are generators of period two. Then:*

(a) *If $a * b \neq H$ then $a * c \neq b$ and $b * c \neq a$,*
(b) *If $a * c \neq H$ then $a * b \neq c$ and $a * d \neq c$,*
(c) *If $b * d \neq H$ then $b * c \neq d$ and $b * a \neq d$,*
(d) *If $d * a \neq H$ then $d * c \neq a$ and $d * b \neq a$,*
(e) *If $d * a \neq H$ then $d * b \neq a$ and $a * b \neq d$,*
(f) *If $d * b \neq H$ then $d * a \neq b$ and $d * c \neq b$,*

Proof. The proof is similar to the proof of Proposition 3.4.

□

4. Non-Isomorphic Commutative Single Power Cyclic Hypergroups of Order 4 and Period 2

In order to determine all the non-isomorphic commutative single power cyclic hypergroups of order 4 and period 2, we use Algorithm 1. Indeed, the algorithm is written base on the following illustrations.

PROPOSITION 4.1. *H_3 and H_2 are isomorphic hypergroups.*

H_3	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	a,b,d
d	c,d	a,b	a,b,d	a,b,c,d

H_2	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	a,b,c
d	c,d	a,b	a,b,c	a,b,c,d

Proof. Define $f : H_3 \longrightarrow H_2$ by $f(a) = b$, $f(b) = a$, $f(c) = d$ and $f(d) = c$. Since H_3 and H_2 are commutative, it is suffices to check the following:

$f(a*a) = f(H) = H$	$f(a) *' f(a) = b *' b = H$
$f(b*b) = f(H) = H$	$f(b) *' f(b) = a *' a = H$
$f(c*c) = f(H) = H$	$f(c) *' f(c) = d *' d = H$
$f(d*d) = f(H) = H$	$f(d) *' f(d) = c *' c = H$
$f(a*b) = f(\{a,b\}) = \{a,b\}$	$f(a) *' f(b) = b *' a = \{a,b\}$
$f(a*c) = f(\{a,b\}) = \{a,b\}$	$f(a) *' f(c) = b *' d = \{a,b\}$
$f(a*d) = f(\{c,d\}) = \{c,d\}$	$f(a) *' f(d) = b *' c = \{c,d\}$
$f(b*c) = f(\{c,d\}) = \{c,d\}$	$f(b) *' f(c) = a *' d = \{c,d\}$
$f(b*d) = f(\{a,b\}) = \{a,b\}$	$f(b) *' f(d) = a *' c = \{a,b\}$
$f(c*d) = f(\{a,b,d\}) = \{a,b,c\}$	$f(c) *' f(d) = d *' c = \{a,b,c\}$

□

PROPOSITION 4.2. *H_{32} and H_{18} are isomorphic hypergroups.*

H_{32}	a	b	c	d
a	a,b,c,d	a,b	a,b	a,c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	c,d
d	a,c,d	a,b	c,d	a,b,c,d

H_{18}	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b,d
c	a,b	c,d	a,b,c,d	c,d
d	c,d	a,b,d	c,d	a,b,c,d

Proof. Define $f : H_{32} \longrightarrow H_{18}$ by $f(a) = d$, $f(b) = c$, $f(c) = a$ and $f(d) = b$. Since H_{32} and H_{18} are commutative, it is suffices to check the following:

$f(a * a) = f(H) = H$	$f(a) *' f(a) = d *' d = H$
$f(b * b) = f(H) = H$	$f(b) *' f(b) = c *' c = H$
$f(c * c) = f(H) = H$	$f(c) *' f(c) = a *' a = H$
$f(d * d) = f(H) = H$	$f(d) *' f(d) = b *' b = H$
$f(a * b) = f(\{a, b\}) = \{c, d\}$	$f(a) *' f(b) = d *' c = \{c, d\}$
$f(a * c) = f(\{a, b\}) = \{c, d\}$	$f(a) *' f(c) = d *' a = \{c, d\}$
$f(a * d) = f(\{a, c, d\}) = \{a, b, d\}$	$f(a) *' f(d) = d *' b = \{a, b, d\}$
$f(b * c) = f(\{c, d\}) = \{a, b\}$	$f(b) *' f(c) = c *' a = \{a, b\}$
$f(b * d) = f(\{a, b\}) = \{c, d\}$	$f(b) *' f(d) = c *' b = \{c, d\}$
$f(c * d) = f(\{c, d\}) = \{a, b\}$	$f(c) *' f(d) = a *' b = \{a, b\}$

□

PROPOSITION 4.3. *H_1, H_2, H_4, H_5 and H_7 are non-isomorphic hypergroups.*

H_1	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	a,b
d	c,d	a,b	a,b	a,b,c,d

H_2	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	a,b,c
d	c,d	a,b	a,b,c	a,b,c,d

H_4	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	c,d
d	c,d	a,b	c,d	a,b,c,d

H_5	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	a,c,d
d	c,d	a,b	a,c,d	a,b,c,d

H_7	a	b	c	d
a	a,b,c,d	a,b	a,b	c,d
b	a,b	a,b,c,d	c,d	a,b
c	a,b	c,d	a,b,c,d	a,b,c,d
d	c,d	a,b	a,b,c,d	a,b,c,d

THEOREM 4.4. *Among the 183398 commutative single power cyclic hypergroups of order four and period two, there are 7906 non-isomorph commutative single power cyclic hypergroups.*

Proof. By using Algorithm 1 presented in Appendix and a computer program. □

5. Appendix

In Algorithm 1, we set $H = \{a, b, c, d\}$ and M the family of all non-empty subsets of H. We investigate all hyperoperations on H with desired properties. In order to do this, we assume that A is a 4×4 matrix as the table of hyperoperations on H. Note that the entries of matrix A are the elements of M. The matrix A must be symmetric and each entry on the main diameter is equal to H. Then,

AllHyperList:= the list of all A such that A is a hypergroup with desired properties.

UniqueHyperList:= The list of all desired hypergroups up to isomorphism.

At the beginning, we investigate all possible cases to put the elements of M above the main diameter. Then, we apply Proposition 3.1 to check the associativity. If all 24 conditions of Proposition 3.1 are satisfied, then (H, A) is one of the desired hypergroups and we put it inside AllHyperList. Afterward, the new A is compared to the elements of AllHyperList to determine the isomorphic. If A is not isomorphic with any element of AllHyperList, then we set A inside UniqueHyperList. We repeat the above process for all cases.

Isomorphism investigation: Let A be a new matrix. We want to check that A is isomorphic to other matrices or not. Suppose that $\mathbb{S}_4$ is the symmetric group of order 4.We check the isomorphism between A and every element in AllHyperList. If under one of the elements of $\mathbb{S}_4$, we obtain an isomorphism, then A is not new. In otherwise, A is non-isomorphic to other matrices in AllHyperList and so we put A inside UniqueHyperList.

In the following we present the algorithm.

Conclusion.

A hypergroup $(H, *)$ is cyclic if there exists $h \in H$ and a natural number s such that $H = h \cup h^2 \cup \cdots h^s \cup \cdots$. If $H = h \cup h^2 \cup \cdots h^s$ then H is cyclic hypergroup with finite period. A hypergroup $(H, *)$ is single power cyclic hypergroup if $H = h \cup h^2 \cup \cdots h^s \cup \cdots$ and $h \cup h^2 \cup \cdots h^{m-1} \subseteq h^m$ for every m. In this paper we enumerated the commutative single power cyclic hypergroups of order 4 and period 2, continuing the study published in [4]. Among the 183398 commutative single power cyclic

Algorithm 1: This algorithm finds the list of hypergroups up to isomorphism for $n = 4$.

Algorithm `SolveN4()`

- $H \leftarrow \{1, 2, 3, 4\}$
- ▷M `is the space of all possibility for elements` a_{ij}:
- $M \leftarrow PowerSet(H) \setminus \{\emptyset\}$
- $AllHypersList \leftarrow []$
- $UniqueHypersList \leftarrow []$
- **foreach** $(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34})$ *in* M^6 **do**
  - Set $a_{ij} = a_{ji}$ for every $1 \leq j < i \leq 4$ ▷A `is symmetric`
  - Set $a_{ii} = H$ for every $1 \leq i \leq 4$ ▷`elements on the diameter are` H.
  - ▷A, `a matrix` 4 `by` 4, `is the table of operation. It is symmetric and all elements on the diameter are` H.
  - $A \leftarrow (a_{ij})$
  - ▷`Checking 24 conditions to be` A `be a table of a hypergroup.`
  - **if** *Check24Conditions(A)* **then**
    - ▷`Checking that weather` A `is isomorphic to the previous founded hypergroups.`
    - **if** *NOT IsRepeated(HypersList,A)* **then**
      - Add A to UniqueHypersList
    - **end**
    - Add A to AllHypersList
  - **end**
- **end**
- **return** $UniqueHypersList$

Procedure `IsRepeated(`$HypersList$,A`)`

- **Input** : $HypersList$, a list of hypergroups;
  A, a matrix of the action table of a hypergroup.
- **Output:** Returns true, if the hypergroup corresponded to A is isomorphic to a hypergroup in the list $HypersList$; Otherwise returns false.
- $S \leftarrow Sym(\{1, 2, 3, 4\})$ ▷S `is the symmetric group over` $\{1, 2, 3, 4\}$.
- **foreach** $C = (c_{ij})$ *in* $HypersList$ **do**
  - **foreach** f *in* S **do**
    - ▷`Define` $b_{f(i)f(j)}$ `as the image of` c_{ij} `under permutation` f.
    - Set $b_{f(i)f(j)} := f(c_{ij})$ for every $1 \leq i, j \leq 4$
    - $B \leftarrow (b_{ij})$ ▷`Note that` $b_{f(i)f(j)} = \{f(x) : x \in c_{ij}\}$.
    - **if** $A = B$ **then**
      - **return** *true*
    - **end**
  - **end**
- **end**
- **return** *false*

hypergroups of order four and period two, we obtained that there are 7906 non-isomorph commutative single power cyclic hypergroups.

For future work, we intend to continue our study of cyclicity in hypergroups with other orders and periods.

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