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SOME RESULTS OF RELATIVE L-ORDER AND GENERALIZED RELATIVE L-ORDER OF ANALYTIC FUNCTIONS IN THE UNIT CISC

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Abstract. Some basic properties in connection with generalized relative order and generalized relative lower order of analytic functions in the unit disc have been dicussed in this article.

1. Introduction, Definitions and Notations

Consider an analytic function f defined in the unit disc $U = \{z : |z| < 1\} \subset \mathbb{C}$, the set of all finite complex numbers. Let $T_f(r)$ be the Nevanlinna's Characteristic function, defined by

$$T_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta$$

where $\log^+ x = \max(\log x, 0)$ for all $x \geqslant 0$.

The maximum modulus of f is defined by

$$M_f(r) = \max_{|z|=r} |f(z)| .$$

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If $T_f(r) < (1-r)^{-\mu}$ for all r in $0 < r_0(\mu) < r < 1$, the Nevanlinna order [3] $\rho(f)$ of f is given by

$$\rho(f) = \limsup_{r \to 1} \frac{\log T_f(r)}{-\log(1-r)}.$$

Banerjee and Dutta [1] extended this notions and defined the relative Nevanlinna order (relative Nevanlinna lower order) of a analytic function f with respect to an entire function g defined as:

Definition 1.1. An entire function g is said to have the *property (A)*, if for any $\sigma > 1, \lambda > 0$ and for all r, $0 < r < 1$ sufficiently close to 1 $\left[G\left(\left(\frac{1}{1-r}\right)^{\lambda}\right)\right]^2 < G\left(\left(\left(\frac{1}{1-r}\right)^{\lambda}\right)^{\sigma}\right)$, where $G(r) = \max_{|z|=r} |g(z)|$.

Definition 1.2. If f be analytic in U and g be entire, then relative order of f with respect to g, denoted by $\rho_g(f)$ is defined by,

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(\exp r^{\mu}) \text{ for all } 0 < r_0(\mu) < r < 1.\} \\ &= \limsup_{r \to 1} \frac{\log T_g^{-1} T_f(r)}{-\log(1-r)} \end{aligned}$$

In the line of Banerjee and Dutta [1] we may give the following definitions:

Definition 1.3. If $l \geq 1$ is a positive integer, then the l-th generalized relative order and l-th generalized relative lower order of an analytic function f in U with respect to an entire function g, denoted by $\rho_f^{[l]}(g)$ is defined by

$$\begin{aligned} \rho_g^{[l]}(f) &= \inf\left\{\mu > 0 : T_f(r) < T_g\left(\exp^{[l-1]} r^{\mu}\right) \text{ for all } 0 < r_0(\mu) < r < 1.\right\} \\ &= \limsup_{r \to 1} \frac{\log^{[l]} T_g^{-1} T_f(r)}{-\log(1-r)} \ . \end{aligned}$$

and

$$\lambda_g^{[l]}(f) = \liminf_{r \to 1} \frac{\log^{[l]} T_g^{-1} T_f(r)}{-\log(1-r)}$$

where $\log^{[n]} x = \log(\log^{[n-1]} x)$ for $n = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

For $n = 2$, the quantities $\rho_g^{[2]}(f) = \overline{\rho_g}(f)$ and $\lambda_g^{[2]}(f) = \overline{\lambda_g}(f)$ are respectively called relative Nevanlinna hyper order and relative Nevanlinna hyper lower order of an analytic function f in U with respect to another entire function g.

Somasundaram and Thamizharasi [4] introduced the notion of L-order for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly
i.e $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a.
Their definitions are as follows:

Definition 1.4. The relative L-order $[\rho_g(f)]^L$ and relative L-lower order $[\lambda_g(f)]^L$ of an analytic function f in U with respect to another entire function g are defined as

$$[\rho_g(f)]^L = \limsup_{r\to 1} \frac{\log T_g^{-1} T_f(r)}{\log(\frac{1}{1-r} L(\frac{1}{1-r}))}$$

and

$$[\lambda_g(f)]^L = \liminf_{r\to 1} \frac{\log T_g^{-1} T_f(r)}{\log(\frac{1}{1-r} L(\frac{1}{1-r}))}.$$

Definition 1.5. The relative generalised L-order $\left[\rho_g^p(f)\right]^L$ and relative generalised L-lower order $\left[\lambda_g^p(f)\right]^L$ of an analytic function f in U with respect to another entire function g are defined as:

$$\left[\rho_g^p(f)\right]^L = \limsup_{r\to 1} \frac{\log^p T_g^{-1} T_f(r)}{\log(\frac{1}{1-r} L(\frac{1}{1-r}))}$$

and

$$\left[\lambda_g^p(f)\right]^L = \liminf_{r\to 1} \frac{\log^p T_g^{-1} T_f(r)}{\log(\frac{1}{1-r} L(\frac{1}{1-r}))}.$$

2. Lemmas

In this section we introduced some preliminary Lemmas which will be needed in the sequel.

Lemma 2.1. [1] *Let g be an entire function which has the* property (A). *Then for any positive integer n and for all $\sigma > 1$, $\lambda > 0$,*

$$\left[G\left(\left(\frac{1}{1-r}\right)^{\lambda}\right)\right]^n < G\left(\left(\left(\frac{1}{1-r}\right)^{\lambda}\right)^{\sigma}\right)$$

holds for all $r, 0 < r < 1$, sufficiently close to 1.

Lemma 2.2. [1] *If g is entire then*

$$T_g(\frac{1}{1-r}) \leq \log G\left(\frac{1}{1-r}\right) \leq 3T_g\left(\frac{2}{1-r}\right)$$

for all $r, 0 < r < 1$, sufficiently close to 1.

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. *Let f be analytic in U of generalised relative L-order $\left[\rho_g^p(f)\right]^L$, where g is entire. Let $\epsilon > 0$ is arbitrary then $T_f(r) = O\left(\log G\left(\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{\left[\rho_g^p(f)\right]^L+\epsilon}\right)\right)$ holds for all $r, 0 < r < 1$, sufficiently close to 1. Conversely, if for an analytic f in U and entire g having the* property (A), $T_f(r) = O\left(\log G\left(\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{k+\epsilon}\right)\right)$ *holds for all $r, 0 < r < 1$, sufficiently close to 1, and $T_f(r) = O\left(\log G\left(\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{k-\epsilon}\right)\right)$ does not hold for all $r, 0 < r < 1$, sufficiently close to 1, then $k = \left[\rho_g^p(f)\right]^L$.*

Proof. From the definition of relative L-order, we have

$$\begin{aligned} T_f(r) &\leq T_g\left[\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{\left[\rho_g^p(f)\right]^L+\epsilon}\right] \text{ ,for } 0 < r_0 < r < 1. \\ &< \log G\left(\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{\left[\rho_g^p(f)\right]^L+\epsilon}\right) \text{ , by Lemma2.2} \end{aligned}$$

Therefore,

$$\therefore T_f(r) = O\left(\log G\left(\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{\left[\rho_g^p(f)\right]^L+\epsilon}\right)\right)$$

Conversely, if $T_f(r) = O\left(\log G\left(\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{k+\epsilon}\right)\right)$ holds for all $r, 0 < r < 1$, sufficiently close to 1, then

$$\begin{aligned} T_f(r) &< [\alpha]\log G\left(\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{k+\epsilon}\right), \alpha > 1 \\ &= \frac{1}{3}\log\left(G\left(\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{k+\epsilon}\right)\right)^{3[\alpha]} \end{aligned}$$

$$T_f(r) \leq T_g\left(2\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{(k+\epsilon)}\right)^{\sigma}, \text{ by Lemma2.2 and 2.1}$$

For any $\sigma > 1$.

$$\begin{aligned} T_g^{-1}T_f(r) &\leq \left(2\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{(k+\epsilon)}\right)^{\sigma} \\ \log^p T_g^{-1}T_f(r) &\leq \sigma\log^p\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{k+\epsilon} + O(1) \\ &= \sigma\,(k+\epsilon)\log\left(\frac{1}{1-r}L(\frac{1}{1-r})\right) + O(1) \end{aligned}$$

So,

$$\limsup_{r\to 1}\frac{\log^p T_g^{-1}T_f\,(r)}{\log(\frac{1}{1-r}L(\frac{1}{1-r}))} \leq \sigma(k+\epsilon)$$

Since $\epsilon > 0$ is arbitrary and let $\sigma \to 1+$ we get

$$\limsup_{r\to 1}\frac{\log^p T_g^{-1}T_f\,(r)}{\log(\frac{1}{1-r}L(\frac{1}{1-r}))} \leq k \tag{1}$$

Again there exists a sequence $\{r_n\}$ of values r tending to 1 for which

$$\begin{aligned} T_f(r) &\geq \log G\left(\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{(k-\epsilon)}\right) \\ &\geq T_g\left(\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{(k-\epsilon)}\right), \text{ by Lemma2.2} \end{aligned}$$

and so,

$$\frac{\log^{p} T_{g}^{-1} T_{f}\left(r\right)}{\log(\frac{1}{1-r}L(\frac{1}{1-r}))} \geq k-\epsilon$$

for $r = r_n \to 1$. Since $\epsilon > 0$ is arbitrary then

$$\limsup_{r\to 1} \frac{\log^{p} T_{g}^{-1} T_{f}\left(r\right)}{\log(\frac{1}{1-r}L(\frac{1}{1-r}))} \geq k \tag{2}$$

combining(1) and (2), we obtain,

$$k = \left[\rho_g^p(f)\right]^L$$

□

Corollary 3.2. *Let f be analytic in U of relative L-order $\left[\rho_g(f)\right]^L$, where g is entire. Let $\epsilon > 0$ is arbitrary then*

$$T_f(r) = O\left(\log G\left(\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{\left[\rho_g(f)\right]^L+\epsilon}\right)\right)$$

holds for all $r, 0 < r < 1$, sufficiently close to 1. Conversely, if for an analytic f in U and entire g having the property(A)

$$, T_f(r) = O\left(\log G\left(\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{k+\epsilon}\right)\right)$$

holds for all $r, 0 < r < 1$, sufficiently close to 1, and

$$T_f(r) = O\left(\log G\left(\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{k-\epsilon}\right)\right)$$

does not hold for all $r, 0 < r < 1$, sufficiently close to 1, then $k = \left[\rho_g(f)\right]^L$.

Theorem 3.3. *Let f_1 and f_2 be analytic in U having generalised relative L-orders $\left[\rho_g^p(f_1)\right]^L$ and $\left[\rho_g^p(f_2)\right]^L$ respectively, where g is entire having the* property (A). *Then*

$$\text{(a)} \quad \left[\rho_g^p(f_1 \pm f_2)\right]^L \leq \max\left\{\left[\rho_g^p(f_1)\right]^L, \left[\rho_g^p(f_2)\right]^L\right\}$$

and

$$\text{(b)} \quad \left[\rho_g^p(f_1.f_2)\right]^L \leq \max\left\{\left[\rho_g^p(f_1)\right]^L, \left[\rho_g^p(f_2)\right]^L\right\}.$$

The same inequality holds for the quotient. The equality holds in (b) if $\left[\rho_g^p(f_1)\right]^L \neq \left[\rho_g^p(f_2)\right]^L$.

Proof. Suppose that $\left[\rho_g^p(f_1)\right]^L$ and $\left[\rho_g^p(f_2)\right]^L$ both are finite, because if one of them or both are infinite, the inequalities are evident. Let $\rho_1 = \left[\rho_g^p(f_1)\right]^L$ and $\rho_2 = \left[\rho_g^p(f_2)\right]^L$ and $\rho_1 \leq \rho_2$.

For arbitrary $\epsilon > 0$ and for all $r, 0 < r < 1$, sufficiently close to 1, we have

$$\begin{aligned} T_{f_1}(r) &< T_g\left[\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{\rho_1+\epsilon}\right] \\ &\leq \log G\left[\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{\rho_1+\epsilon}\right] \\ T_{f_2}(r) &< T_g\left[\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{\rho_2+\epsilon}\right] \\ &\leq \log G\left[\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{\rho_2+\epsilon}\right] \end{aligned}$$

Now for all $r, 0 < r < 1$, sufficiently close to 1,

$$\begin{aligned} T_{f_1 \pm f_2}(r) &\leq T_{f_1}(r) + T_{f_2}(r) + O(1) \\ &\leq \log G\left[\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{\rho_1+\epsilon}\right] \\ &\quad + \log G\left[\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{\rho_2+\epsilon}\right] + O(1) \\ &\leq 3\log G\left[\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{\rho_2+\epsilon}\right] \\ &= \frac{1}{3}\log\left(G\left[\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{\rho_2+\epsilon}\right]\right)^9 \\ &\leq \frac{1}{3}\log G\left[\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{(\rho_2+\epsilon)}\right]^\sigma, \text{ by Lemma2.1} \end{aligned}$$

for any $\sigma > 1$

$$\leq T_g\left[2\exp^{p-1}\left(\frac{1}{1-r}L(\frac{1}{1-r})\right)^{(\rho_2+\epsilon)}\right]^\sigma, \text{ by Lemma2.2}$$

$$\begin{aligned}\log^{p} T_{g}^{-1} T_{f_1 \pm f_2}(r) &\leq \sigma \log^{p}\left[\exp^{p-1}\left(\frac{1}{1-r} L(\frac{1}{1-r})\right)^{(\rho_2+\epsilon)}\right] + O(1) \\ &= \sigma(\rho_2+\epsilon)\log\left(\frac{1}{1-r} L(\frac{1}{1-r})\right) + O(1)\end{aligned}$$

$$\therefore \left[\rho_g^p\left(f_1 \pm f_2\right)\right]^L = \limsup_{r\to 1} \frac{\log^{p} T_{g}^{-1} T_{f_1 \pm f_2}(r)}{\log(\frac{1}{1-r} L(\frac{1}{1-r}))} \leq (\sigma\rho_2 + \sigma\epsilon)$$

since $\epsilon > 0$ is arbitrary, we obtain by letting $\sigma \to 1+$

$$\left[\rho_g^p\left(f_1 \pm f_2\right)\right]^L \leq \rho_2 = \max\left\{\left[\rho_g^p(f_1)\right]^L, \left[\rho_g^p(f_2)\right]^L\right\}.$$

which proves (a).

For (b), since

$$T_{f_1.f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r).$$

We obtain similarly as above,

$$\left[\rho_g^p(f_1.f_2)\right]^L \leq \max\left\{\left[\rho_g^p(f_1)\right]^L, \left[\rho_g^p(f_2)\right]^L\right\}.$$

Let $f = f_1.f_2$ and $\left[\rho_g^p(f_1)\right]^L \leq \left[\rho_g^p(f_2)\right]^L$.

Then applying (b), $\left[\rho_g^p(f)\right]^L \leq \left[\rho_g^p(f_2)\right]^L$.

Again since $f_2 = f/f_1$, applying the first part of (b), we have

$$\left[\rho_g^p(f_2)\right]^L \leq \max\left\{\left[\rho_g^p(f)\right]^L, \left[\rho_g^p(f_1)\right]^L\right\}.$$

Since $\left[\rho_g^p(f_1)\right]^L \leq \left[\rho_g^p(f_2)\right]^L$, we have

$$\left[\rho_g^p(f)\right]^L = \left[\rho_g^p(f_2)\right]^L = \max\left\{\left[\rho_g^p(f_1)\right]^L, \left[\rho_g^p(f_2)\right]^L\right\}.$$

when $\left[\rho_g^p(f_1)\right]^L \neq \left[\rho_g^p(f_2)\right]^L$. This proves the theorem. □

Corollary 3.4. *Let f_1 and f_2 be analytic in U having relative L-orders $[\rho_g(f_1)]^L$ and $[\rho_g(f_2)]^L$ respectively, where g is entire having the* property (A). *Then*

$$\text{(a)}\quad [\rho_g(f_1 \pm f_2)]^L \leq \max\left\{[\rho_g(f_1)]^L, [\rho_g(f_2)]^L\right\}$$

and

$$\text{(b)}\quad [\rho_g(f_1.f_2)]^L \leq \max\left\{[\rho_g(f_1)]^L, [\rho_g(f_2)]^L\right\}.$$

The same inequality holds for the quotient. The equality holds in (b) if $[\rho_g(f_1)]^L \neq [\rho_g(f_2)]^L$.

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