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# A STUDY ON UNDERSTANDING OF DEFINITE INTEGRAL AND RIEMANN SUM

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Abstract. Conceptual and procedural knowledge of integration is necessary not only in calculus but also in real analysis, complex analysis, and differential geometry. However, students show not only focused understanding of procedural knowledge but also limited understanding on conceptual knowledge of integration. So they are good at computation but don't recognize link between several concepts. In particular, Riemann sum is helpful in solving applied problem, but students are poor at understanding structure of Riemann sum. In this study, we try to investigate understanding on conceptual and procedural knowledge of integration and to analyze errors. Conducting experimental class of Riemann sum, we investigate the understanding of Riemann sum structure and so present the implications about improvement of integration teaching.

## 1. Introduction

Integration with differentiation is an important research subject which applied widely in mathematics, science, engineerings. Physics and engineering textbooks regularly use integrals to define and compute nature phenomena like work, force, mass center, tension, and aspects of kinetics [11].

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Integration is important in differential equation, complex function analysis as well as in defining several nature phenomena and calculating in physics and engineering. However, students are concentrated in acquiring with skill to solve definite integral and so they have difficulty in understanding of concept of definite integral. Integration is one of two important concepts in calculus. In order to improve students' understanding of integration concept, it is important to decide level of conceptual and procedural knowledge. Important procedural knowledge is integration technique which used in finding anti-derivative of function. On students' procedural, conceptual ability of integration, [17] showed most students can apply procedure and basic technique of integral. [1] found many students accomplished common procedure in solving area below the curve. But a few of them said they didn't know well why they did such. [15] and [13] found out students had difficulty in conceptual understanding of integration. For successful mathematical learning, balance of procedural knowledge and conceptual knowledge to emphasize concept and relation is more important than technique of problem solving.

To interpret conceptual meaning of integration, we depend on definition of definite integral which is the limit of Riemann sum, but we become to use fundamental theorem of calculus in calculating the definite integral. Fundamental theorem of calculus can make definite integral solve by using process to change integrand into one anti-derivative. However, students are familiar with using the fundamental theorem of calculus without understanding definition of definite integral.

Many practical applied problems include functions which do not have anti-derivative which expressed with elementary function. These functions cannot apply fundamental theorem of calculus. Other method to solve definite integral is necessary. To approximate the definite integral, the method of Riemann sum is necessary [11]. Since students cannot understand the structure of Riemann sum deeply, they have some difficulties in solving definite integral of applied problem. [22] investigated students have problems in analysis of definite integral using Riemann sum.

Thus in this study, we try to investigate understanding on conceptual and procedural knowledge of integration and to analyze conceptual and procedural errors. We try to investigate mathematics historical background of definite integral. In addition, conducting experimental class of structure of Riemann sum, we investigate the understanding Riemann sum structure and so present the implications about improvement of integration teaching.

## 2. Theoretical Background

**2.1. Investigation of conceptual and procedural knowledge.** Important principle of understanding is an ability of connecting relation of conceptual and procedural knowledge [11]. [10] defined conceptual knowledge as follows; "It is rich knowledge in relation and can be considered as network of knowledge." Students' conceptual knowledge is discovered by relation establishment between information pieces, and procedural knowledge is divided into following two parts. "One part is constructed into formal language, symbolic representative system of mathematics, and other part is constructed into algorithm, law for ending mathematics work." According to [10], understanding is occurred when new information is connected through proper relation with existing knowledge. Students who can connect conceptual knowledge and procedural knowledge are able to understand and develop mathematics knowledge deeply. Students who are poor at any knowledge among two knowledge, or students who develop two knowledge separately become to understand concepts of mathematics limitedly.

Since learning of conceptual knowledge require serious mental activities, students prefer learning to memorize procedural law and algorithm [15]. According to [18], students are poor at concept which involved in procedure. Procedural knowledge which conceptual understanding isn't premised is the limited understanding [17].

Flexibility which required in dealing with several kinds of problems can be got only by conceptual understanding. Moreover, as shown in the study of [5], conceptual understanding will increase procedural ability and efficiency. Students who exposed in concept-centered learning environment get very high point than students who exposed in procedure-centered learning environment in evaluation to estimate conceptual knowledge as well as procedural knowledge.

[17] investigated which problems one hundred ten students have calculating area surrounded by curve. [8] announced there was discordance between items of procedural knowledge and understanding of conceptual knowledge. [14] investigated students' understanding on basic concept of

definite integral and fundamental theorem of calculus. In [3], study is based on the relation between infinite series and definite integral. [20] investigated whether students understand the definition of definite integral or not. [6] covered understanding on limit formula of Riemann sum and definite integral symbol.

However, there is few study of undergraduate students' understanding on procedural and conceptual knowledge of integration. Also, there is few study on definite integral of practical applied problems which do not have anti-derivative. Thus, this study tries to investigate definite integral in which the method of Riemann sum is necessary.

**2.2. Historic investigation of definite integral.** In calculus, one of important concepts is definite integral of function. Historically, integration formula starts from grasping area as infinitesimal sum and discovering the formula. Modern text books introduce calculus as limit process instead of infinitesimal concept. Thus, in this study we try to investigate developmental step of definite integral as before and after of Newton and Leibniz.

**2.2.1.** *Eudoxos, Archimedes, and Cavalieri.* The first person to attempt to compute area and volume of geometric figure is known as Eudoxos. Eudoxos found method of exhaustion which is a strong and very exquisite method to caculate area and volume. Archimedes computed areas and volumes of many figures by using the method of Eudoxos. The method of exhaustion is acted by increasing the number of segment gradually in the process of approximation and solving better approximation value about area below curve [7].

Cavalieri calculated area and volume using method of indivisibles. Followings are considered in this approach method. Geometric figure is consisted with infinitely many atoms of area or volume and if these were added up, then wanted area or volume are found. If there are only finite number of rectangles, approximate value of area under parabola is obtained by adding all their area [7]. If there are infinitely many rectangles with infinitesimal width and implement such infinite calculation, the sum will be real area [7]. Thus modern textbook comprehends Riemann sum as sum of close rectangles and define real area by the limit of Riemann sum.

**2.2.2.** *Newton and Leibniz.* The thing to explain area and distance by using infinitesimal idea is substituted in terms of explanation of differentiation by Newton and Leibniz, and manipulation on function appeared by means to explain relation of area and distance instead of geometric proportional relation [12]. Newton and Leibniz established not only the fact integration and differentiation are each in relation with inverse operation but also calculus which can make integral formula deduce from differentiation formula.

According to Leibniz, $\int ydx$ is the sum of infinitesimal rectangles whose each area is $ydx$. Indefinite integral is chosen into integral by mixing Netwon and Leibniz's idea and double meaning is given to integral symbol by using differentiation. It is at time when calculus was established and developed explosively, that definite and indefinite integral were not separated clearly and existed with one fusion form [12].

**2.2.3.** *After 18th century-Fourier, Cauchy, and Riemann.* As deal with partial differential equation after mid 18th century, function hard to find anti-derivative appeared and so started to appear limitations on aspect which consider integration as inverse of differentiation [16]. Thus, mathematicians observed area gradually, especially Fourier became to explain integration by using area. He thought since there exists the ordinate f(x) for each x, this ordinate decides domain of plane and this domain appears definite area.

Canchy established definite integral to correspond perspective Fourier posed by using continuous function. By establishing definite integral, he could form fundamental theorem of calculus of modern form, accordingly his theory appeared to handle indefinite integral strictly through definite integral [12]. Cauchy defined definite integral of $f(x)$ as $\int_{x_0}^{X} f(x)dx$ which is continuous on the interval $[x_0, X]$ by using the limit in [4] and [9]. Here, Cauchy represents definite integral as $\int_{x_0}^{X} f(x)dx$ and he emphasized $\int f(x)$ is represented as the limit of sum not as sum of product. Fundamental theorem of calculus of modern form was basically originated from Cauchy and these results are proved for continuous functions. He represented the relation of indefinite and definite integral through this. Cauchy proved that $F(x) = \int_{x_0}^{X} f(x)dx$ and $F'(x) = f(x)$ when $f(x)$ is continuous on $[x_0, X]$. Through fundamental theorem of calculus, Cauchy connected the concept of definite integral with integration as inverse process of differentiation in 18th [8].

Riemann's explanation on integration dealt with much wider kinds of functions than continuous functions. When he posed Riemann integrable function which discontinuous points within the interval are infinitely many, generalization of Riemann integral was appeared apparently. In the form of $f(x_i)(x_{i+1} - x_i)$, the length of interval $[x_i, x_{i+1}]$ all may be different and choose the value of $f$ from arbitrary number $x_i^*$ belonging in ith interval $[x_i, x_{i+1}]$ as i-th height of rectangle instead of left ending point or right ending point. The formula $\sum_{i=1}^{n} f(x_i^*)(x_{i+1} - x_i)$ which defined in interval [a, b] is called Riemann sum, when n is large, the limit of Riemann sum is defined by definite integral [2].

## 3. Method and procedure of study

**3.1. Object of study.** Objects of study are second and third year students of university which are 20 students and 30 students, respectively. They solved problems on procedural, conceptual knowledge of definite integral and we analyzed error on procedural, conceptual knowledge on students' understanding of definite integral. All the students learned calculus, 2nd-year students were learning real analysis and 3rd-year students already learned and were learning complex analysis. Among 2nd year students, 7 people got A grade in real analysis, 9 people got B grade in real analysis.

As a result of the first survey, students show they have difficulty in conceptual question on understanding of integration. Thus by adding question which related with Riemann sum and deepening questions of conceptual and procedural knowledge of definite integral, we let students conduct the 2nd survey. The object of second survey was 20 sophomores and answers were collected after the first survey.

**3.2. Survey question.** Questions of first survey are questions related with typical understanding of definite integral and questions were chosen from [21] and [13]. Questions are directed to solve in different paper and students can use extra paper and calculator. In questions, No.1 question is a question for procedural knowledge, No.3 is for conceptual knowledge, No.2 and No.4 are for conceptual and procedural knowledge. First survey was conducted in mid first semester for one hour, and survey papers ware submitted.

**3.3. Data analysis.**

**3.3.1.** *The first analysis.* We collect survey papers which students complete and use as data. Errors which appeared on understanding conceptual and procedural knowledge are used as data analysis. The numbers of 2nd-year, 3rd-year students who get the correct answer, the form of error, the number of non-responders, and percentage on No.1-No.4 questions excluding No.3(1) are presented in Table 1. Because No.2 question can be solved by conceptual or procedural knowledge, the number of students who got correct is additionally presented in Table 2 after analyzing the form of knowledge which students who got correct use. Here, percentage is percentage of students who got correct for students to solve questions excluding none.

No.1 question is a question to verify the concept of integration as relation of anti-derivative and coordinate. Integration of gradient is good but there are students who get wrong by doing carelessly and a few students who don't write coordinate. No.2 question is a question which can use conceptual or procedural knowledge. In the case of 3rd-year, the number of students who solve by using conceptual knowledge was 13. 12 students among these got the correct answer, and 1 student got the wrong answer. The number of students who solve by using procedural knowledge is 5. Two among them got wrong because they didn't recognize the part of curve $y = x(x-4)$ $x$ from 0 to 4 is below $x$-axis, $x$ from 4 to 5 is above $x$-axis. The number of students who got correct answer was 3.

| | NCS | | C | | P | | none | | T |
|---|---|---|---|---|---|---|---|---|---|
| grade | 2 | 3 | 2 | 3 | 2 | 3 | 2 | 3 | 2, 3 |
| question | person | person | person | person | person | person | person | person | person |
| 1 | 13 | 15 | 0 | 0 | 3 | 3 | 4 | 2 | 20 |
| 2 | 17 | 15 | 0 | 1 | 0 | 2 | 3 | 2 | 20 |
| 3.(2) | 11 | 14 | 2 | 2 | 5 | 3 | 2 | 1 | 20 |
| 4.(1) | 4 | 8 | 1 | 1 | 8 | 13 | 7 | 1 | 20 |
| 4.(2) | 12 | 5 | 0 | 4 | 6 | 9 | 2 | 2 | 20 |
| 4.(3) | 13 | 12 | 2 | 2 | 2 | 3 | 3 | 3 | 20 |

NCS: number of correct students C: conceptual error
P:procedural error T: total number of students

TABLE 1. The number of sophomore, junior to get correct and the form of error

The percentage of sophomores who use conceptual knowledge and the percentage of sophomores who use procedural knowledge are equal to

| form | NS(person) | | PCS(person) | | PCS(%) | | NWS(person) | |
|---|---|---|---|---|---|---|---|---|
| grade | 2 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| NC | 13 | 13 | 13 | 12 | 100 | 92.3 | 0 | 1 |
| NP | 4 | 5 | 4 | 3 | 100 | 60 | 0 | 2 |

NC: number of students to use conceptual knowledge NP: number of students to use procedural knowledge PCS: Percentage of Correct Students NWS: Number of wrong students NS: Number of students

Table 2. The form of knowledge to use in No.2

| | area | anti-derivative | limit of R-sum | D-integral |
|---|---|---|---|---|
| 2nd | 9(pic), 9(no pic) | 3 | 3, 1(R-sum) | 0 |
| 3rd | 7(pic), 4(no pic) | 2 | 4 | 4 |

Table 3. Four categories (Allow repetition, unit: person)

100%. However, the percentage of juniors who use conceptual knowledge is 92.3% and the percentage of juniors who use procedural knowledge is 60%. We know from Table 2 that regardless of grade, the percentage of students who use conceptual knowledge is higher than students who use procedural knowledge.

We classify students' respondence in No.3(1) question as follows. If respondence appears in at least one category, then divide into two more categories. If respondence doesn't belong to any of three categories, then classify as none. As a result, No.3(1) question can be divided into four categories which are area, anti-derivative, Riemann sum, and Darboux integral. In the case that students' respondences don't belong to only one category but belong to two categories, we allow repetition of all the categories. The number of students to belong to each category is shown in table 3, except wrong and none answers(sophomore 2). Only 9 persons are able to present various expressions.

(1) Integral to represent area: The students of this category describe definite integral as area. Area is most popular image of definite integral which used by students. Students who don't say how the area under curve is calculated and answer with shaded picture are sophomore 9 persons, junior 7 persons. Students to represent as area which surrounded by $x = a, x = b, x$-axis, $f(x)$ without picture are sophomore 9 persons, junior 4 persons.

(2) Integral as anti-derivative: Students of this category answered mainly as $F(b) - F(a)$. They think primitive function which become $f(x)$ through the derivative. But they don't explain well that chain which connects differentiation with integration is fundamental theorem of calculus.

(3) Integral which represented as the limit of Riemann sum: The limit process to divide and add up infinitely many infinitesimal pieces is described as means to measure area under curve. Figures students draw present traditional pictures which related with Riemann sum. There is one sophomore to represent as Riemann sum, there are three sophomores, four juniors to explain as the limit of Riemann sum.

(4) Darvoux integral: There are four juniors to answer by using upper sum and lower sum. Only when new concept is related with previous concept represent strong understanding. But since they forgot previous concept, they used the definition of Darvoux integral.

We found that conceptualization of definite integral is related with area under curve, value of anti-derivative, and the limit of Riemann sum through No.3(1) question. Especially through over half students choose category of area and anti-derivative, we can know students' conceptualization of definite integral is concentrated on procedural knowledge to solve area. No.3(2) question is a question to solve the definite integral when function takes all the values of +, -, i.e. the sign of function is changed. This is to verify that it can be related with other expression of definite integral. Only 11 sophomores explained exactly as area above x-axis − area below x-axis after drawing the graph when the function takes all the values of +, -. Students with misunderstanding of definite integral was regarded as having error of conceptual knowledge. And students with only picture but without explanation was regarded as having procedural error.

Only 4 sophomores got correct anwers, 8 sophomores made procedural errors in No.4(1) question. Solving Riemann sum, student who make conceptual error using absolute value was one person. We can know from this result only four students understand definite integral. Only 5 juniors got correct answers, 4 juniors made conceptual errors in No.4(2). This is compared with that sophomores made no conceptual error and only procedural error. It seems that since juniors confuse the definition of Riemann sum with solving area, they find solution by using absolute value.

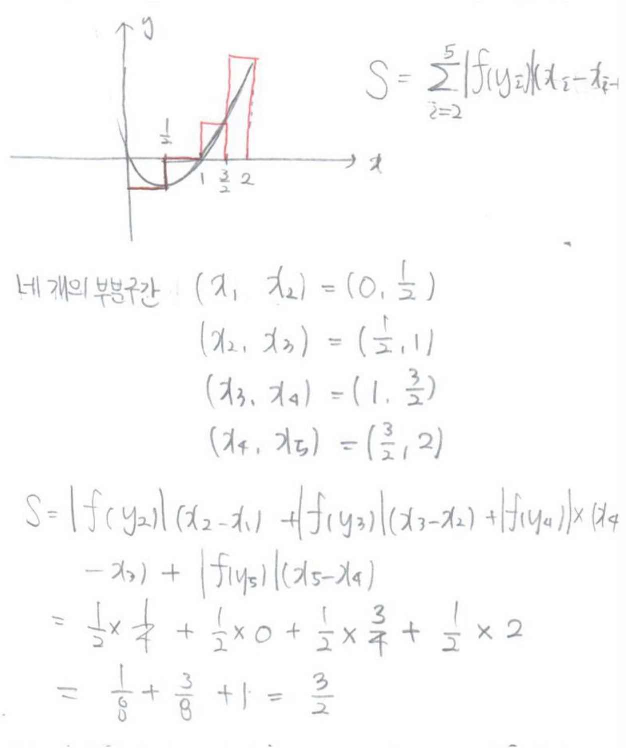



FIGURE 1. Incorrect answer of No.4(1)

| question | 1.(3) | 1.(4) | 2 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| procedural error | 8(40) | 2(10) | 0(0) | 9(45) | 1(5) | 1(5) |
| conceptual error | 0(0) | 0(0) | 3(15) | 0(0) | 0(0) | 0(0) |
| none | 5(25) | 5(25) | 5(25) | 5(25) | 5(25) | 5(25) |

TABLE 4. Form of error of 2nd survey (unit: person(%))

**3.3.2.** *The second analysis.* In the first survey, over half students were concentrated with procedural knowledge rather than concept of definite integral in No.3(1). No.4(1) is a question which related with Riemann sum and percentages of correct answers of sophomores, juniors are 20%, 40% respectively. These show the lowest percentage of correct answer in the first survey. Thus by adding question which related with Riemann sum and deepening questions of conceptual and procedural knowledge of definite integral, we let students conduct the 2nd survey.

Students were very poor at answering No.8 question and so after conducting experimental class, students' answers were collected. Questions of 2nd survey are questions which extracted from [19] and [15]. Here, No.1(1),(2), No.3, and No.4 are questions for procedural knowledge of integral, No.5, No.6, No.7, and No.8 for conceptual knowledge, No.1(3),(4), and No.2 for conceptual and procedural knowledge.

Time of survey is adjusted and conducted in several times for reliability of survey. Non-understandable parts among answers were asked additively with interview. Interview questions of 2nd survey were presented in appendix.

1. Understanding of conceptual and procedural knowledge

No.1(1),(2), No.3, and No.4 can be solved by integral formula and technique. These questions are questions to estimate students procedural technique and the correct percentage of these is 100%. Table 4 is the classified table of form of error on conceptual and procedural knowledge of No.1(3),(4), No.2, No.5, No.6, and No.7. No.1(3),(4), and No.5 questions are questions which can be solved by two ways of conceptual and procedural knowledge. Students to use conceptual knowledge solved them by using relation of integral and area. Students to use procedural knowledge solved by using coordinate or integral formula. On these questions the number of students to use conceptual and procedural knowledge, the number of students to have correct and incorrect answer are shown in Table 5. The percentage of this table is the percentage of students who got correct answers via students to solve excluding non-respondence. We can know from No.1(3),(4), and No.5 in table 5, the students to use conceptual knowledge got higher percentage in correct answers than the students to use procedural knowledge. And students to use conceptual knowledge have higher achievement in questions to use procedural knowledge, too. In fact, they appeared to get grade A in real analysis.

In No.1(3), students to solve by using conceptual knowledge of area-integral relation were 2 persons and the answers of them were all correct. Students to use integral formula was 13 persons, 8 persons among them made mistake and so 5 persons got correct answers. In No.1(4), students to solve by using conceptual knowledge were 6 persons, whose answers were all correct. But students who solve by using integral formula are 9 persons, 2 persons among them made mistakes and so the number of students who got correct answers was 7 persons. This represents students to solve by using conceptual knowledge get higher correct answer.

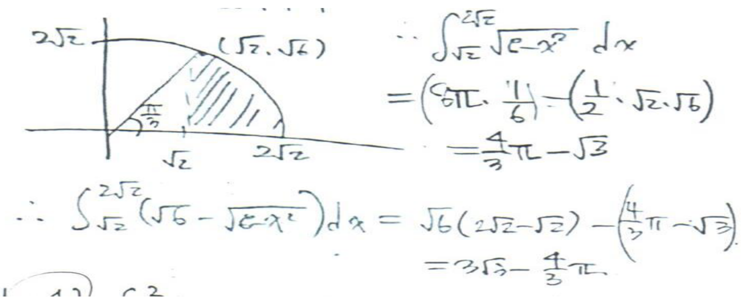


Figure 2. Correct answer of No.1(3) using graph

| qn | form | no. students(person) | NCS(person) | PCS(%) | NWS(person) |
|---|---|---|---|---|---|
| 1.(3) | NSCK | 2 | 2 | 100 | 0 |
| | NSPK | 13 | 5 | 38.5 | 8 |
| 1.(4) | NSCK | 6 | 6 | 100 | 0 |
| | NSPK | 9 | 7 | 77.8 | 2 |
| 5 | NSCK | 14 | 5 | 35.7 | 9 |
| | NSPK | 2 | 0 | 0 | 2 |

NCS: Number of correct students NWS: Number of wrong students PCS: Percentage of correct students NSCK: Number of students to use conceptual knowledge NSPK: Number of students to use procedural knowledge

Table 5. The correct percentage of students to use conceptual and procedural knowledge

No.5 is a question which can be solved by integrating many concepts. That's because it is related to fundamental theorem of calculus, integral, and area. So they can solve the integral of function by using algebraic sum of area. Thus, ability to solve No.5 represents strongly that one has conceptual knowledge. In No.5, the number of students to solve by using conceptual knowledge were 14 persons. 5 persons got correct answers, 9 persons got incorrect. In spite of using conceptual knowledge, there were many students who got wrong answers. That was because they didn't know area below x-axis was not dealt with minus. Students to solve questions using integral formula are 2 persons, who all got wrong answers. Table 5 represents these. The results of No.1(3),(4), and No.5 say students like typical calculation and try to avoid approach of hard concept. This becomes evidence students don't have enough conceptual knowledge of integral. No.6 and No.7 are the conceptual questions to connect derivative with integral using fundamental theorem of calculus, but students had no difficulty in connecting derivative with integral.

2. Understanding of structure of Riemann sum

Though Riemann sum is often means to calculate area of applied question, students' focus of definite integral is confined to understanding of anti-derivative and so students are poor at using the concept of Riemann sum. No.8 is a question to check understanding of students' structure of Riemann sum by using function not having anti-derivative which expressed in terms of elementary function. This function is based on normal distribution and used regularly in statistics. But since the function like normal distribution cannot apply to fundamental theorem of calculus, the method of Riemann sum is required to evaluate definite integral.

Any students didn't talk the method of Riemann sum in No.8. Here meaningful analysis of integral can not be done without adding area of pieces, but they couldn't use the concept of Riemann sum usefully. They thought only integration by parts, Taylor series for anti-derivative and then said they can not integrate. Thus we gave them experimental class of dam problem in order to develop understanding of structure of Riemann sum.

2.A Experimental class

Dam problem: Pressure P is uniformly applied to thin horizontal plate which is surface area A, and the force F which added to horizontal plate is F=PA. Since the density of water $6.25lb/ft^3$, the pressure of every direction in the depth of water d changes according to the depth d, i.e. P= 62.5d. The shape of dam is rectangle, height is 50ft, and width is 100ft. When the dam is full of water, approximate the force of the water which exerted on this dam.

The aim of this class is to approximate the force which based on the structure of Riemann sum. Students took long time to know what pressure P and force F are in formula F=PA. They didn't know the need of Riemann sum or approximation and so we conduct dividing this class into three processes.

(1) the process of multiplication: Since students are not familiar with the concept of physics, they have big difficulty in this process. Especially

they have difficulty in finding area A which needed for F=PA and so they cannot find the proper multiplication. Multiplication of two terms which are length and height is necessary in order to find area. The idea of dividing of the area of dam into small horizontal pieces is also necessary. If they decide how the piece is formed, then they can form $f(x_i)$ and $\Delta x_i$.

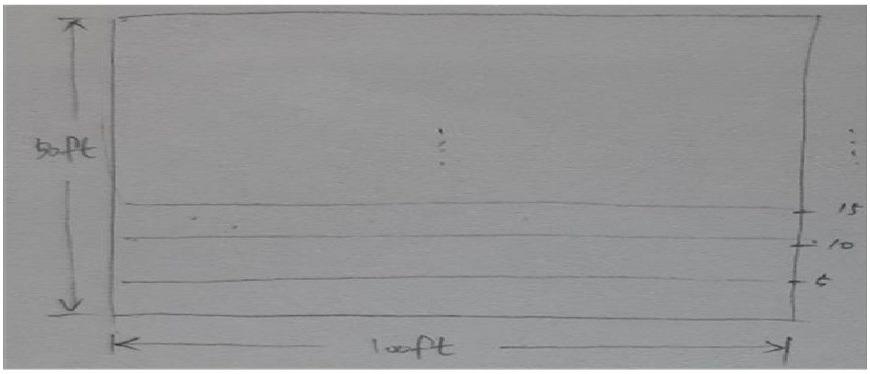



FIGURE 3. Dam which drawn backwards

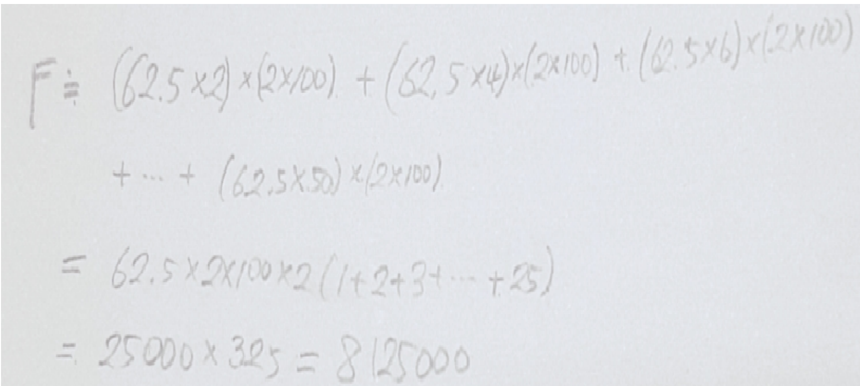



FIGURE 4. Force which applied to dam when the interval of depth is 2

A few students thought area A as total area of dam 5000 which multiplied maximum height 50ft by width 100ft. Since pressure is not constant and increases according to height, it is not a simple problem but they don't recognize. They understood do not as verifying quantity but only as depth of dam to the bottom. They didn't understand this is not a necessary area for F=PA and the need to find approximation by partitioning.

Other student used area which increased as depth changes for area of each strip. He used 100ft * depth as area of each strip. From $2ft$ to $4ft$ depth, used area in $4ft$ depth($100ft$width*$4ft$depth=$400ft^2$). From $4ft$ to $6ft$ depth, used area in $6ft$ depth($100ft$width*$6ft$depth=$600ft^2$).

Since in this case the area increases as the depth of water becomes deeper, this isn't the necessary area in F=PA, either.

Pressure is almost equal in each horizontal strip. Since pressure is almost constant in thin horizontal strip, we can get good approximation on force which applied to each strip if dam is partitioned into horizontal strip. Thus it is necessary to make all the strips disjoint and observe the set of strip in order to cover dam. Most of students knew they need to find approximation of force corresponding to each strip in order to solve problem. But because they were familiar with partition vertically, they had difficulty in thinking horizontal strip. Thus, we let them implement following activity.

We have them partition dam into pieces of long, thin rectangle and draw and so find area of this horizontal strip. And then let them draw strip of long, thin, same size of rectangle below it. They continue this process to the bottom of dam.

In this process they have to choose surface level of water as base line. However, like figure 3 there were a few students who think the bottom as base line backwardly. This is because to use existing base line is more familiar than to use the surface of water. Answer to find area when width of depth of water is 2 is presented in figure 4.

(2)The process of sum: Since pressure verifies according to depth, students need to find area not by the formula of force but by Riemann sum or definite integral. And they have to estimate with based on Riemann sum. They have to form sum not only in each strip(sub interval) but also in dam(entire interval), which corresponds to Riemann sum. They partition dam into horizontal strip and find force which exerted to each strip and thus find force which exerted to dam by using discrete sum of these.

When they find the pressure which applied to each strip, if they use the pressure acting on the bottom of each strip the pressure increases as the depth increases, which corresponds to upper sum of pressure.

In depth 2ft-4ft, pressure(upper sum) is 62.5*4, the area of strip is 100*2=200, force becomes 62.5*4*100*2. In depth 4ft-6ft, pressure(upper sum) is 62.5*6, the area of strip is 100*2=200, force becomes 62.5*6*100*2. Figure 4 is the upper sum of force to find by using upper sum of pressure.

(3)The process of limit and definite integral: To approximate more accurately and fastly, many horizontal strips need. To partition into small pieces makes a proper approximation. They can find pressure accurately by minifying the distance of horizontal strip and making a thin horizontal strip. Thus they can find approximation of force fastly and accurately.

2.B Analysis of Riemann sum and definite integral after experimental class

Dam problem and No.8 question are to approximate the force which based on the structure of Riemann sum. To develop understanding on the structure of Riemann sum, we let students solve No.8 after doing experimental class.

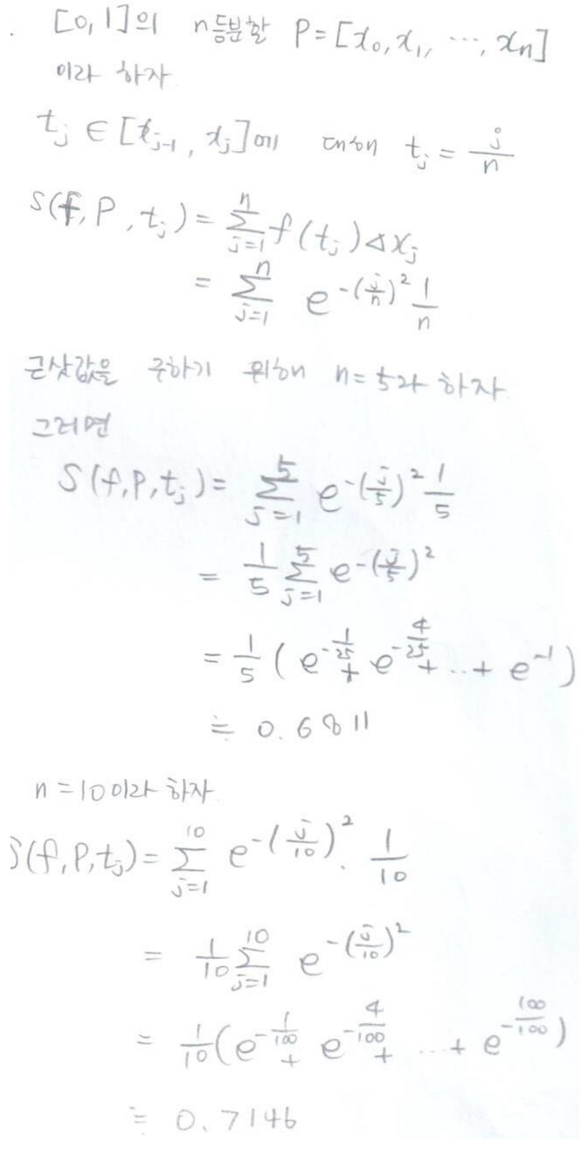
Students tried to approach No.8 question as approximation of Riemann sum after experimental class of dam problem. To find sum of area of rectangle which surrounded by the curve and x-axis in an interval for given curve on coordinate plane, they implemented the processes of product, sum, and limit.

First, they partition the interval into 5, found $\Delta x_i = 1/5$, $f(x_i)$, and computed the area of rectangle, $f(x_i)\Delta x_i$. Adding up the area of rectangles and doing the process of finding the total area, they calculated by adding discrete values of function. The function value at 1/5, 2/5, 3/5, 4/5, 1 were found using calculator. They solved upper sum, lower sum, and limit of error. Then they tried to get more accurate approximation by using more subinterval.

They thought real value existing between lower sum and upper sum and how approximation was approached the definite integral. They said as divide the interval to integrate into smaller, they saw the value of integral was closer to definite integral.

They were surprised at finding definite integral of function hard to solve realistically. Several students' responses are as follows. 4 persons among them got grade A, 1 person B in real analysis.

-Estimate the problem which is hard to solve by formula as Riemann sum so I seem to know what Riemann sum means. Having this process without just computing thoughtlessly is useful.



[0,1]의 n등분할 $P=[x_0, x_1, \cdots, x_n]$
이라 하자.

$t_j \in [x_{j-1}, x_j]$에 대하여 $t_j = \frac{j}{n}$

$$S(f, P, t_j) = \sum_{j=1}^{n} f(t_j)\Delta x_j = \sum_{j=1}^{n} e^{-(\frac{j}{n})^2}\frac{1}{n}$$

근삿값을 구하기 위해 $n=5$라 하자.
그러면

$$S(f,P,t_j) = \sum_{j=1}^{5} e^{-(\frac{j}{5})^2}\frac{1}{5} = \frac{1}{5}\sum_{j=1}^{5} e^{-(\frac{j}{5})^2} = \frac{1}{5}\left(e^{-\frac{1}{25}} + e^{-\frac{4}{25}} + \cdots + e^{-1}\right) \fallingdotseq 0.6811$$

$n=10$이라 하자.

$$S(f,P,t_j) = \sum_{j=1}^{10} e^{-(\frac{j}{10})^2}\frac{1}{10} = \frac{1}{10}\sum_{j=1}^{10} e^{-(\frac{j}{10})^2} = \frac{1}{10}\left(e^{-\frac{1}{100}} + e^{-\frac{4}{100}} + \cdots + e^{-\frac{100}{100}}\right) \fallingdotseq 0.7146$$

FIGURE 5. Riemann sum when n=5, n=10

-I became to know principle and so felt wonderful and useful.

-I can see the approximation is close to definite integral in the process of solving upper sum and lower sum − the smaller the distance of interval, the more accurate the approximation is.

-After I divided the interval directly and computed, I compared the result. As I divided the interval more, the approximation became close to the definite integral. I felt Riemann was great and point of view to look at definite integral changed.

In addition, we can analyze their process as follows:

-When they are hard to find the definite integral by area surrounded by curve, they became to find the definite integral using Riemann sum.

-They were very poor at the process of product in dam problem. But they could decide the process of proper product in No.8 question since they had no difficulty in understanding of concept of physics.

-Though they spend much time in computing sum, they have no cognitive difficulty in grasping.

-They tend to feel difficult in using Riemann sum and relating it to definite integral.

-They didn't think to increase the number of subinterval would increase sum infinitely.

-They aren't asked to find the accurate value of the limit but are asked to approach the limit, calculate upper sum, lower sum, and then get approximation.

This experimental class was focused to approximation to approach to the definite integral, not to get the exact value of limit. Through this process, students became basic understanding to get the definite integral after approximating Riemann sum, and they experienced complicated calculation directly in its process. This says Riemann sum using area can be the cause of conceptualization letting know the meaning of definite integral.

## 4. Conclusion

To perceive mathematical concept as not only efficiency of mathematical knowledge, technique, but also picture, relation is the essential element to consist conceptual understanding of mathematical thinking [11]. When students solve the problem of integral, they prefer solving by using formula which is procedural knowledge to solving by using geometric method. Though geometric method is more effective, students don't know the object which related to problem because of the shortage of flexibility and so can't connect properly. Thus they don't seem to make sure to make correct image of mathematics problem. When they solve integral problems, they tend to concentrate only technique and be unable to relate concept as graph and relation.

When integrand is explicit, students implemented fundamental theorem of calculus successfully. They solved problems which based on area, too. However, they had some difficulties in problems which curve passes x-axis or to understand the relation between the definite integral and area under curve. Moreover, they had difficulties in interpreting the problem to compute area and the define integral in wide context. This represents that they can solve integral problems by using procedural knowledge but basic understanding of integration is limited.

Many students can present only one expression of context and cannot present various expressions. They have some difficulties in connecting other expressions of definite integral. Since perfect understanding of context require proper connection between procedural knowledge and conceptual knowledge, it is necessary to develop ability which related various expression of definition of concept.

Riemann sum is very useful in applied field. Especially when we fail to find anti-derivative, area under curve, we have to find approximation of area by approximating the domain into rectangles and using sum of areas of rectangles. If we increase the number of strips, we can get more accurate approximation. Since conceptualization of Riemann sum can provide the meaning of area under curve, it is helpful in understanding definite integral. However, most of students can't explain how the integral acts with using Riemann sum in the case of function which doesn't have anti-derivative which represented by elementary function. This says that students don't understand the structure of Riemann sum deeply in the realistic applied problem which cannot be integrated by fundamental theorem of calculus.

Thus after we develop understanding of the structure of Riemann sum through experimental class of dam problem, we let them solve No.8. Only after then they can connect definite integral with Riemann sum. Students knew when there is a view which look at definite integral into structure of Riemann sum, definite integral become a powerful tool in settling applied problem which the concept of integral is included. To apply integration rightly, it is necessary to connect between ability to understand approximation of Riemann sum and ability to construct definite integral. So we can resolve imbalance between conceptualization of definite integral to claim in calculus and the most useful definite integral in applied field [11].

An experimental class of structure of Riemann sum became chance to understand the need to find approximation, observe historic background of definite integral, and experience difficulty of complicated calculation directly. It became the chance to realize value which fundamental theorem of calculus has and served as an opportunity to understand conceptual meaning of definite integral.

Though this study doesn't have enough sample and there is a limitation to apply the result generally, we supplemented survey by various and deep question and individual's interview. When you teach Riemann sum and the concept of definite integral, you remember students have difficulty in finding product of two terms in applied problem and have to give chance to allow act to use these terms effectively in definite integral. When such effort is reflected in improving teaching of integration, we expect this study will contribute to improve students' conceptual understanding and achievement of definite integral.

## Appendix

THE FIRST SURVEY

1. When the curve $\frac{dy}{dx} = \frac{6}{(2x-3)^2}$ passes through point(3,5), find the coordinate of x-intercept of curve.
2. Find the area surrounded by $y = x(x-4)$, $x$-axis, $x = 0, x = 5$.
3. (1) When $f(x) \geq 0$, interpret $\int_a^b f(x)dx$ geometrically.
   (2) When $f(x)$ assures positive and negative value, interpret $\int_a^b f(x)dx$.
4. (1) Find Riemann sum of following function by choosing 4 subinterval and sample point as right-end point $f(x) = x^2 - x (0 \leq x \leq 2)$.
   (2) Calculate following integral value by using definition of definite integral (using right-end point). $\int_0^2 (x^2 - x)dx$.
   (3) Check the answer of (2) by using fundamental theorem of calculus.

THE SECOND SURVEY

1. Evaluate the following.

   (1) $\displaystyle\int_2^3 \frac{x+3}{x-1}dx$ (2) $\displaystyle\int_0^6 \frac{1}{(x-4)^{2/3}}dx$

(3) $\int_{\sqrt{2}}^{2\sqrt{2}} (\sqrt{6} - \sqrt{8 - x^2})dx$  (4) $\int_0^2 (1 - |x - 1|)dx$

2. When function $f(x) = x - |x|$ is given, find the area of parts surrounded by $f(x)$, $x$-axis, $x = 0, x = 3$.
3. When function is as follows $f(x) = \sin x$, on $[0, 2\pi]$, find the area of parts surrounded by $f(x)$, $x$-axis, $x = 0, x = 2\pi$.
4. When function is as follows, find $\int_0^1 f(x)dx$.

$$f(x) = \begin{cases} 2x, & \text{if } x \leq 1/2 \\ 2x - 2, & \text{if } x > 1/2 \end{cases}$$

5. The graph of $f$ is given. When $\int_{-2}^{5} f(x)dx = \frac{39}{8}$, Find $\alpha$.

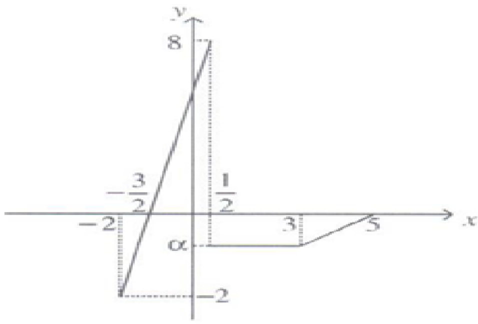



FIGURE 6. Graph of No.5

6. The graph of derivative of $f$ is given. When area of A, B, C domain is 20, 8, 5 respectively and $f(0) = -5$, find $f(6)$.

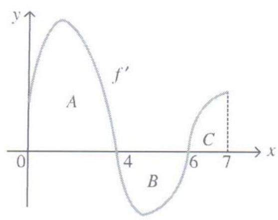



FIGURE 7. Graph of No.6

7. Graph is for $y + 2x = 8$ and $y = x^2 - 6x + 8$. Calculate ratio of A area for B area.

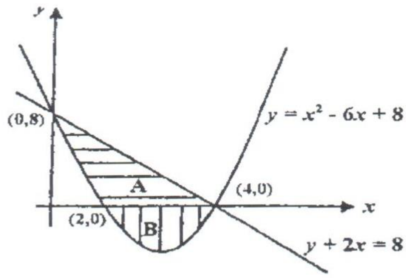



FIGURE 8. Graph of No.7

8. Integrate $\int_0^1 e^{-x^2} dx$.

## References


[1] M. Artigue, *Analysis in Advanced Mathematical Thinking*, D. Tall, ed., Kluwer, Boston, 167–198, 1991.

[2] D. M. Bressoud, *Historical reflections on Teaching the fundamental theorem of integral calculus*, The American Mathematical Monthly **118** (2) (2011), 99–115.

[3] Jeongsun Byun, *Understanding on concept of definite integral through measuration by division*, graduate school of education at Ehwa woman's University, 1992.

[4] A. Cauchy, *Resume des Logons donne"es 1'Ecolo royale Polyteehniquo sur le Galcul infinitesimal*, publiees sous la direction scientifique de l'Academie des sciences et sous les auspices de M. le ministre de 1'Instruction publique, Oeuvres completes d'Augustin Cauchy, 1899.

[5] K.K. Chapell and K. Killpatrick, *Effects of concept-based instruction on Students' conceptual understanding and procedural knowledge of calculus*, Primus **13** (1) (2003), 17–37.

[6] Jeonghyun Choi, *Characteristic of understanding on definite integral symbol and didactical strategy*, Korean journal of mathematical history, **24** (3) (2011), 77–94.

[7] K. Devlin, *MATHEMATICS: The Science of Patterns*, Scientific American Library, 1994.

[8] J. Ferrini-Mundy and M. Guardard, *Preparation of pitfall in the study of college calculus* , J. Res. Math. Educ. **23** (1) (1992), 56–71.

[9] J. Grabiner, *The Origins of Cauchy's Rigorous Calculus*, Dover Publications, 2005.

[10] J. Hiebert and P. Lefevre, *Conceptual and procedural knowledge in mathematics: An introductory analysis, in Conceptual and Procedural Knowledge: The Case of Mathematics*, J. Hiebert, ed., Lawrence Erlbaum, Hilllsdale (NJ) (1986), 1–27.

[11] S. Jones, *Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts*, Journal of Mathematical Behavior **38** (2015), 9–28.

[12] Youn Joon Joung and Kyeong Hwa Lee, *A study on the Relationship between indefinite integral and definite integral*, Journal of Korea Society of Educational Studies in Mathematics, School Mathematics **11** (2) (2009), 301–316.

[13] Seah Eng Kiat, *Analysis of Students' Difficulties in Solving Integration Problem*, The Mathematics Educator **9** (1) (2005), 39-59.

[14] Ohsung Kwon, *Inclination understanding on integration chapter*, master thesis Korea University, 2011.

[15] N. Mahir, *Conceptual and procedural performance of undergraduate students in integration.*,International Journal of Mathematical Education in Science and Technology **40** (2) (2009), 201–211.

[16] F. A. Medvedev, *Floer homology of Lagrangian foliation and noncommutative mirror symmetry*, Preprint 98-08, Kyoto Univ, 1998.

[17] A. Orton, *Students' understanding of integration*, Educational Studies in Mathematics **14** (1) (1983), 1–18.

[18] A.B. Oaks, *The effects of the interaction of conception of mathematics and affective constructs on college students in remedial mathematics*, Ph.D.diss., University of Rochester, 1987, Dissertation Abstracts International, 49, 54A.

[19] S. Rasslan and D. Tall, *Definitions and images for the definite integral concept*, Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education. Norwich, UK, 2002.

[20] Bomi Sin, *High school students' understanding on concept of definite integral*, Journal of Korea Society of Educational Studies in Mathematics, School Mathematics **11** (1) (2009), 93–100.

[21] J. Stewart, *Calculus*, $6^{th}$ ed, Belmont: Brooks Cole, 2008.

[22] Joseph F. Wagner, *Students' Obstacles to Using Riemann Sum Interpretations of the Definite Integral*, Springer International Publishing, 2017.

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