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ALMOST PERIODIC SOLUTIONS OF PERIODIC SECOND ORDER LINEAR EVOLUTION EQUATIONS

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Abstract. The paper is concerned with periodic linear evolution equations of the form $x''(t) = A(t)x(t) + f(t)$, where $A(t)$ is a family of (unbounded) linear operators in a Banach space X, strongly and periodically depending on t, f is an almost (or asymptotic) almost periodic function. We study conditions for this equation to have almost periodic solutions on $\mathbb{R}$ as well as to have asymptotic almost periodic solutions on $\mathbb{R}^+$. We convert the second order equation under consideration into a first order equation to use the spectral theory of functions as well as recent methods of study. We obtain new conditions that are stated in terms of the spectrum of the monodromy operator associated with the first order equation and the frequencies of the forcing term f.

1. Introduction

In this paper we first consider the existence and uniqueness of almost periodic solutions with the same structure of spectrum as f to periodic second order evolution equations of form

$$x''(t) = A(t)x(t) + f(t), \quad x \in \mathbb{X}, t \in \mathbb{R}, \tag{1.1}$$

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where the family of (unbounded) linear operators $A(t)$ depend periodically on t and generate an evolution operator, and f is an almost (or asymptotic) periodic function. Then, we consider the asymptotic almost periodic solutions of the equation on the half line $\mathbb{R}^+$ when so is f. There are many works devoted to the asymptotic behavior, and in particular, the almost periodicity of solutions of first order linear evolution equations. We refer the reader to the monographs [2, 8] and [6] for more complete accounts of the study. In the finite dimensional case, second order ordinary differential equations can be converted to a first order equations easily. However, in the infinite dimensional case this procedure looks more complicated. For first order periodic evolution equations, the asymptotic behavior of mild solutions has been studied in many works, for instance, [1, 6, 12, 15, 19]. In these works, the spectral theory of functions play an important role in understanding the behavior of solutions. One of the most difficult steps in using this theory is to estimate the spectrum of a bounded mild solution as the equation under consideration is not autonomous. We note that the concept of circular spectrum and the associated transform introduced in [12] appear to be a useful tool to capture the Beurling spectrum of the solutions, and thus, its asymptotic behavior. To our best knowledge, there is no similar study for second order equations (1.1).

In this paper we will go further in the direction of [12] to use the circular spectrum and its associated transform to estimate the Beurling spectrum of mild solutions to the second order equation Eq. (1.1). We assume that the homogeneous equation associated with Eq.(1.1) generates an evolution operator (see e.g. [4, 5]) and is well-posed. This assumption allows us to use the ideas of [12] to estimate the spectrum of a bounded mild solution of Eq.(1.1) in Lemma 3.1. This is the first step to apply the decomposition procedure method in [16] to prove Theorem 3.2, an analog of the Massera's Theorem on the existence of an almost periodic mild solution if the non-homogeneous equation has a bounded mild solution on the real line. We then prove Theorem 3.6 on the asymptotic almost periodicity of bounded mild solutions based. We give an example of application at the end of the paper to a hyperbolic partial differential equation that is well posed and generates an evolution operator. The obtained results in the present paper are new. They complement the known results for first order evolution equations (see e.g. [2, 6]).

2. Preliminaries

2.1. Notation. Throughout the paper we will use the following notations: $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ stand for the sets of natural, integer, real, complex numbers, respectively. Γ denotes the unit circle in the complex plane $\mathbb{C}$. For any complex number z the notation $\Re z$ stands for its real part. $\mathbb{X}$ will denote a given complex Banach space. Given two Banach spaces $\mathbb{X}, \mathbb{Y}$ by $L(\mathbb{X}, \mathbb{Y})$ we will denote the space of all bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$. As usual, $\sigma(T), \rho(T), R(\lambda, T)$ are the notations of the spectrum, resolvent set and resolvent of the operator T. The notations $BC(\mathbb{R}, \mathbb{X}), BUC(\mathbb{R}, \mathbb{X}), AP(\mathbb{X})$ will stand for the spaces of all $\mathbb{X}$-valued bounded continuous, bounded uniformly continuous functions on $\mathbb{R}$ and its subspace of almost periodic (in Bohr's sense) functions, respectively. For functions on the half line we will use $BUC(\mathbb{R}^+, \mathbb{X}), AP(\mathbb{R}^+, \mathbb{X}), AAP(\mathbb{R}^+, \mathbb{X})$ to denotes the spaces of all bounded and uniformly continuous functions, of all almost periodic functions, and of all asymptotic almost periodic functions on $\mathbb{R}^+$, respectively.

2.2. Circular Spectrum of Functions on the line $\mathbb{R}$. Below we will introduce a transform of a function $g \in BUC(\mathbb{R}, \mathbb{X})$ on the real line that leads to a concept of spectrum of a function. This spectrum coincides with the set of $\overline{e^{isp(g)}}$ (overlining means closure in the complex plane topology) if in addition g is uniformly continuous, where $sp(g)$ denotes the Beurling spectrum of g. All results mentioned below on the circular spectrum of a function could be found in [12].

Let $g \in BUC(\mathbb{R}, \mathbb{X})$. Consider the complex function $\mathcal{S}g(\lambda)$ in $\lambda \in \mathbb{C}\backslash\Gamma$ defined as

$$\mathcal{S}g(\lambda) := R(\lambda, S)g, \quad \lambda \in \mathbb{C}\backslash\Gamma. \tag{2.1}$$

Since S is a translation, this transform is an analytic function in $\lambda \in \mathbb{C}\backslash\Gamma$.

Definition 2.1. The *circular spectrum* of $g \in BUC(\mathbb{R}, \mathbb{X})$ is defined to be the set of all $\xi_0 \in \Gamma$ such that $\mathcal{S}g(\lambda)$ has no analytic extension into any neighborhood of ξ_0 in the complex plane. This spectrum of g is denoted by $\sigma(g)$ and will be called for short *the spectrum of g* if this does not cause any confusion. We will denote by $\rho(g)$ the set $\Gamma\backslash\sigma(g)$.

Proposition 2.2. *Let $\{g_n\}_{n=1}^{\infty} \subset BUC(\mathbb{R}, \mathbb{X})$ such that $g_n \to g \in BUC(\mathbb{R}, \mathbb{X})$ in the uniform topology on $\mathbb{R}$, and let Λ be a closed subset of the unit circle. Then the following assertions hold:*

i) $\sigma(g)$ *is closed.*
ii) If $\sigma(g_n) \subset \Lambda$ *for all* $n \in \mathbb{N}$, *then* $\sigma(g) \subset \Lambda$.
iii) $\sigma(\mathcal{A}g) \subset \sigma(g)$ *for every bounded linear operator* $\mathcal{A}$ *acting in* $BUC(\mathbb{R}, \mathbb{X})$ *that commutes with* S.
iv) If $\sigma(g) = \emptyset$, *then* $g = 0$.

Proof. For i), ii) and iv) the proofs are given in [12]. For iii) the proof is obvious from the definition of the circular spectrum. □

Below we will recall the concept of Beurling spectrum of a function. We denote by F the Fourier transform, i.e.

$$(Ff)(s) := \int_{-\infty}^{+\infty} e^{-ist} f(t)dt \tag{2.2}$$

$(s \in \mathbb{R}, f \in L^1(\mathbb{R}))$. Then the *Beurling spectrum* of $u \in BUC(\mathbb{R}, \mathbb{X})$ is defined to be the following set

$$\begin{aligned} sp(u) \ := \ & \{\xi \in \mathbb{R} : \forall \epsilon > 0 \ \ \exists f \in L^1(\mathbb{R}), \\ & \qquad supp Ff \subset (\xi - \epsilon, \xi + \epsilon), f * u \neq 0\} \end{aligned}$$

where

$$f * u(s) := \int_{-\infty}^{+\infty} f(s-t)u(t)dt.$$

The following result is a consequence of the Weak Spectral Mapping Theorem that relates the circular spectrum and Beurling spectrum of a uniformly continuous function.

Corollary 2.3. *Let $g \in BUC(\mathbb{R}, \mathbb{X})$. Then*

$$\sigma(g) = \overline{e^{isp(g)}}. \tag{2.3}$$

Next, we recall some concepts and results in [16]. Let us consider the subspace $\mathcal{N} \subset BUC(\mathbb{R}, \mathbb{X})$ (or $AP(\mathbb{X})$, respectively) consisting of all functions $v \in BUC(\mathbb{R}, \mathbb{X})$ (or $AP(\mathbb{X})$, respectively) such that

$$\sigma(v) \subset S_1 \cup S_2 \ , \tag{2.4}$$

where S_1, S_2 are disjoint closed subsets of the unit circle Γ.

LEMMA 2.4. *Under the above notations and assumptions the function space $\mathcal{N}$ can be split into a direct sum $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ such that $v \in \mathcal{N}_i$ if and only if $\sigma(v) \subset S_i$ for $i = 1, 2$. Moreover, any bounded linear operator in $BUC(\mathbb{R}, \mathbb{X})$ (or $AP(\mathbb{X})$, respectively), that commutes with the translation S, leaves invariant $\mathcal{N}$ as well as $\mathcal{N}_j$, $j = 1, 2$.*

2.3. Circular spectrum of a function on the half line. Consider the quotient space $BUC(\mathbb{R}^+, \mathbb{X})/AAP(\mathbb{R}^+, \mathbb{X})$ and the induced semigroup of translations $(\bar{S}(t))_{t\geq 0}$. As is well known this semigroup is extendable to a group of isometries (see [2, 9]) we can define the circular "quotient spectrum" of a function $x(\cdot) \in BUC(\mathbb{R}^+, \mathbb{X})$ as follows:

DEFINITION 2.5. The *circular spectrum* of $g \in BUC(\mathbb{R}^+, \mathbb{X})$ with respect to $AAP(\mathbb{R}^+, \mathbb{X})$ is defined to be the set of all $\xi_0 \in \Gamma$ such that $R(\lambda, \bar{S}\bar{g})$ has no analytic extension into any neighborhood of ξ_0 in the complex plane. This spectrum of g is denoted by $\sigma_{AAP}(g)$.

Before proceeding we introduce a new notation: let $0 \neq z \in \mathbb{C}$ such that $z = re^{i\varphi}$ with reals $r = |z|, \varphi$, and let $F(z)$ be any complex function. Then, (with s larger than r) we define

$$\lim_{\lambda \downarrow z} F(\lambda) := \lim_{s \downarrow r} F(se^{i\varphi}).$$

The proof of the following can be found in [9]

THEOREM 2.6. *Let $g \in BUC(\mathbb{R}^+, \mathbb{X})$ such that the set $\sigma_{AAP}(g)$ is countable, and let the following condition hold for each $\xi_0 \in \sigma_{AAP}(g)$*

$$\lim_{\lambda \downarrow \xi_0} (\lambda - \xi_0) R(\lambda, \bar{S})\bar{g} = 0. \tag{2.5}$$

Then, $g \in AAP(\mathbb{R}^+, \mathbb{X})$.

2.4. Almost periodic functions. A subset $E \subset \mathbb{R}$ is said to be *relatively dense* if there exists a number $l > 0$ (*inclusion length*) such that every interval $[a, a + l]$ contains at least one point of E. Let f be a continuous function on $\mathbb{R}$ taking values in a complex Banach space $\mathbb{X}$. f is said to be *almost periodic in the sense of Bohr* if to every $\epsilon > 0$ there corresponds a relatively dense set $T(\epsilon, f)$ (*of ϵ-periods*) such that

$$\sup_{t \in \mathbb{R}} \|f(t + \tau) - f(t)\| \leq \epsilon, \ \forall \tau \in T(\epsilon, f).$$

If f is almost periodic function, then (approximation theorem [8, Chap. 2]) it can be approximated uniformly on $\mathbb{R}$ by a sequence of trigonometric polynomials, i.e., a sequence of functions in $t \in \mathbb{R}$ of the form

$$P_n(t) := \sum_{k=1}^{N(n)} a_{n,k} e^{i\lambda_{n,k} t}, \quad n = 1, 2, ...; \lambda_{n,k} \in \mathbb{R}, a_{n,k} \in \mathbb{X}, t \in \mathbb{R}. \tag{2.6}$$

Of course, every function which can be approximated by a sequence of trigonometric polynomials is almost periodic. Specifically, the exponents of the trigonometric polynomials (i.e., the reals $\lambda_{n,k}$ in (2.6)) can be chosen from the set of all reals λ (*Fourier exponents*) such that the following integrals (*Fourier coefficients*)

$$a(\lambda, f) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\lambda t} dt$$

are different from 0. As is known, there are at most countably such reals λ, the set of which will be denoted by $\sigma_b(f)$ and called *Bohr spectrum* of f. Throughout the paper we will use the relation $sp(f) = \overline{\sigma_b(f)}$.

If $g \in BUC(\mathbb{R}, \mathbb{X})$ with countable $\sigma(g)$, then its Beurling spectrum $sp(g)$ is also countable by Corollary 2.3. Therefore, if $\mathbb{X}$ does not contain any space isomorphic to c_0 (the space of all numerical sequences converging to zero), the function g is almost periodic (see e.g. [8]). If $\mathbb{X}$ is convex it does not contain c_0.

2.5. Evolution processes associated with a homogeneous linear evolution equation of second order. Assume that $(A(t))_{t\in\mathbb{R}}$ is a family of (generally, unbounded) linear operators in a Banach space $\mathbb{X}$ with the same domain D such that the map $t \mapsto A(t)x$ is continuous for each $x \in D$.

DEFINITION 2.7. Let $(S(t,s))_{t\geq s}$ be a two-parameter family of bounded operators in a Banach space $\mathbb{X}$. Then, it is called an *evolution operator* associated with the second order equation

$$u''(t) = A(t)u(t), \; t \in \mathbb{R}, \tag{2.7}$$

if

(D1) The map $(t,s) \mapsto S(t,s)x$ is continuously differentiable for every fixed $x \in \mathbb{X}$, and
  (a) $S(t,t) = 0$ for all $t \in \mathbb{R}$,
  (b) For all $t, s \in \mathbb{R}$, if $x \in \mathbb{X}$ and each $x \in D$, then

$$\frac{\partial S(t,s)x}{\partial t}|_{t=s} = x, \; \frac{\partial S(t,s)x}{\partial s}|_{t=s} = -x$$

(D2) For all $t, s \in \mathbb{R}$, if $x \in D$, then $S(t,s)x \in D$, the map $(t,s) \mapsto S(t,s)x$ is of class C^2, and

(a)
$$\frac{\partial^2 S(t,s)x}{\partial t^2}x = A(t)S(t,s)x;$$

(b)
$$\frac{\partial^2 S(t,s)x}{\partial s^2}x = S(t,s)A(s)x;$$

(c)
$$\frac{\partial^2 S(t,s)x}{\partial s \partial t}x|_{t=s} = 0.$$

(D3) For all $t, s \in \mathbb{R}$, if $x \in D$, then $\frac{\partial S(t,s)}{\partial t}x \in D$. Further, there exists $\frac{\partial^3}{\partial t^2 \partial s}$ and $\frac{\partial^3}{\partial s^2 \partial t}$ such that

(a)
$$\frac{\partial^3}{\partial t^2 \partial s}S(t,s)x = A(t)\frac{\partial}{\partial s}S(t,s)x;$$

and the mapping $(t,s) \mapsto A(t)\frac{\partial}{\partial s}S(t,s)x$ is continuous.

(b)
$$\frac{\partial^3}{\partial s^2 \partial t}S(t,s)x = A(t)\frac{\partial}{\partial t}S(t,s)x,$$

Definition 2.8. Eq. (2.7) is said to be *well posed* if

i) There exists an evolution operator $(S(t,s)_{t \geq s})$ associated with it;
ii) There exists a dense subspace $\tilde{D}$ of $\mathbb{X}$ invariant under $(S(t,s)_{t \geq s})$ such that for every $y, z \in \tilde{D}$, $s \in \mathbb{R}$, the function $u(t) := S(t,s)x$ is the unique solution on the interval $[s, \infty)$ of the equation with initial condition $u(s) = y, u'(s) = z$.

Let $(S(t,s)_{t \geq s})$ be an evolution operator associated with Eq.(2.7). A differentiable function u on $\mathbb{R}$ is said to be a *mild solution* of Eq.(1.1) if for all $t \geq s$

$$u(t) = C(t,s)u(s) + S(t,s)u'(s) + \int_s^t S(t,\xi)f(\xi)d\xi. \tag{2.8}$$

Let us consider a first order evolution equation

$$u'(t) = B(t)u(t), \ t \in \mathbb{R}, \tag{2.9}$$

where $A(t)$ is an (unbounded) linear operator with $D := D(A(t))$ independent of t. We say that a two parameter family of bounded linear operators is an evolution process if it satisfies

i) $U(t,t) = I$, for all $t \in \mathbb{R}$;
ii) $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r$;
iii) $(t,s) \mapsto U(t,s)x$ is continuous for every fixed $x \in \mathbb{X}$;
iv) There exists a dense subspace $\tilde{D} \subset D$ such that for each $x \in \tilde{D}$ the function $x(t) := U(t,s)x$ is the unique solution of Eq. (2.9) on $[s,\infty)$.

LEMMA 2.9. *Let Eq.(2.7) be well posed. Assume further that $(S(t,s)_{t\geq s})$ is the evolution operator associated with Eq.(2.7). Then, the family of operators*

$$U(t,s) := \begin{bmatrix} C(t,s) & S(t,s) \\ \frac{\partial C(t,s)}{\partial t} & \frac{\partial S(t,s)}{\partial t} \end{bmatrix} \tag{2.10}$$

is an evolution process associated with

$$\begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ A(t) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}. \tag{2.11}$$

Proof. First,

$$U(t,t) := \begin{bmatrix} C(t,s)|_{s=t} & S(t,s)|_{s=t} \\ \frac{\partial C(t,s)}{\partial t}|_{s=t} & \frac{\partial S(t,s)}{\partial t}|_{s=t} \end{bmatrix} = \begin{bmatrix} I_{\mathbb{X}} & 0 \\ 0 & I_{\mathbb{X}} \end{bmatrix} = I_{\mathbb{X}\times\mathbb{X}}.$$

Second, the map $(t,s) \mapsto U(t,s)x$ is continuous because the map $(t,s) \mapsto S(t,s)x$ is for every fixed $x \in \mathbb{X}$. As the Cauchy Problem (CP) for Eq.(2.7) has a unique solution

$$x(t) = C(t,r)y + S(t,s)z$$

with initial condition $x(r) = y \in \tilde{D}, x'(r) = z \in \tilde{D}$, then the solution $x(t)$ is the unique solution of (CP) with initial

$$\begin{aligned} x(s) &= C(s,r)x(r) + S(s,r)x'(r), \\ x'(s) &= \frac{\partial C(t,s)}{\partial t}x(r) + \frac{\partial S(t,s)}{\partial t}x'(r). \end{aligned}$$

This means, since $\tilde{D}$ is dense everywhere in $\mathbb{X}$ for all $t \geq s \geq r$

$$U(t,s)U(s,r) = U(t,r)$$

Next, let us define the space $\tilde{D}_B := \tilde{D} \times \tilde{D}$. By the definition of the well-posedness of the Eq.(2.7), there exists a unique solution $x(\cdot)$ such that

$x(s) = y, x'(s) = z$ for any given $y, z \in \tilde{D}$. This yields that the function $u(t) := (x(t), x'(t))^T$ is a solution of Eq.(2.11). Note that the uniqueness of $u(\cdot)$ follows from that of $x(\cdot)$. The lemma's proof is completed. $\square$

Let $F(\cdot)$ be a continuous function on $\mathbb{R}$. A function $u(\cdot)$ on $\mathbb{R}$ is said to be a mild solution to

$$u'(t) = B(t)u(t) + F(t) \tag{2.12}$$

if for all $t \geq s$,

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)F(\xi)d\xi, \quad t \geq s. \tag{2.13}$$

LEMMA 2.10. *Let Eq.(2.7) be well posed with the associated evolution operator $(S(t,s))_{t\geq s}$. Then, if $x(\cdot)$ is a mild solution of Eq.(1.1) on $\mathbb{R}$, then $(x(\cdot), x'(\cdot))^T$ is a mild solution of the following equation*

$$\begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ A(t) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}. \tag{2.14}$$

Conversely, if $(x(\cdot), y(\cdot))^T$ is a mild solution of Eq. (2.14), then $x(\cdot)$ is a mild solution of Eq.(1.1).

Proof. Since

$$x(t) = C(t,s)x(s) + S(t,s)x'(s) + \int_s^t S(t,\xi)f(\xi)d\xi$$

we have

$$\begin{aligned} x'(t) &= \frac{\partial C(t,s)}{\partial t}x(s) + \frac{\partial S(t,s)}{\partial t}x(s) + S(t,t)f(t) + \int_s^t \frac{\partial S(t,\xi)}{\partial t}f(\xi)d\xi \\ &= \frac{\partial C(t,s)}{\partial t}x(s) + \frac{\partial S(t,s)}{\partial t}x(s) + \int_s^t \frac{\partial S(t,\xi)}{\partial t}f(\xi)d\xi. \end{aligned}$$

Therefore,

$$\begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} = \begin{bmatrix} C(t,s) & S(t,s) \\ \frac{\partial C(t,s)}{\partial t} & \frac{\partial S(t,s)}{\partial t} \end{bmatrix} \begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} + \int_s^t \begin{bmatrix} C(t,\xi) & S(t,\xi) \\ \frac{\partial C(t,\xi)}{\partial t} & \frac{\partial S(t,\xi)}{\partial t} \end{bmatrix} \begin{bmatrix} 0 \\ f(\xi) \end{bmatrix} d\xi. \tag{2.15}$$

That is, if we denote by $u(t) := (x(t), x'(t))^T$, and $F(t) := (0, f(t))^T$, then for all $t \geq s$

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)F(\xi)d\xi. \tag{2.16}$$

Conversely, let $(x(\cdot), y(\cdot))^T$ be a mild solution of Eq.(2.14). Then, by definition,

(2.17)

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} C(t,s) & S(t,s) \\ \frac{\partial C(t,s)}{\partial t} & \frac{\partial S(t,s)}{\partial t} \end{bmatrix} \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} + \int_s^t \begin{bmatrix} C(t,\xi) & S(t,\xi) \\ \frac{\partial C(t,\xi)}{\partial t} & \frac{\partial S(t,\xi)}{\partial t} \end{bmatrix} \begin{bmatrix} 0 \\ f(\xi) \end{bmatrix} d\xi.$$

Therefore, for all $t \geq s$,

$$x(t) = C(t,s)x(s) + S(t,s)y(s) + \int_s^t S(t,\xi)f(\xi)d\xi. \tag{2.18}$$

This yields that $x(\cdot)$ is differentiable, and $x'(\cdot)$ satisfies for all $t \geq s$

$$x'(t) = \frac{\partial C(t,s)}{\partial t}x(s) + \frac{\partial}{\partial t}S(t,s)y(s) + \int_s^t \frac{\partial}{\partial t}S(t,\xi)f(\xi)d\xi. \tag{2.19}$$

This shows that $x'(t) = y(t)$ for all t, and thus, $x(\cdot)$ is a mild solution of Eq.(1.1).

□

LEMMA 2.11. *Let Eq.(2.7) be well posed, and $A(t+1) = A(t)$ for all $t \in \mathbb{R}$. Then, for all $t \geq s$ the following holds true*

$$U(t+1, s+1) = U(t,s). \tag{2.20}$$

Proof. Suppose that $y, z \in \tilde{D}$. Set $v(t) := C(t,s)y + S(t,s)z$, and $w(t) := C(t+1,s+1)y + S(t+1,s+1)z$. Then, by definition of $S(t,s)$ and by the 1-periodicity of $A(\cdot)$,

$$\frac{d^2v(t)}{dt^2} = A(t+1)v(t) = A(t)v(t), t \geq s.$$

and $v(s) = y, v'(s) = z$. Note that $w(\cdot)$ also satisfies the equation and the same initial data. Due to the density of $\tilde{D}$ in $\mathbb{X}$ and the arbitrary nature of $y, z \in \tilde{D}$, this yields that $S(t,s) = S(t+1,s+1), C(t,s) = C(t+1,s+1)$, so $U(t+1,s+1) = U(t,s)$ for all $t \geq s$. □

Below we denote by $P(t) := U(t,t-1)$ for all $t \in \mathbb{R}$ and by $\mathcal{P}$ the operator of multiplication by $P(t)$, and $(U(t)_{t\in\mathbb{R}})$ the translation group in $BUC(\mathbb{R},\mathbb{X})$, with $S := S(1)$. By the periodicity of $(S(t,s)_{t\geq s})$, the following lemma is true (see [15]):

LEMMA 2.12. *Under the notation as above the following assertions hold:*

i) *$P(t+1) = P(t)$ for all t; characteristic multipliers are independent of time, i.e. the nonzero eigenvalues of $P(t)$ coincide with those of P,*

ii) *$\sigma(P(t))\backslash\{0\} = \sigma(P)\backslash\{0\}$, i.e., it is independent of t,*

iii) *If $\lambda \in \rho(P)$, then the resolvent $R(\lambda, P(t))$ is strongly continuous,*

iv) *If $\mathcal{P}$ denotes the operator of multiplication by $P(t)$ in any one of the function spaces $BUC(\mathbb{R}^+, \mathbb{X})$ or $AP(\mathbb{R}^+, \mathbb{X})$, then*

$$\sigma(\mathcal{P})\backslash\{0\} \subset \sigma(P)\backslash\{0\}. \tag{2.21}$$

3. Almost Periodic Mild Solutions

3.1. Mild solutions and an analog of the Massera's Theorem.

LEMMA 3.1. *Let $u \in BUC(\mathbb{R}, \mathbb{X})$ be a mild solution of the equation (1.1). Then,*

$$\sigma(u) \subset \sigma(f) \cup \sigma_\Gamma(P). \tag{3.1}$$

Proof. By Lemma 2.10 the function $t \mapsto (u(t), u'(t))^T$ is a mild solution of the first order equation

$$\begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ A(t) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}. \tag{3.2}$$

That means, for all $t \geq s$, $w(t) := (u(t), u'(t))^T$ satisfies

$$w(t) = U(t,s)w(s) + \int_s^t U(t,\xi)F(\xi)d\xi. \tag{3.3}$$

In particular, for all t

$$w(t) = U(t,t-1)w(t-1) + \int_{t-1}^t U(t,\xi)F(\xi)d\xi. \tag{3.4}$$

By [12, Lemma 4.1] (3.1) holds because $\sigma(F) = \sigma(f)$ and $\sigma(u) \subset \sigma(w)$. □

As a consequence of Lemma 3.1 we have the following as the main result of the paper. Before stating the theorem we recall the notation $\sigma_\Gamma(T)$ stands for the part of the spectrum of an operator T on the unit circle.

THEOREM 3.2. *Let Eq.(2.7) be well posed, and $A(t+1) = A(t)$ for all $t \in \mathbb{R}$. Further, assume that the following condition are satisfied* *i) Eq.(1.1) has a mild solution $u \in BUC(\mathbb{R}, \mathbb{X})$ (or in $AP(\mathbb{X})$, respectively)*

ii)

$$\sigma_\Gamma(P) \setminus \sigma(f) \text{ be closed.} \tag{3.5}$$

Then there exists a mild solution $x(\cdot)$ of Eq.(1.1) in $BUC(\mathbb{R}, \mathbb{X})$ (or $AP(\mathbb{X})$, respectively) such that

$$\sigma(\sigma(x(\cdot)) \subset \sigma(f), \tag{3.6}$$

that is unique if

$$\sigma_\Gamma(P) \cap \sigma(f) = \emptyset. \tag{3.7}$$

Proof. Set $v(t) := (u(t), v'(t))^T$. Note that $\sigma(u') \subset \sigma(u)$, so $\sigma(v) = \sigma(u)$. Let us denote by $S_1 := \sigma(f),\ S_2 := \sigma_\Gamma(P)$. Then by [16, Theorem 3.1], $\mathcal{M} = N_1 \oplus N_2$, where $\mathcal{M} := \{g \in BUC(\mathbb{R}, \mathbb{X} \times \mathbb{X}) | \sigma(g) \subset \sigma(u)$, $N_1 := \{g \in \mathcal{M} | \sigma(u) \subset S_1\},\ N_2 := \{g \in \mathcal{M} | \sigma(u) \subset S_2\}$. Moreover, the projections P_1, P_2 on these subspaces N_1, N_2 are defined by the Riesz projections of the restriction S to $\mathcal{M}$ corresponding to the spectral sets S_1, S_2. As a consequence, these projections commute with any bounded operator that commutes with the translation S. Further, note that for every fixed $h > 0$ the operators of multiplication by $U(t, t-h)$ (as a 1-periodic function of t) commutes with P_1, P_2 because they commute with the translation S. It is also noted that in the proof of [12, Lemma 5.3] the operator G mapping F to the function $t \mapsto \int_{t-h}^t U(t,\xi)F(\xi)(\xi)d\xi$ also commutes with S. Therefore, for every fixed $h > 0$, and all $t \in \mathbb{R}$

$$P_1 v(t) = U(t, t-h)P_1 v(t-h) + \int_{t-h}^t U(t,\xi)P_1 F(\xi)d\xi. \tag{3.8}$$

Since $P_1 F = F$ this yields $w := P_1 v$ is a mild solution of Eq.(2.14) with $\sigma(w) \subset \sigma(u)$. Let $w = (x(\cdot), y(\cdot))^T$. Then, $x(\cdot)$ is a mild solution of Eq.(1.1). Obviously $\sigma(x(\cdot)) \subset \sigma(w) \subset \sigma(u)$.

Next, let x_1, x_2 be two mild solutions of Eq.(1.1) with $\sigma(x_1) \subset \sigma(f)$ and $\sigma(x_1) \subset \sigma(f)$. Then the function $x(\cdot) := x_1(\cdot) - x_2(\cdot)$ is a mild solution of Eq.(1.1) with $f = 0$. Hence, $\sigma(x(\cdot)) \subset \sigma_\Gamma(P)$. Finally, this shows that $\sigma(x(\cdot)) = \emptyset$, or $x(\cdot) = 0$. □

COROLLARY 3.3. *Let $\mathbb{X}$ be a Banach space that does not contain c_0. Further let all assumptions of Theorem 3.2 be satisfied, and the spectrum $\sigma(f)$ is countable. Then, if there exists a bounded uniformly continuous* *mild solution to Eq. (1.1), there exists an almost periodic mild solution w to Eq. (1.1) such that $\sigma(w) \subset \sigma(f)$.*

Proof. By Theorem 3.2 there exists a uniformly continuous mild solution w to Eq. (1.1) such that $\sigma(w) \subset \sigma(f)$. Next, if $\sigma(w)$ is countable, then its Beurling spectrum $sp(f)$ is also countable due to the relation from the Weak Spectral Mapping Theorem (the overline below means closure in the complex plane's topology)

$$\overline{e^{i\cdot sp(f)}} = \sigma(f).$$

Since $\mathbb{X}$ does not contain c_0 this yields that w is an almost periodic function. $\square$

Remark 3.4. In his famous paper [10] Massera proved that for a periodic linear non-homogeneous ODE to have a periodic solution it is sufficient that it has bounded solution on the positive half line. There are several extensions and analogs of this theorem to different classes of evolution equations in [6]. Corollary 3.3 may be seen as an analog for a second order evolution equations for almost periodic solutions.

3.2. Asymptotical almost periodicity of mild solutions on the half line. Consider the quotient space $\mathbb{Y} := BUC(\mathbb{R}, \mathbb{X}^+)/AAP(\mathbb{R}^+, \mathbb{X})$. The translation semigroup $(S(t)_{t\geq 0})$ will induce a group of isometries in $\mathbb{Y}$ ([2], that we will denote by $(\bar{S}(t), t \in \mathbb{R})$. For every $g \in BUC(\mathbb{R}^+, \mathbb{X})$ we will denote by $\sigma_{AAP}(g)$ as the set of all $\xi \in \Gamma$ such that the complex function

$$h(\lambda) := R(\lambda, \bar{S})\bar{g}, \ (|\lambda| \neq 1) \tag{3.9}$$

has no analytic extension to any neighborhood of ξ.

Lemma 3.5. *Let u be a bounded uniformly continuous mild solution of Eq.(1.1) on the half positive line $\mathbb{R}^+$. Furthermore, assume that f is an asymptotically almost periodic function on $\mathbb{R}^+$. Then,*

$$\sigma_{AAP}(u) \subset \sigma(P) \cap \Gamma. \tag{3.10}$$

Proof. By Lemma 2.10 the function $v(\cdot) := (u(\cdot), u'(\cdot))^T$ is a mild solution to the periodic equation (2.14), that is, for all $t \geq s$

$$v(t) = U(t,s)v(s) + \int_s^t U(t,\xi)F(\xi)d\xi, \tag{3.11}$$

where $U(t,s)$ and $F(\xi)$ are defined in the Lemma 2.10. In particular,

$$v(t) = U(t,t-1)v(t-1) + \int_{t-1}^{t} U(t,\xi)F(\xi)d\xi. \tag{3.12}$$

In other words,

$$v = \mathcal{P}Sv + GF, \tag{3.13}$$

where the operator G maps F to the function defined as (for all $t \in \mathbb{R}^+$ such that $t \geq 1$)

$$GF(t) := \int_{t-1}^{t} U(t,\xi)F(\xi)d\xi,$$

that is also asymptotically almost periodic. Therefore,

$$\bar{S}\bar{v} = \bar{\mathcal{P}}\bar{v},$$

and thus, for $|\lambda| \neq 1$,

$$(\lambda - \bar{S})\bar{v} = (\lambda - \bar{\mathcal{P}})\bar{v}. \tag{3.14}$$

Suppose that $\lambda_0 \in \Gamma$ such that $\lambda_0 \notin \sigma(P)$. Then, $\lambda_0 \notin \sigma(\bar{\mathcal{P}})$, so $R(\lambda, \bar{\mathcal{P}})$ exists for λ around λ_0, and $|\lambda| \neq 1$

$$R(\lambda, \bar{S})\bar{v} = R(\lambda, \bar{\mathcal{P}})\bar{v}. \tag{3.15}$$

This shows that $R(\lambda, \bar{S})\bar{v}$ has an analytic extension around λ_0, so the lemma is proved. □

As a consequence of this lemma we have the following

THEOREM 3.6. *Assumed that Eq.(2.7) is well posed and 1-periodic with $\sigma(P) \cap \Gamma$ being countable. Further, assume that at every $\xi_0 \in \sigma(P) \cap \Gamma$*

$$\lim_{\lambda \downarrow \xi_0} (\lambda - \xi_0) R(\lambda, \mathcal{P}) g = 0 \tag{3.16}$$

for all $g \in BUC(\mathbb{R}^+, \mathbb{X})$. Then, every mild solution of Eq. (1.1) on the positive line $\mathbb{R}^+$ is asymptotically almost periodic provided that its derivative is bounded and uniformly continuous.

Proof. Let u be such a mild solution of Eq.(1.1). Then, the function $v(t) := (u(t), u'(t))^T$ is a bounded and uniformly continuous mild solution of Eq.(2.14). By Lemma 3.5 and its proof,

$$\sigma_{AAP}(v) \subset \sigma(P) \cap \Gamma,$$

so it is countable. Moreover, by (3.15) $R(\lambda,\bar{S})\bar{v} = R(\lambda,\bar{\mathcal{P}})\bar{v}$

$$\lim_{\lambda\downarrow\xi_0}(\lambda-\xi_0)R(\lambda,\bar{S})\bar{v} = \lim_{\lambda\downarrow\xi_0}(\lambda-\xi_0)R(\lambda,\bar{P})\bar{v} = 0. \tag{3.17}$$

Therefore, by Theorem 2.6, $v \in AAP(\mathbb{R}^+,\mathbb{X})$. And thus, u is asymptotically almost periodic. □

4. Examples and Applications

As an example of a second order evolution equation we consider the following periodic initial value problem

$$\frac{\partial^2 w(t,\xi)}{\partial t^2} = \frac{\partial^2 w(t,\xi)}{\partial \xi^2} + b(t)\frac{\partial w(t,\xi)}{\partial \xi}, \tag{4.1}$$

$$w(t,0) = w(t,2\pi) = 0,\ t\in\mathbb{R}, \tag{4.2}$$

$$w(s,\xi) = \alpha(\xi),\ \frac{\partial w(s,\xi)}{\partial t} = \beta(\xi),\ 0\le\xi\le 2\pi. \tag{4.3}$$

We model this problem with the space $\mathbb{X} := L^2(\mathbb{T},\mathbb{C})$, where the group $\mathbb{T}$ is defined as $\mathbb{R}/2\pi\mathbb{Z}$. Every 2π-periodic function will be identified with a function on $\mathbb{T}$. $H^2(\mathbb{T},\mathbb{C})$ denotes the Sobolev space of 2π periodic functions $f:\mathbb{R}\to\mathbb{C}$ such that $f''\in L^2(\mathbb{T},\mathbb{C})$. The operator A is defined as

$$(Au)(\xi) = \frac{d^2u(\xi)}{d\xi^2} \tag{4.4}$$

with domain

$$D(A) := \{u\in\mathbb{X}:\ u\in H^2(\mathbb{T},\mathbb{C}), u'(0) = u'(2\pi = 0\}.$$

$B(t)$ is defined as $B(t)u = b(t)u'(t)$ on $H^1(\mathbb{T},\mathbb{C})$. Then, $A(t) := A+B(t)$. As is shown in [4, Theorem 2.2], there exists an evolution operator $S(t,s)$ for initial values problem.

$$x''(t) = A(t)x(t),\ t\ge s \tag{4.5}$$

$$x(s) = y, \tag{4.6}$$

$$x'(s) = z, \tag{4.7}$$

where y,z are any elements of $\mathbb{X}$ and any $s\in\mathbb{R}$. If we assume that $b(t)$ is periodic with period 1, then the evolution process $(U(t,s)_{t\ge s})$ that is associated with the equation

$$\begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ A + B(t) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

is 1-periodic. By [4, Corollary 2.2] and [4, Application 2] the Cauchy Problem (4.5,4.6,4.7) has a unique solution for every $(y, z) \in \mathbb{X}_\nu \times \mathbb{X}_\nu$, where $\mathbb{X}_\nu$ is the subspace formed by the exponential vectors x such that $\|A^k\| \leq c\nu^k$ for all $k \in \mathbb{N}$ with norm

$$\|x\|_{\mathbb{X}_\nu} := \sup_{k \geq 0} \frac{\|A^k x\|}{\nu^k}.$$

This being said, the equation (4.1) with boundary value conditions (4.2) gives rise to a second order evolution equation that is well posed. Therefore, for a function $f(t, \xi)$ that is 2π-periodic in ξ and $f(t, \cdot)$ is an almost periodic function in t taking values in $\mathbb{X}$ we can apply our results presented above to the equation

$$\frac{\partial^2 w(t,\xi)}{\partial t^2} = \frac{\partial^2 w(t,\xi)}{\partial \xi^2} + b(t)\frac{\partial w(t,\xi)}{\partial \xi} + f(t,\xi), \tag{4.8}$$

$$w(t,0) = w(t,2\pi) = 0, \ t \in \mathbb{R}. \tag{4.9}$$

References

[1] C. J. K. Batty, W. Hutter, and F. Räbiger, *Almost periodicity of mild solutions of inhomogeneous periodic Cauchy problems*, J. Differential Equations **156** (1999), 309–327.

[2] W. Arendt, C.J.K. Batty, M. Hieber, and F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems.* Second edition. Monographs in Mathematics, 96. Birkhauser/Springer Basel AG, Basel, 2011.

[3] A.M. Fink, *Almost Periodic Differential Equations*, Lecture Notes in Math., **377**, Springer, Berlin-New York, 1974.

[4] H.R. Henriquez, *Existence of solutions of the nonautonomous abstract Cauchy problem of second order*, Semigroup Forum **87** (2) (2013), 277–297.

[5] H.R. Henriquez, V. Poblete, and J.C. Pozo, *Mild solutions of non-autonomous second order problems with nonlocal initial conditions*, J. Math. Anal. Appl. **412** (2) (2014), 1064–1083.

[6] Y. Hino, T. Naito, N.V. Minh, and J.S. Shin, *Almost Periodic Solutions of Differential Equations in Banach Spaces*, Taylor & Francis. LOndon & New York 2001.

[7] M. Kozak, *A fundamental solution of a second-order differential equation in a Banach space*, Univ. Iagel. Acta Math., **32** (1995), 275–289.

[8] B. M. Levitan, and V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Moscow Univ. Publ. House 1978. English translation by Cambridge University Press 1982.

[9] Vu Trong Luong, Nguyen Huu Tri, and Nguyen Van Minh, *Asymptotic behavior of solutions of periodic linear partial functional differential equations on the half line* Submitted. Preprint available at https://arxiv.org/abs/1807.03828

[10] J.L. Massera, *The existence of periodic solutions of systems of differential equations*, Duke Math. J. **17** (1950). 457–475.

[11] Nguyen Van Minh, *Asymptotic behavior of individual orbits of discrete systems*, Proceedings of the A.M.S. **137** (9) (2009), 3025–3035.

[12] Nguyen Van Minh, G. N'Guerekata, and S. Siegmund, *Circular spectrum and bounded solutions of periodic evolution equations*, J. Differential Equations **246** (8) (2009), 3089–3108.

[13] R. Miyazaki, D. Kim, T. Naito and J.S. Shin, *Fredholm operators, evolution semigroups, and periodic solutions of nonlinear periodic systems*, J. Differential Equations, **257** (2014), 4214–4247.

[14] S. Murakami, T. Naito, and Nguyen Van Minh, *Massera's theorem for almost periodic solutions of functional differential equations*, Journal of the Math Soc. of Japan, **47** (2004) (1), 247–268.

[15] T. Naito, and N. V. Minh, *Evolution semigroups and spectral criteria for almost periodic solutions of periodic evolution equations*, J. Differential Equations **152** (1999), 358–376.

[16] T. Naito, N. V. Minh, and J. S. Shin, *New spectral criteria for almost periodic solutions of evolution equations*, Studia Mathematica, **145** (2001), 97–111.

[17] W. M. Ruess, and Q. P. Vu, *Asymptotically almost periodic solutions of evolution equations in Banach spaces*, J. Differential Equations **122** (2) (1995), 282–301.

[18] J.S. Shin and T. Naito, *Semi-Fredholm operators and periodic solutions for linear functional differential equations*, J. Differential Equations **153** (1999), 407–441.

[19] Q. P. Vu, *Stability and almost periodicity of trajectories of periodic processes*, J. Differential Equations, **115** (1995), 402–415.

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