Korean J. Math. **27** (2019), No. 3, pp. 779–791
https://doi.org/10.11568/kjm.2019.27.3.779

A NOTE ON DERIVATIONS OF ORDERED Γ-SEMIRINGS

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Abstract. In this paper, we consider derivation of an ordered Γ-semiring and introduce the notion of reverse derivation on ordered Γ-semiring. Also, we obtain some interesting related properties. Let I be a nonzero ideal of prime ordered Γ-semiring M and let d be a nonzero derivation of M. If Γ-semiring M is negatively ordered, then d is nonzero on I.

1. Introduction

A semiring is an algebraic structure with two binary operations called addition and multiplication where one of them distributive over the other. A semiring is a common generalization of rings and distributive lattices and was first introduced by Vandiver([10)] 1934 but nontrivial examples of semiring have appeared in the earlier studies on the theory of commutative ideals of rings by Richard Dedekind 19th centrary The notion of a Γ-ring was introduced by Nobusawa([7)] as a generalization of ring 1981. Sen([9]) introduced the concept of a Γ semigroup in 1981. In 1995, M. K. Rao([4, 5]) introduced the notion of Γ-semiring which is a generalization Γ-ring, ring and semiring. Over the last few decades serval authors have investigates the relationship between the commutativity of ring R and the existence of certain specified derivation of R. The first result in this relation is due to Posner([8)] in 1957. In the 1990,

Received July 16, 2019. Revised August 5, 2019. Accepted August 6, 2019.
2010 Mathematics Subject Classification: 16Y30, 06B35, 06B99.
Key words and phrases: Semiring, ordered Γ-semiring, reverse derivation, positively ordered, idempotent, $Fix_d(M)$.

Bresar and Vukman([1]) established that a prime ring must be commutative if it admits a nonzero left derivation. Kim([2, 3]) studied right derivations and generalized right derivations of incline algebras. M. K. Rao([6]) introduced the notion of right derivation in ordered Γ-semirings and generalized right derivations of ordered Γ-semirings. In this paper, we consider derivations of ordered Γ-semirings and introduced the notion of reverse derivations on ordered Γ-semirings. Also, we obtain some interesting related properties. Let I be a nonzero ideal of prime ordered Γ-semiring M and let d be a nonzero derivation of M. If Γ-semigroup M is negatively ordered, then d is nonzero on I.

2. Preliminaries

Definition 2.1. A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called a *semiring* if

(1) : $(S, +)$ is commutative.
(2) : $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$ for all $x, y, z \in S$
(3) : there exists $0 \in S$ such that $x + 0 = x$ and $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.

Definition 2.2. Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. Then M is called a Γ-*semiring* if there exists a mapping $M \times \Gamma \times M \to M$, where $(x, \alpha, y) = x\alpha y$ such that it satisfies the following axioms for any $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

(1) : $x\alpha(y+z) = x\alpha y + x\alpha z$
(2) : $(x+y)\alpha z = x\alpha z + y\alpha z$
(3) : $x(\alpha+\beta)y = x\alpha y + x\beta y$
(4) : $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Every semiring S is a Γ-semiring with $\Gamma = S$, where the ternary operation is the usual semiring multiplication.

Example 2.3. Let S be a semiring and $M_{p,q}(S)$ denote the addition abelian semigroup of all $p \times q$ matrices with identity element whose entries are from S. Then $M_{p,q}(S)$ is a Γ-semiring with $\Gamma = M_{p,q}(S)$. A ternary operation is defined by $x\alpha z = x(\alpha^t)z$ as the usual matrix multiplication, where α^t denotes the transpose of the matrix α, for all x, y and $\alpha \in \Gamma$.

A Γ-semiring M is said to have a *zero element* if there exists an element $0 \in M$ such that $0 + x = x + 0 = x$ and $0\alpha x = x\alpha 0 = 0$ for all $x \in M$ and $\alpha \in \Gamma$. A Γ-semiring M is said to be *commutative* if $x\alpha y = y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. An element $a \in M$ is said to be *idempotent* if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$ and $a+a = a$. If every element of M is an idempotent of M, then M is called an *idempotent* Γ-*semiring*. An element $1 \in M$ is said to be *unity* if for each $x \in M$, there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

DEFINITION 2.4. Let M be an ordered Γ-semiring.

(1) : $(M, +)$ is *positively ordered* if $a + b \geq a, b$ for all $a, b \in M$
(2) : $(M, +)$ is *negatively ordered* if $a + b \leq a, b$ for all $a, b \in M$
(3) : A Γ-semigroup M is *positively ordered* if $a\alpha b \geq a, b$ for all $a, b \in M$ and $\alpha \in \Gamma$
(4) : A Γ-semigroup M is *negatively ordered* if $a\alpha b \leq a, b$ for all $a, b \in M$ and $\alpha \in \Gamma$.

DEFINITION 2.5. An Γ-semiring M is called an *ordered* Γ-*semiring* if it admits a compatible relation $\leq$, that is, $\leq$ is a partial ordering on M which satisfies the following conditions,

(1) : If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$
(2) : If $a \leq b$ and $c \leq d$, then $a\alpha c \leq b\alpha d$
(3) : If $a \leq b$ and $c \leq d$, then $c\alpha a \leq d\alpha b$, for all $a, b, c \in M$ and $\alpha \in \Gamma$

EXAMPLE 2.6. Let $M = [0, 1], \Gamma = N, x + y = \max\{x, y\}$ and $x\alpha y = \min\{x, \alpha, y\}$ for all $x, y \in M$ and $\alpha \in \Gamma$. Then M is an ordered Γ-semiring with respect to the usual ordering (see[6]).

DEFINITION 2.7. A nonempty subset A of ordered Γ-semiring M is called a Γ-*subsemiring* if $(A, +)$ is a subsemigroup of $(M, +)$ and $a\alpha b \in A$ for all $a, b \in A$ and $\alpha \in \Gamma$. A nonempty subset I of ordered Γ-semiring M is called a left ideal (*right ideal*) of M if for any $a \in M$ and $b \in I$,

(1) : I is closed under addition
(2) : $M\Gamma I \subseteq I (A\Gamma M \subseteq I)$
(3) : $a \leq b$ and $b \in I$ implies $a \in I$.

A nonempty subset I of ordered Γ-semiring M is called *ideal* of M if it is both a left ideal and a right ideal of M. A nonempty subset I of ordered Γ-semiring M is called *k-ideal* of M if I is an ideal and $x + y \in I$ and $y \in I$ implies $x \in I$ for any $x \in M$.

Definition 2.8. Let M be an ordered Γ-semiring. A Γ-subsemiring P of M is said to be *prime* ideal of M if

(1) : $a \leq b$ and $b \in P$ implies $a \in P$ for any $a \in M$
(2) : $a\alpha b \in P$ implies $a \in P$ or $b \in P$ for all $a, b \in M$ and $\alpha \in \Gamma$.

Definition 2.9. Let M be an ordered Γ-semiring. An element $a \in M$ is said to be *additively left cancellative* if for all $b, c \in M$, $a+b = a+c \Rightarrow b = c$. An element $a \in M$ is said to be *additively right cancellative* if for all $b, c \in M$, $b+a = c+a \Rightarrow b = c$. It is said to be *additively cancellative* if it is both left and right cancellative. If every element of M is additively left cancellative, it is said to be *additively left cancellative.* If every element of M is additively right cancellative, it is said to be *additively right cancellative.*

3. Derivations in ordered Γ-semirings

In what follows, let M denote an ordered Γ-semiring unless otherwise specified.

Definition 3.1. Let M be an ordered Γ-semiring. If the mapping $d : M \to M$ satisfies the following conditions

(1) : $d(x + y) = d(x) + d(y)$
(2) : $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$
(2) : If $x \leq y$, then $d(x) \leq d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$,

then d is called a *derivation* on M.

Example 3.2. Let $M = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in Q \right\}$, where Q is the set of rational numbers and $\Gamma = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} : a, b, c \in N \right\}$, where N is the set of natural numbers. Then M and Γ are additive abelian semigroups with respect to the usual matrix addition of 2×2 matrices and a ternary operation, which is defined as $M \times \Gamma \times M$ by $(x, \alpha, y) \to x\alpha y$ using the usual matrix multiplication for all $x, y \in M$ and $\alpha \in \Gamma$. Let $A = (a_{ij})$ and $B = (b_{ij}) \in M$, we define $A \leq B \Leftrightarrow a_{ij} \leq b_{ij}$ for all i, j. Then M is an ordered Γ-semiring. Define a map $d : M \to M$ given by

$$d\left(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$$

Then d is a derivation on M.

Proposition 3.3. *Let M be a commutative ordered Γ-semiring. If M is additive idempotent, then for a fixed $a \in M$ and $\alpha \in \Gamma$, the mapping $d_a : M \to M$ given by $d_a(x) = x \circ a$ for all $x \in M$, where $x \circ a = x\alpha a + a\alpha x$ is a derivation of M.*

Proof. Let M be a commutative ordered Γ-semiring. Then for a fixed $a \in M$ and $\alpha \in \Gamma$,

$$\begin{aligned} d_a(x+y) &= (x+y)\circ a = (x+y)\alpha a + a\alpha(x+y) \\ &= x\alpha a + y\alpha a + (x+y)\alpha a \\ &= x\alpha a + y\alpha a + x\alpha a + y\alpha a \\ &= x\alpha a + y\alpha a + a\alpha x + a\alpha y \\ &= (x\alpha a + a\alpha x) + (y\alpha a + a\alpha y) \\ &= x\circ a + y\circ a \\ &= d_a(x) + d_a(y) \end{aligned}$$

for all $x, y \in M$. Also we have for all $x, y \in M$ and $\alpha \in \Gamma$,

$$\begin{aligned} d_a(x\alpha y) &= (x\alpha y)\circ a = (x\alpha y)\alpha a + a\alpha(x\alpha y) \\ &= (x\alpha y)\alpha a + (x\alpha y)\alpha a \\ &= (x\alpha y)\alpha a + (x\alpha y)\alpha a = (x\alpha y)\alpha a \end{aligned}$$

and

$$\begin{aligned} d_a(x)\alpha y + x\alpha d_a(y) &= (x\alpha a + a\alpha x)y + x\alpha(y\alpha a + \alpha a y) \\ &= (x\alpha a + a\alpha x)\alpha y + (y\alpha a + a\alpha y)\alpha x \\ &= (x\alpha a)\alpha y + (a\alpha x)\alpha y + (y\alpha a)\alpha x + (a\alpha y)\alpha x \\ &= (x\alpha y)\alpha a + (x\alpha y)\alpha a + (x\alpha y)\alpha a + (x\alpha y)\alpha a, \\ &= (x\alpha y)\alpha a, \end{aligned}$$

which implies $d_a(x\alpha y) = d_a(x)\alpha y + x\alpha d_a(y)$.

Finally, let $x, y \in M$ be such that $x \leq y$. Then we have for any $\alpha \in \Gamma$, we have

$$\begin{aligned} x \leq y &\Rightarrow x\alpha a \leq y\alpha a \\ &\Rightarrow x\alpha a + x\alpha a \leq y\alpha a + y\alpha a \\ &\Rightarrow x\alpha a + a\alpha x \leq y\alpha a + a\alpha y \\ &\Rightarrow d_a(x) \leq d_a(y). \end{aligned}$$

Hence d_a is a derivation of M. □

PROPOSITION 3.4. *Let M be a commutative ordered Γ-semiring. Then $d_{a+b} = d_a + d_b$ for all $a, b \in M$ and $\alpha \in \Gamma$.*

Proof. Let M be a commutative ordered Γ-semiring and $a, b \in M$. Then for all $c \in M$ and $\alpha \in \Gamma$, we have

$$\begin{aligned} d_{a+b}(c) &= (a+b) \circ c = (a+b)\alpha c + c\alpha(a+b) \\ &= (a+b)\alpha c + (a+b)\alpha c = a\alpha c + b\alpha c + a\alpha c + b\alpha c \\ &= a\alpha c + b\alpha c + c\alpha a + c\alpha b = (a\alpha c + c\alpha a) + (b\alpha c + c\alpha b) \\ &= (a \circ c) + (b \circ c) = d_a(c) + d_b(c) \\ &= (d_a + d_b)(c). \end{aligned}$$

□

PROPOSITION 3.5. *Let M be an ordered Γ-semiring. If d is a derivation of M, then we have $d(0) = 0$.*

Proof. Let M be an ordered Γ-semiring. For any $\alpha \in \Gamma$, we have

$$d(0) = d(0\alpha 0) = d(0)\alpha 0 + 0\alpha d(0) = 0 + 0 = 0.$$

This completes the proof. □

PROPOSITION 3.6. *Let M be a commutative ordered Γ-semiring. A sum of two derivations of M is again a derivation of M.*

Proof. Let d_1 and d_2 be two derivations of M, respectively. Then we have for all $a, b \in M$ and $\alpha \in \Gamma$,

$$\begin{aligned} (d_1 + d_2)(a+b) &= d_1(a+b) + d_2(a+b) \\ &= d_1(a) + d_1(b) + d_2(a) + d_2(b) \\ &= (d_1(a) + d_2(a)) + (d_1(b) + d_2(b)) \\ &= (d_1 + d_2)(a) + (d_1 + d_2)(b) \end{aligned}$$

and

$$\begin{aligned} (d_1 + d_2)(a\alpha b) &= d_1(a\alpha b) + d_2(a\alpha b) \\ &= d_1(a)\alpha b + a\alpha d_1(b) + d_2(a)\alpha b + a\alpha d_2(b) \\ &= d_1(a)\alpha b + d_2(a)\alpha b + a\alpha d_1(b) + a\alpha d_2(b) \\ &= (d_1 + d_2)(a)\alpha b + a\alpha(d_1 + d_2)(b). \end{aligned}$$

Clearly, $x \leq y$ implies $(d_1 + d_2)(x) \leq (d_1 + d_2)(y)$ for any $x, y \in M$. This completes the proof. □

THEOREM 3.7. *Let M be a commutative ordered Γ-semiring let d_1, d_2 be derivations of M, respectively. Define $d_1d_2(x) = d_1(d_2(x))$ for all $x \in K$. If $d_1d_2 = 0$, then d_2d_1 is a derivation of M.*

Proof. Let $d_1d_2 = 0$. For every $x, y \in M$ and $\alpha \in \Gamma$, then we have

$$\begin{aligned} 0 &= d_1d_2(x\alpha y) = d_1(d_2(x)\alpha y + x\alpha d_2(y)) \\ &= d_1d_2(x)\alpha y + d_2(x)\alpha d_1(y) + d_1(x)\alpha d_2(y) + x\alpha d_1(d_2(y)) \\ &= d_2(x)\alpha d_1(y) + d_1(x)\alpha d_2(y). \end{aligned}$$

Then

$$\begin{aligned} d_2d_1(x\alpha y) &= d_2(d_1(x)\alpha y + x\alpha d_1(y)) \\ &= d_2d_1(x)\alpha y + d_1(x)\alpha d_2(y) + d_2(x)\alpha d_1(y) + x\alpha d_2(d_1(y)) \\ &= d_2d_1(x)\alpha y + x\alpha d_2d_1(y). \end{aligned}$$

Also, for all $x, y \in K$, we get

$$d_2d_1(x + y) = d_2(d_1(x) + d_1(y)) = d_2d_1(x) + d_2d_1(y).$$

Finally, $x \leq y$ implies $d_2(d_1(x)) \leq d_2(d_1(x))$. This implies that d_2d_1 is a derivation of M. □

PROPOSITION 3.8. *Let d be a derivation of the idempotent commutative ordered Γ-semiring M. If M is negatively ordered, then $d(x) \leq x$ for all $x \in M$.*

Proof. Let d be a derivation of the idempotent commutative ordered Γ-semiring M. Then we have

$$\begin{aligned} d(x) &= d(x\alpha x) = d(x)\alpha x + x\alpha d(x) \\ &= d(x)\alpha x + d(x)\alpha x = d(x)\alpha x \leq x \end{aligned}$$

for all $x \in M$ and $\alpha \in \Gamma$. □

PROPOSITION 3.9. *Let d be a derivation of a prime ordered Γ-semiring M and $a \in M$. If $a\alpha d(x) = 0$ for all $x \in M$ and $\alpha \in \Gamma$, then either $a = 0$ or $d(x) = 0$.*

Proof. Let $a\alpha d(x) = 0$ for all $x \in M$ and $\alpha \in \Gamma$. Replacing x by $x\alpha y$, then we have

$$\begin{aligned} 0 &= a\alpha d(x\alpha y) = a\alpha(d(x)\alpha y + x\alpha d(y)) \\ &= a\alpha d(x)\alpha y + a\alpha x\alpha d(y) \\ &= a\alpha x\alpha d(y) \end{aligned}$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Since M is prime, if $d(y) \neq 0$ for some $y \in M$, we have $a = 0$. □

Proposition 3.10. *Let M be an idempotent prime ordered Γ-semiring and let d be a derivation on M. Define $d^2(x) = d(d(x))$ for all $x \in M$. If $d^2 = 0$, then d is zero.*

Proof. Let $x, y \in M$ and $\alpha \in \Gamma$. Then we have

$$\begin{aligned} 0 &= d^2(x\alpha y) = d(d(x)\alpha y + x\alpha d(y)) \\ &= d^2(x)\alpha y + d(x)\alpha d(y) + d(x)\alpha d(y) + x\alpha d^2(y) \\ &= d(x)\alpha d(y) + d(x)\alpha d(y) \\ &= d(x)\alpha d(y) \end{aligned}$$

By Proposition 3.9, we have $d = 0$. □

Proposition 3.11. *Let M be an additively cancellative ordered Γ-semiring and let d_1 and d_2 be derivations of M. Define $d_1d_2(x) = d_1(d_2(x))$ for all $x \in M$. If d_1d_2 is also a derivation of M, then*

$$d_2(x)d_1(y) + d_1(x)d_2(y) = 0$$

for all $x, y \in M$ and $\alpha \in \Gamma$.

Proof. Since d_1 and d_2 are derivations of M, we have for all $x, y \in M$ and $\alpha \in \Gamma$,

$$\begin{aligned} (1)\quad d_1d_2(x\alpha y) &= d_1(d_2(x\alpha y)) \\ &= d_1(d_2(x)\alpha y + x\alpha d_2(y)) \\ &= d_1d_2(x)\alpha y + d_2(x)\alpha d_1(y) + d_1(x)\alpha d_2(y) + x\alpha d_1d_2(y). \end{aligned}$$

Since d_1d_2 is also a derivation of M, we have

$$(2)\qquad d_1d_2(x\alpha y) = d_1d_2(x)\alpha y + x\alpha d_1d_2(y).$$

Combining (1) and (2) yields

$$d_2(x)\alpha d_1(y) + d_1(x)\alpha d_2(y) = 0$$

for all $x, y \in M$ and $\alpha \in \Gamma$. □

Proposition 3.12. *Let I be a nonzero ideal of prime ordered Γ-semiring M and let d be a nonzero derivation of M. If an Γ-semigroup M is negatively ordered, then d is nonzero on I.*

Proof. Let $d = 0$ be on I and $x \in I$. Then $d(x) = 0$ for all $x \in I$. Also, let $y \in M$. Since $x\alpha y \leq x$ and I is an ideal of M, we have $x\alpha y \in I$, Therefore, $d(x\alpha y) = 0$, that is,

$$0 = d(x\alpha y) = d(x)\alpha y + x\alpha d(y) = x\alpha d(y).$$

Since M is prime, we get $x = 0$ for all $x \in I$ or $d(y) = 0$ for all $y \in M$. Since $I \neq 0$, we have $d(y) = 0$ for all $y \in M$. This is a contradiction by hypothesis. So, d is nonzero on I. □

THEOREM 3.13. *Let M be an additively cancellative prime ordered Γ-semiring and let d be a nonzero derivation on M. If $[a, d(M)] = (0)$, where $[a, x]_\alpha = a\alpha x - x\alpha a$ for all $x, a \in M$, then $a \in Z$, the center of M.*

Proof. By hypothesis, we have $[a, d(x)]_\alpha = 0$ for all $x \in M$. Replacing x by $a\alpha x$ for all x and $\alpha \in \Gamma$, we have $[a, d(a\alpha x)] = 0$. Hence we get

$$\begin{aligned} 0 &= [a, d(a)\alpha x + a\alpha d(x)]_\alpha \\ &= [a, d(a)\alpha x]_\alpha + [a, a\alpha d(x)]_\alpha \\ &= d(a)\alpha[a, x]_\alpha + [a, d(a)]_\alpha \alpha x + a\alpha[a, d(x)]_\alpha + [a, a]_\alpha \alpha d(x). \end{aligned} \tag{3}$$

By using the hypothesis and the fact that $[a, a]_\alpha = 0$ for all $a \in M$, we have $d(a)\alpha[a, x]_\alpha = 0$. Also, replacing x by $x\beta y$, we have $d(a)\Gamma M\Gamma[a, y]_\alpha = (0)$ for all $y \in M$. Since M is prime and $d \neq 0$, we have $[a, y]_\alpha = 0$ for all $y \in M$. Hence we have $a \in Z$, the center of M. □

THEOREM 3.14. *Let M be an additively cancellative prime ordered Γ-semiring and let d be a nonzero derivation on M. Then M is commutative ordered Γ-semiring.*

Proof. Let $a, b \in M$ and $\alpha \in \Gamma$. Then we have

$$\begin{aligned} d(a\alpha b\alpha a) &= d(a)\alpha b\alpha a + a\alpha d(b\alpha a) \\ &= d(a)\alpha b\alpha a + a\alpha(d(b)\alpha a + b\alpha d(a)) \\ &= d(a)\alpha b\alpha a + a\alpha d(b)\alpha a + b\alpha a\alpha d(a) \end{aligned} \tag{4}$$

and

$$\begin{aligned} d(a\alpha b\alpha a) &= d(a\alpha b)\alpha a + a\alpha b\alpha d(a) \\ &= (d(a)\alpha b + a\alpha d(b))\alpha a + a\alpha b\alpha d(a) \\ &= d(a)\alpha b\alpha a + a\alpha d(b)\alpha a + a\alpha b\alpha d(a) \end{aligned} \tag{5}$$

From (4) and (5), we have $a\alpha b\alpha d(a) = b\alpha a\alpha d(a)$, that is, $[a, b]_\alpha \alpha d(a) = 0$. Also, replacing b by $c\alpha b$ in this relation, we have $[a, c]_\alpha \alpha b\alpha d(a) = 0$ for all $a, b, c \in M$ and $\alpha \in \Gamma$. Since M is prime and $d \neq 0$, we get $[a, c]_\alpha = 0$. This implies that M is a commutative ordered Γ-semiring. $\square$

4. Reverse derivations in ordered Γ-semirings

DEFINITION 4.1. Let M be an ordered Γ-semiring. If the mapping $d : M \to M$ satisfies the following conditions

(1) : $d(x + y) = d(x) + d(y)$
(2) : $d(x\alpha y) = d(y)\alpha x + y\alpha d(x)$
(3) : If $x \leq y$, then $d(x) \leq d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$,

then d is a *reverse derivation* of M.

EXAMPLE 4.2. Let $M = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} : a, b \in Q \right\}$, where Q is the set of rational numbers and $\Gamma = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in N \right\}$, where N is the set of natural numbers. Then M and Γ are additive abelian semigroups with respect to the usual matrix addition of 2×2 matrices and a ternary operation, which is defined as $M \times \Gamma \times M$ by $(x, \alpha, y) \to x\alpha y$ using the usual matrix multiplication for all $x, y \in M$ and $\alpha \in \Gamma$. Let $A = (a_{ij})$ and $B = (b_{ij}) \in M$, we define $A \leq B \Leftrightarrow a_{ij} \leq b_{ij}$ for all i, j. Then M is an ordered Γ-semiring. Define a map $d : M \to M$ given by

$$d\left(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$$

Then d is a reverse derivation on M.

EXAMPLE 4.3. Let $M = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} : a, b \in Q \right\}$, where Q is the set of rational numbers and $\Gamma = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a \in N \right\}$, where N is the set of natural numbers. Then M and Γ are additive abelian semigroups with respect to the usual matrix addition of 2×2 matrices and a ternary operation, which is defined as $M \times \Gamma \times M$ by $(x, \alpha, y) \to x\alpha y$ using the usual matrix multiplication for all $x, y \in M$ and $\alpha \in \Gamma$. Let $A = (a_{ij})$

and $B = (b_{ij}) \in M$, we define $A \leq B \Leftrightarrow a_{ij} \leq b_{ij}$ for all i, j. Then M is an ordered Γ-semiring. Define a map $d : M \to M$ given by

$$d\left(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$$

Then d is a derivation on M but not a reverse derivation of M.

THEOREM 4.4. *Let d be a reverse derivation of M. If M is of characteristic 2, then d^2 is a derivation of M.*

Proof. Let d be a reverse derivation of M and let M is of characteristic 2. For any $x, y \in M$ and $\alpha \in \Gamma$, we have

$$\begin{aligned} d^2(x\alpha y) &= d(d(x\alpha y)) = d(d(y)\alpha x + y\alpha d(x)) \\ &= d(x)\alpha d(y) + x\alpha d^2(y) + d^2(x)\alpha y + d(x)\alpha d(y) \\ &= d^2(x)\alpha y + x\alpha d^2(y). \end{aligned}$$

Hence d^2 is a derivation of M. □

PROPOSITION 4.5. *Let M be an idempotent ordered Γ-semiring and additively cancellative. If d is a reverse derivation of M, then $a\alpha d(a)\alpha a = 0$ for all $\alpha \in \Gamma$.*

Proof. Let M be an idempotent ordered Γ-semiring and additively cancellative. Hence $a\alpha a = a$ for any $a \in M$ and $\alpha \in \Gamma$. Since d is a reverse derivation of M, we have $d(a)\alpha a + a\alpha d(a) = d(a)$. Premultiplying by a, we have $a\alpha d(a)\alpha a + a\alpha a\alpha d(a) = a\alpha d(a)$. That is, $a\alpha d(a)\alpha a + a\alpha d(a) = a\alpha d(a) + 0$. Since M is additively cancellative, we get $a\alpha d(a)\alpha a = 0$. □

PROPOSITION 4.6. *Let d be a reverse derivation of an ordered Γ-semiring and $a \in M$. If a is a commuting idempotent element, then $d(a) = 0$.*

Proof. Let $a \in M$ be a commuting idempotent element. That is, $b\alpha a = a\alpha b$ for all $b \in M$ and $\alpha \in \Gamma$. In particular, $a\alpha d(a) = d(a)\alpha a$. Postmultiplying by a, we have $a\alpha d(a)\alpha a = d(a)\alpha a\alpha a = d(a)\alpha a$. By Proposition 4.5, we get $d(a)\alpha a = 0$. Therefore,

$$\begin{aligned} d(a) &= d(d(a\alpha a)) = d(a)\alpha a + a\alpha d(a) \\ &= d(a)\alpha a + d(a)\alpha a = d(a)\alpha a = 0 \end{aligned}$$

That is, $d(a) = 0$. □

Theorem 4.7. *Let d be a reverse derivation of an additively cancellative commutative idempotent ordered Γ-semiring M in which $(M, +)$ is positively ordered. Define a set $Fix_d(M)$ by*

$$Fix_d(M) = \{x \in M | d(x) = x\}.$$

Then $Fix_d(M)$ is an ideal of M.

Proof. Let $x, y \in Fix_d(M)$ and $\alpha \in \Gamma$. Then we have $d(x) = x$ and $d(y) = y$, which implies $d(x + y) = d(x) + d(y) = x + y$. That is, $x + y \in Fix_d(M)$. Also, $d(x\alpha y) = d(y)\alpha x + y\alpha d(x) = y\alpha x + y\alpha x = y\alpha x = x\alpha y$. Therefore, $x\alpha y \in Fix_d(M)$. So, $Fix_d(M)$ is a ordered Γ-subsemiring of M. Let $x \leq y$ and $y \in Fix_d(M)$. Then $x \leq y$ implies $x + y \leq y + y$, so $x+y \leq y \leq x+y$, which means $x+y = y$. Hence $d(x+y) = x+y$ implies $d(x) + d(y) = x + y$, that is, $d(x) + y = x + y$. Since M is additively cancellative, we have $d(x) = x$. This completes the proof. □

Corollary 4.8. *Let d be a reverse derivation of an additively cancellative commutative idempotent ordered Γ-semiring M in which $(M, +)$ is positively ordered. Then $Fix_d(M)$ is an k-ideal of M.*

Proof. Let $x+y \in Fix_d(M)$ and $y \in Fix_d(M)$. Then $d(x+y) = x+y$ and $d(y) = y$. So, $d(x)+d(y) = x+y$ implies $d(x)+y = x+y$. Therefore, $d(x) = x$. By Theorem 4.7, $Fix_d(M)$ is an ideal of M. Hence $Fix_d(M)$ is a k-ideal of M. □

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