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SOME NEW ESTIMATES FOR EXPONENTIALLY $(\hbar, \mathfrak{m})$-CONVEX FUNCTIONS VIA EXTENDED GENERALIZED FRACTIONAL INTEGRAL OPERATORS

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Abstract. In the article, we present several new Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for the exponentially $(\hbar, \mathfrak{m})$-convex functions via an extended generalized Mittag-Leffler function. As applications, some variants for certain type of fractional integral operators are established and some remarkable special cases of our results are also have been obtained.

1. Introduction

Fractional calculus involving integral or differential operator of fractional order is very close to classical calculus. To see the historical backgrounds of fractional calculus, one can refer to the papers [15, 42]. Integral inequalities that are established by fractional calculus are important in proving the uniqueness of solutions for fractional differential equations. They also offer some new estimate for the solutions of boundary value problems for fractional order. Several mathematicians have investigated expansions and improvements of inequalities which include fractional calculus, see [7, 12, 16, 19, 28, 29, 40, 45]. Several new fractional

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integral operator have been introduced to consider the applications. Recently, several new fractional operators and some of their applications are presented. These operators are known as left-sided and right-sided generalized conformable fractional operators. These operators are such that they contain several operators of fractional calculus. These can be viewed as the generalization of Katugampola fractional operators, Hadamard fractional operators, Riemann-Liouville fractional operators, conformable fractional operators and ordinary derivative and integral operators. These operators enjoy some basic properties such as linearity, continuity and boundedness.
The fractional differential calculus technique has contributed to the interpretation of physical phenomena as well as a new dimension to the mathematical approaches for explaining physical phenomena. The order of the differential equations describing physical phenomena determines the rate of change in the physical event discussed. At this point, the fractional order differential has an powerful affect in understanding the character of the physical phenomenon, although it loses the weaknesses of the integer order differential equations to explain some physical events.

The classification of functions can be done with various features such as continuity, convexity, monotony and differentiability. The concept of convexity in mathematics is known to play an important role in the development of various branches. Hermite-Hadamard's inequality is associated with the concept of convexity. Now we recall the some well known definition related to convexity as follows:
A mapping $\psi : \mathcal{K} \subseteq \mathbb{R} \to \mathbb{R}$ is called convex, if

$$\psi(\varsigma_1\tau + (1-\tau)\varsigma_2) \leq \tau\psi(\tau) + (1-\tau)\psi(\tau), \quad \varsigma_1, \varsigma_2 \in \mathcal{K}, \tau \in [0,1].$$

Studies on inequalities are based on exploring new inequalities and strengthening classical approaches. Modern inequality theory continues to be an active area of mathematical sciences. Inequality theory continues to be a field that is continuously studied and still active in research and enchanting. The following famous inequality among these, the HHI [13] is one of the most celebrated variants, which can be stated as follows:

Let $\mathcal{K} \subseteq \mathbb{R}$ be an interval and $\psi : \mathcal{K} \longrightarrow \mathbb{R}$ be a convex function. Then the double inequality

$$\psi\left(\frac{\varsigma_1 + \varsigma_2}{2}\right) \leq \frac{1}{\varsigma_2 - \varsigma_1}\int_{\varsigma_1}^{\varsigma_2} \psi(x)dx \leq \frac{\psi(\varsigma_1) + \psi(\varsigma_2)}{2} \tag{1.1}$$

satisfies for convex mappings and is called as the HHI. This inequality has long been known as Hadamard's inequality. This inequality of Hermite was discovered by Mitrinovic, who he had found earlier than Hadamard and is known now in the literature as Hermite- Hadamard's inequality. The importance of this inequality is due to the fact that it is equivalent to the definition of convexity under certain conditions. The inequality (1.1) has been studied by several researchers due to its usefulness and applications. See [2–4, 6–14, 16–18, 20, 22, 25, 26, 30–39, 41, 43, 44, 46] and the references therein.

On other hand, the minimum of the differentiable convex functions can be characterized by variational inequalities. These two aspects of the convexity theory have far reaching applications and have provided powerful tools for studying difficult problems. In recent years, integral inequalities are being derived via fractional analysis, which has emerged as another interesting technique.

To the best of our knowledge, a comprehensive investigation of exponentially convex functions as as an extended Mittag Leffler functions in the present paper is new one. The class of exponentially convex functions was introduced by Bernstein [5], Dragomir and Gomm [10], Noor and Noor [23, 24] and Rashid et al. [33]. Motivated by these facts, Awan et al. [2] introduced and investigated another class of convex functions, which is called exponentially convex function and is significantly different from the class introduced by [5, 10]. The growth of research on big data analysis and deep learning has recently increased the interest in information theory involving exponentially convex functions. The smoothness of exponentially convex function is exploited for statistical learning, sequential prediction and stochastic optimization, see [1, 5, 27] and the references therein.

In [33], it is known that a function ψ is exponentially convex, if and only if, ψ satisfies the inequality

$$e^{\psi\left(\frac{\varsigma_1+\varsigma_2}{2}\right)} \leq \frac{1}{\varsigma_2-\varsigma_1}\int_{\varsigma_1}^{\varsigma_2} e^{\psi(x)}dx \leq \frac{e^{\psi(\varsigma_1)}+e^{\psi(\varsigma_2)}}{2}. \tag{1.2}$$

The inequality (1.4) is called the Hermite-Hadamard inequality and provides the upper and lower estimates for the exponential integral.

Now, we recall and introduce some preliminaries for exponentially convex functions.

Definition 1.1. [10, 23, 33]. A positive real-valued function $\psi : \mathcal{K} \subseteq \mathbb{R} \longrightarrow (0, \infty)$ is said to be exponentially convex on $\mathcal{K}$ if the inequality

$$e^{\psi(\tau\varsigma_1+(1-\tau)\varsigma_2)} \leq \tau e^{\psi(\varsigma_1)} + (1-\tau)e^{\psi(\varsigma_2)}$$

grips for $\varsigma_1, \varsigma_2 \in \mathcal{K}$ and $\tau \in [0, 1]$.

Exponentially convex functions are used to manipulate for statistical learning, sequential prediction and stochastic optimization, see [1, 27] and the references therein.

Next, we use the concept of exponentially $\hbar$-convex function which is explored by Rashid et al. [34].

Definition 1.2. [34] Let $J \subseteq \mathbb{R}$ be an interval such that $(0, 1) \subseteq \mathcal{J}$ and $\hbar : \mathcal{J} \to \mathbb{R}$ be a nonnegative real-valued function. Then $\psi : \mathcal{K} \longrightarrow \mathbb{R}$ is said to be exponentially $\hbar$-convex if ψ is non-negative such that the inequality

$$e^{\psi(\tau\varsigma_1+(1-\tau)\varsigma_2)} \leq \hbar(\tau)e^{\psi(\varsigma_1)} + \hbar(1-\tau)e^{\psi(\varsigma_2)}$$

holds for all $\varsigma_1, \varsigma_2 \in \mathcal{K}$ and $\tau \in [0, 1]$.

Definition 1.3. [25] Let $\mathfrak{m} \in (0, 1]$ and $\mathcal{K} \subseteq \mathbb{R}$ be an interval. Then the real-valued function $\psi : \mathcal{K} \to \mathbb{R}$ is said to be exponentially $\mathfrak{m}$-convex if the inequality

$$e^{\psi[\tau\varsigma_1+\mathfrak{m}(1-\tau)\varsigma_2]} \leq \tau e^{\psi(\varsigma_1)} + \mathfrak{m}(1-\tau)e^{\psi(\varsigma_2)}$$

grips for all $\varsigma_1, \varsigma_2 \in \mathcal{K}$ and $\tau \in [0, 1]$.

Now we introduce the concept of exponentially $(\hbar, \mathfrak{m})$-convex functions as follows:

Definition 1.4. Let $\mathcal{J} \subseteq \mathbb{R}$ be an interval such $(0, 1) \subseteq \mathcal{J}$ and $\hbar : \mathcal{J} \to \mathbb{R}$ be a nonnegative real-valued function. Then the nonnegative real-valued function $\psi : \mathcal{K} \longrightarrow [0, \infty)$ is said to be exponentially $(\hbar, \mathfrak{m})$-convex if the inequality

$$e^{\psi[\tau\varsigma_1+\mathfrak{m}(1-\tau)\varsigma_2]} \leq \hbar(\tau)e^{\psi(\varsigma_1)} + \mathfrak{m}\hbar(1-\tau)e^{\psi(\varsigma_2)}$$

grips for all $\varsigma_1, \varsigma_2 \in \mathcal{K}$ and $\tau \in [0, 1]$.

Definition 1.5. [16] Let $\psi \in L_1[\varsigma_1, \varsigma_2]$. The left and right sided Riemann-Liouville fractional integrals of order $u > 0$ with $\varsigma_1 \geq 0$ are given by

$$I^u_{\varsigma_1^+}\psi(\tau) = \frac{1}{\Gamma(u)}\int_{\varsigma_1}^{\tau}(\tau - \xi)^{u-1}\psi(\xi)d\xi \quad (\tau > \varsigma_1) \tag{1.3}$$

and

$$I^u_{\varsigma_2^-}\psi(\tau) = \frac{1}{\Gamma(u)}\int_{\tau}^{\varsigma_2}(\xi - \tau)^{u-1}\psi(\xi)d\xi \quad (\tau < \varsigma_2), \tag{1.4}$$

where $\Gamma(x) = \int_0^\infty \tau^{x-1}e^{-\tau}d\tau$ is the classical Gamma function.

We now give the definition of the extended generalized Mittag-Leffler functions which is mainly due to [15]:

Definition 1.6. Let $\nu, \theta, \jmath, \gamma, c \in \mathbb{C}$ such that $\Re(\nu), \Re(\theta), \Re(\jmath) > 0$ and $\Re(\mathfrak{c}) > \Re(\gamma) > 0$, $\rho \geq 0, \eta > 0$ and $0 < \zeta \leq \eta + \Re(\nu)$. Then the extended generalized Mittag-Leffler function $E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(t;\rho)$ is defined by

$$E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(\tau;\rho) = \sum_{n=0}^{\infty}\frac{\beta_\rho(\gamma + n\zeta, c-\gamma)(c)_{n\zeta}}{\beta(\gamma, c-\gamma)\Gamma(\nu n + \theta)}\frac{\tau^n}{(\jmath)_{n\eta}}, \tag{1.5}$$

where the generalized beta function defined by

$$\beta_\rho(\chi_1, \chi_2) = \int_0^1 \tau^{\chi_1 - 1}(1-\tau)^{\chi_2 - 1}e^{-\frac{\rho}{\tau(1-\tau)}}d\tau$$

and $(c)_{\mathfrak{n}\zeta} = \Gamma(c + \mathfrak{n}\zeta)/\Gamma(c)$ is the Pochhammer symbol.

Definition 1.7. Let $\psi \in L[\varsigma_1, \varsigma_2]$, $\nu, \theta, \jmath, \gamma, c \in \mathbb{C}$ such that $\Re(\nu), \Re(\theta)$, $\Re(\jmath) > 0, \Re(\mathfrak{c}) > \Re(\gamma) > 0$, $\rho \geq 0, \eta > 0$, $0 < \zeta \leq \eta + \Re(\nu)$. Then the extended generalized fractional integral operators $\epsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w,\varsigma_1^+}\psi$ and $\epsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w,\varsigma_2^-}\psi$ are defined by

$$\epsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w,\varsigma_1^+}\psi(x;\rho) = \int_{\varsigma_1}^{x}(x-\tau)^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w(x-\tau)^\nu;\rho)\psi(\tau)d\tau \tag{1.6}$$

and

$$\epsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w,\varsigma_2^-}\psi(x;\rho) = \int_{x}^{\varsigma_2}(\tau-x)^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w(\tau-x)^\nu;\rho)\psi(\tau)d\tau. \tag{1.7}$$

Remark 1.1. Equations (1.6) and (1.7) are the generalization of some fractional integral operators. Indeed, we have

(1) If $\rho = 0$, then we get the fractional integral operators defined by Salim and Faraj in [40];

(2) If $\jmath = \eta = 1$, then we get the fractional integral operators defined by Rahman et al. in [29];

(3) If $\rho = 0$ and $\jmath = \eta = 1$, then then we get the fractional integral operators defined by Srivastava and Tomovski in [42];

(4) If $\rho = 0$ and $\jmath = \eta = \zeta = 1$, then we get the fractional integral operators defined by Prabhakar in [28];

(5) if $\rho = w = 0$, then we get the two sided Riemann-Liouville fractional integrals.

The main motivation of this article is to figure the new HHI and HH-Fejér type inequalities for the exponentially $(\hbar, \mathfrak{m})$-convex functions by the use of an extended Mittag-lefflers function.

2. main results

Theorem 2.1. *Suppose that* $0 < \varsigma_1 < \varsigma_2$, $\mathfrak{m} \in (0,1]$ *and* $\psi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ *is a real-valued function such that* $\psi \in L_1[\varsigma_1, \varsigma_2]$. *If* ψ *is exponentially* $(\hbar, \mathfrak{m})$*-convex and* $\hbar \in L_1[0,1]$, *then the following inequalities for extended generalized fractional integral operators grips*

$$\begin{aligned} & e^{\psi\left(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2}\right)}\left(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',\varsigma_1^+}1\right)(\mathfrak{m}\varsigma_2;\rho) \qquad (2.1)\\ \leq\ & \hbar\left(\frac{1}{2}\right)\left[\left(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',\varsigma_1^+}e^{\psi}\right)\left(m\varsigma_2;\rho\right)+\mathfrak{m}^{\theta+1}\left(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'\mathfrak{m}^{\nu},\varsigma_2^-}e^{\psi}\right)\left(\frac{\varsigma_1}{\mathfrak{m}};\rho\right)\right]\\ \leq\ & \hbar\left(\frac{1}{2}\right)(\mathfrak{m}\varsigma_2-\varsigma_1)^{\theta}\Bigg[\mathfrak{m}\left(\mathfrak{m}e^{\psi\left(\frac{\varsigma_1}{\mathfrak{m}^2}\right)}+e^{\psi(\varsigma_2)}\right)\left(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',0^+}\hbar\right)(1;\rho)\\ & +\left(e^{\psi(\varsigma_1)}+\mathfrak{m}e^{\psi(\varsigma_2)}\right)\left(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',1^-}\hbar\right)(0;\rho)\Bigg], \end{aligned}$$

where $w' = \frac{w}{(\mathfrak{m}\varsigma_2-\varsigma_1)^{\theta}}$.

Proof. It follows the exponentially $(\hbar, \mathfrak{m})$-convexity of ψ that

$$e^{\psi\left(\frac{x\mathfrak{m}+y}{2}\right)} \leq \hbar\left(\frac{1}{2}\right)\left[\mathfrak{m}e^{\psi(x)}+e^{\psi(y)}\right]. \qquad (2.2)$$

Let $x=(1-\tau)\frac{\varsigma_1}{\mathfrak{m}}+\tau\varsigma_2$ and $y=\mathfrak{m}(1-\tau)\varsigma_2+\tau\varsigma_1$. Then (2.2) leads to

$$e^{\psi\left(\frac{\varsigma_2\mathfrak{m}+\varsigma_1}{2}\right)}\leq\hbar\left(\frac{1}{2}\right)\left[\mathfrak{m}e^{\psi\left((1-\tau)\frac{\varsigma_1}{\mathfrak{m}}+\tau\varsigma_2\right)}+e^{\psi\left(\mathfrak{m}(1-\tau)\varsigma_2+\tau\varsigma_1\right)}\right]. \tag{2.3}$$

If we multiply the above inequality by $\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}\big(w\tau^{\nu};\rho\big)$, we get

$$\begin{aligned}&e^{\psi\left(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2}\right)}\int_0^1\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}\big(w\tau^{\nu};\rho\big)d\tau \qquad (2.4)\\ \leq\ &\hbar\left(\frac{1}{2}\right)\left[\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}\big(w\tau^{\nu};\rho\big)\mathfrak{m}e^{\psi\left((1-\tau)\frac{\varsigma_1}{\mathfrak{m}}+\tau\varsigma_2\right)}d\tau\right.\\ &\left.+\int_0^1\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}\big(w\tau^{\nu};\rho\big)e^{\psi\left(\mathfrak{m}(1-\tau)\varsigma_2+\tau\varsigma_1\right)}d\tau\right].\end{aligned}$$

Letting in the above $x=(1-\tau)\frac{\varsigma_1}{\mathfrak{m}}+\tau\varsigma_2$ and $y=\mathfrak{m}(1-\tau)\varsigma_2+\tau\varsigma_1$, then using (1.6) and (1.7), w have

$$\begin{aligned}&e^{\psi\left(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2}\right)}\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',\varsigma_1^+}1\big)\big(\mathfrak{m}\varsigma_2;\rho\big) \qquad (2.5)\\ \leq\ &\hbar\left(\frac{1}{2}\right)\left[\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',\varsigma_1^+}e^{\psi}\big)\big(\mathfrak{m}\varsigma_2;\rho\big)+\mathfrak{m}^{\theta+1}\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'\mathfrak{m}^{\nu},\varsigma_2^-}e^{\psi}\big)\left(\frac{\varsigma_1}{\mathfrak{m}};\rho\right)\right].\end{aligned}$$

Again by the exponentially $(\hbar,m)$-convexity of ψ, we obtain

$$\begin{aligned}&e^{\psi\left(\mathfrak{m}(1-\tau)\varsigma_2+\tau\varsigma_1\right)}+\mathfrak{m}e^{\psi\left((1-\tau)\frac{\varsigma_1}{\mathfrak{m}}+\tau\varsigma_2\right)} \qquad (2.6)\\ \leq\ &\mathfrak{m}^2\hbar(1-\tau)e^{\psi\left(\frac{\varsigma_1}{\mathfrak{m}^2}\right)}+\mathfrak{m}\hbar(\tau)e^{\psi(\varsigma_2)}+\mathfrak{m}\hbar(1-\tau)e^{\psi(\varsigma_2)}+\hbar(\tau)e^{\psi(\varsigma_1)}\\ =\ &\hbar(\tau)\left(\mathfrak{m}e^{\psi(\varsigma_2)}+e^{\psi(\varsigma_1)}\right)+\mathfrak{m}\hbar(1-\tau)\left(\mathfrak{m}e^{\psi\left(\frac{\varsigma_1}{\mathfrak{m}^2}\right)}+e^{\psi(\varsigma_2)}\right).\end{aligned}$$

If we multiply (2.6) by $\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}\big(w\tau^{\nu};\rho\big)$ on both sides, then integrating over $[0,1]$, we have

$$\begin{aligned}&\hbar\left(\frac{1}{2}\right)\left[\int_0^1\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}\big(wt^{\nu};\rho\big)\mathfrak{m}e^{\psi\left((1-\tau)\frac{\varsigma_1}{\mathfrak{m}}+\tau\varsigma_2\right)}d\tau+\int_0^1\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}\big(w\tau^{\nu};\rho\big)e^{\psi\left(\mathfrak{m}(1-\tau)\varsigma_2+\tau\varsigma_1\right)}d\tau\right]\\ \leq\ &\hbar\left(\frac{1}{2}\right)\left[\mathfrak{m}\left(\mathfrak{m}e^{\psi\left(\frac{\varsigma_1}{\mathfrak{m}^2}\right)}+e^{\psi(\varsigma_2)}\right)\int_0^1\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}\big(w\tau^{\nu};\rho\big)\hbar(1-\tau)d\tau\right.\\ &\left.+\big(e^{\psi(\varsigma_1)}+\mathfrak{m}e^{\psi(\varsigma_2)}\big)\int_0^1\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}\big(w\tau^{\nu};\rho\big)\hbar(\tau)d\tau\right]. \qquad (2.7)\end{aligned}$$

By using (1.6) and (1.7), we get

$$\hbar\Big(\frac{1}{2}\Big)\Big[\mathfrak{m}^{\theta+1}\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'\mathfrak{m}^{\nu},\varsigma_2^-}e^{\psi}\big)\big(\frac{\varsigma_1}{\mathfrak{m}};\rho\big)+\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'\mathfrak{m}^{\nu},\varsigma_1^+}e^{\psi}\big)\big(\mathfrak{m}\varsigma_2;\rho\big)\Big] \quad (2.8)$$
$$\leq \hbar\Big(\frac{1}{2}\Big)(\mathfrak{m}\varsigma_2-\varsigma_1)^{\theta}\Big[\mathfrak{m}\Big(\mathfrak{m}e^{\psi\left(\frac{\varsigma_1}{\mathfrak{m}^2}\right)}+e^{\psi(\varsigma_2)}\Big)\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',0^+}\hbar\big)(1;\rho)$$
$$+\big(e^{\psi(\varsigma_1)}+\mathfrak{m}e^{\psi(\varsigma_2)}\big)\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',1^-}\hbar\big)(0;\rho)\Big].$$

From the above inequality and (2.5), establishing the result (2.1). □

Corollary 2.1. *If we put $\rho=0$ in (2.1), then we have*

$$e^{\psi\left(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2}\right)}\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',\varsigma_1^+}1\big)\big(\mathfrak{m}\varsigma_2\big)$$
$$\leq \hbar\Big(\frac{1}{2}\Big)\Big[\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',\varsigma_1^+}e^{\psi}\big)\big(\mathfrak{m}\varsigma_2\big)+\mathfrak{m}^{\theta+1}\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',\varsigma_2^-}e^{\psi}\big)\big(\frac{\varsigma_1}{m}\big)\Big]$$
$$\leq \hbar\Big(\frac{1}{2}\Big)\Big[\mathfrak{m}\Big(\mathfrak{m}e^{\psi\left(\frac{\varsigma_1}{\mathfrak{m}^2}\right)}+e^{\psi(\varsigma_2)}\Big)\int_0^1\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w\tau^{\nu})\hbar(1-\tau)d\tau$$
$$+\big(e^{\psi(\varsigma_1)}+\mathfrak{m}e^{\psi(\varsigma_2)}\big)\int_0^1\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w\tau^{\nu})\hbar(\tau)d\tau\Big],$$

where $w'=\frac{w}{(\mathfrak{m}\varsigma_2-\varsigma_1)^{\theta}}$.

Corollary 2.2. *If we put $\hbar(\tau)=\tau,\mathfrak{m}=1$ and $\rho=0$ in (2.1), then we get*

$$e^{\psi\left(\frac{\varsigma_1+\varsigma_2}{2}\right)}\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',\varsigma_1^+}1\big)\big(\varsigma_2\big)$$
$$\leq \frac{\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',\varsigma_1^+}e^{\psi}\big)\big(\varsigma_2\big)+\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',\varsigma_2^-}e^{\psi}\big)\big(\varsigma_1\big)}{2}$$
$$\leq \frac{e^{\psi(\varsigma_1)}+e^{\psi(\varsigma_2)}}{2}\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',\varsigma_2^-}1\big)\big(\varsigma_1\big),$$

where $w'=\frac{w}{(\mathfrak{m}\varsigma_2-\varsigma_1)^{\theta}}$.

Corollary 2.3. *If we put* $\hbar(\tau) = \tau$ *and* $\rho = 0$ *in* (2.1), *then one has*

$$\begin{aligned} & e^{\psi\left(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2}\right)}\left(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',\varsigma_1^+}1\right)\left(\mathfrak{m}\varsigma_2\right) \\ \leq \ & \mathfrak{m}^{\theta+1}\frac{e^{\psi(\varsigma_2)}+\mathfrak{m}e^{\psi\left(\frac{\varsigma_1}{\mathfrak{m}}\right)}}{2}\left(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta+1,\jmath,w'\mathfrak{m}^\nu,\varsigma_2^-}1\right)\left(\frac{\varsigma_1}{\mathfrak{m}}\right), \end{aligned}$$

where $w' = \frac{w}{(\mathfrak{m}\varsigma_2-\varsigma_1)^\theta}$

Corollary 2.4. *If we put* $\hbar(\tau) = \tau, \mathfrak{m} = 1$ *and* $\rho = w = 0$ *in* (2.1), *then we get*

$$e^{\psi\left(\frac{\varsigma_1+\varsigma_2}{2}\right)} \leq \frac{\Gamma(\theta+1)}{2(\varsigma_2-\varsigma_1)^\theta}\left[J^\theta_{\varsigma_1^+}e^{\psi(\varsigma_2)} + J^\theta_{\varsigma_2^-}e^{\psi(\varsigma_1)}\right] \leq \frac{e^{\psi(\varsigma_1)}+e^{\psi(\varsigma_2)}}{2}$$

for $\theta > 0$.

Remark 2.1. Letting $\hbar(\tau) = \tau$ and $\rho = w = 0$ in (2.1), then we attain Theorem 3.1 in [33].

Remark 2.2. If we put $\hbar(\tau) = \tau, \mathfrak{m} = 1, \theta = 1$ and $\rho = w = 0$ in (2.1), then we get Theorem 2.1 in [22].

Theorem 2.2. *Suppose that* $0 < \varsigma_1 < \varsigma_2$, $\mathfrak{m} \in (0, 1]$ *and* $\psi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ *be a real-valued function such that* $\psi \in L_1[\varsigma_1, \varsigma_2]$. *If* ψ *is an exponentially* $(\hbar, \mathfrak{m})$*-convex function, then the following inequalities for the extended generalized fractional integral operators grips*

$$\begin{aligned} & e^{\psi\left(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2}\right)}\left(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'2^\nu,\left(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2}\right)^+}1\right)(\mathfrak{m}\varsigma_2;\rho) \qquad (2.9) \\ \leq \ & \hbar\left(\frac{1}{2}\right)\left[\left(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'2^\nu,\left(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2}\right)^+}e^\psi\right)(\mathfrak{m}\varsigma_2;\rho) + \mathfrak{m}^{\theta+1}\left(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'(2\mathfrak{m})^\nu,\left(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2\mathfrak{m}}\right)^-}e^\psi\right)\left(\frac{\varsigma_1}{\mathfrak{m}};\rho\right)\right] \\ \leq \ & \hbar\left(\frac{1}{2}\right)\frac{(\mathfrak{m}\varsigma_2-\varsigma_1)^\theta}{2^\theta}\left[\mathfrak{m}\left(\mathfrak{m}e^{\psi\left(\frac{\varsigma_1}{\mathfrak{m}^2}\right)}+e^{\psi(\varsigma_2)}\right)\int_0^1 \tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w\tau^\nu;\rho)\hbar\left(\frac{2-\tau}{2}\right)d\tau \right. \\ & \left. +\left[e^{\psi(\varsigma_1)}+\mathfrak{m}e^{\psi(\varsigma_2)}\right]\int_0^1 \tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w\tau^\nu;\rho)\hbar\left(\frac{\tau}{2}\right)d\tau\right], \end{aligned}$$

where $w' = \frac{w}{(\mathfrak{m}\varsigma_2-\varsigma_1)^\nu}$.

Proof. Since ψ is an exponentially $(\hbar, \mathfrak{m})$-convex function, we obtain

$$e^{\psi\left(\frac{x\mathfrak{m}+y}{2}\right)} \leq \hbar\left(\frac{1}{2}\right)\left[\mathfrak{m}e^{\psi(x)}+e^{\psi(y)}\right].$$

Substituting in the above $x = \frac{(2-\tau)}{2}\frac{\varsigma_1}{\mathfrak{m}} + \frac{\tau}{2}\varsigma_2$ and $y = \mathfrak{m}\frac{(2-\tau)}{2}\varsigma_2 + \frac{\tau}{2}\varsigma_1$, we get

$$e^{\psi\left(\frac{\varsigma_1+m\varsigma_2}{2}\right)} \leq \hbar\Big(\frac{1}{2}\Big)\left[e^{\psi\left(\frac{\tau}{2}\varsigma_1+\frac{2-\tau}{2}\mathfrak{m}\varsigma_2\right)} + \mathfrak{m}e^{\psi\left(\frac{2-\tau}{2\mathfrak{m}}\varsigma_1+\frac{\tau}{2}\varsigma_2\right)}\right]. \quad (2.10)$$

Multiplying (2.10) by $\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w\tau^{\nu};\rho)$ on both sides and then integrating over $[0,1]$, we have

$$e^{\psi\left(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2}\right)}\int_0^1 \tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w\tau^{\nu};\rho)d\tau$$

$$\leq \hbar\Big(\frac{1}{2}\Big)\Bigg[\int_0^1 \tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w\tau^{\nu};\rho)\mathfrak{m}e^{\psi\left(\frac{\tau}{2}\varsigma_1+\frac{2-\tau}{2}\mathfrak{m}\varsigma_2\right)}d\tau$$

$$+\int_0^1 \tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w\tau^{\nu};\rho)e^{\psi\left(\frac{2-\tau}{2}\frac{\varsigma_1}{\mathfrak{m}}+\frac{\tau}{2}\varsigma_2\right)}d\tau\Bigg]. \quad (2.11)$$

Putting $u=\frac{\tau}{2}\varsigma_2+\frac{(2-\tau)}{2}\frac{\varsigma_1}{\mathfrak{m}}$ and $v=\mathfrak{m}\frac{(2-\tau)}{2}\varsigma_2+\frac{\tau}{2}\varsigma_1$, then using (1.6) and (1.7), we get

$$e^{\left(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2}\right)}\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'2^{\nu},(\frac{\varsigma_1+\mathfrak{m}\varsigma_1}{2})^+}1\big)(\mathfrak{m}\varsigma_2;\rho)$$

$$\leq \hbar\Big(\frac{1}{2}\Big)\Big[\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'2^{\nu},(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2})^+}e^{\psi}\big)(\mathfrak{m}\varsigma_2;\rho)+\mathfrak{m}^{\theta+1}\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'(2m)^{\nu},(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2\mathfrak{m}})^-}e^{\psi}\big)\Big(\frac{\varsigma_1}{\mathfrak{m}};\rho\Big)\Big].$$

Again by using the exponentially $(\hbar,\mathfrak{m})$-convexity of ψ, we have

$$\begin{aligned}
& e^{\psi\left(\frac{\tau}{2}\varsigma_1+\frac{2-\tau}{2}\mathfrak{m}\varsigma_2\right)}+\mathfrak{m}e^{\psi\left(\frac{2-\tau}{2\mathfrak{m}}\varsigma_1+\frac{\tau}{2}\varsigma_2\right)} \qquad (2.12)\\
\leq\ & \hbar\Big(\frac{\tau}{2}\Big)e^{\psi(\varsigma_1)}+\mathfrak{m}\hbar\Big(\frac{2-\tau}{2}\Big)e^{\psi(\varsigma_2)}+\mathfrak{m}\hbar\Big(\frac{\tau}{2}\Big)e^{\psi(\varsigma_2)}+\mathfrak{m}^2\hbar\Big(\frac{2-\tau}{2}\Big)e^{\psi\left(\frac{\varsigma_1}{\mathfrak{m}^2}\right)}\\
=\ & \big[e^{\psi(\varsigma_1)}+\mathfrak{m}e^{\psi(\varsigma_2)}\big]\hbar\Big(\frac{\tau}{2}\Big)+\mathfrak{m}\hbar\left(\frac{2-\tau}{2}\right)\left[\mathfrak{m}e^{\psi\left(\frac{\varsigma_1}{\mathfrak{m}^2}\right)}+e^{\psi(\varsigma_2)}\right].
\end{aligned}$$

Multiplying (2.12) by $\hbar(\frac{1}{2})\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w\tau^{\nu};\rho)$ on both sides, then integrating over $[0,1]$, we have

$$\hbar\Big(\frac{1}{2}\Big)\Big[\int_0^1 \tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w\tau^{\nu};\rho)e^{\psi\left(\frac{\tau}{2}\varsigma_1+\frac{2-\tau}{2}\mathfrak{m}\varsigma_2\right)}d\tau+\mathfrak{m}\int_0^1 \tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w\tau^{\nu};\rho)e^{\psi\left(\frac{2-\tau}{2\mathfrak{m}}\varsigma_1+\frac{\tau}{2}\varsigma_2\right)}d\tau\Big]$$
$$\leq \hbar\Big(\frac{1}{2}\Big)\Big[\big[e^{\psi(\varsigma_1)}+\mathfrak{m}e^{\psi(\varsigma_2)}\big]\int_0^1 \tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w\tau^{\nu};\rho)\hbar\Big(\frac{\tau}{2}\Big)d\tau$$
$$+\mathfrak{m}\int_0^1 \tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w\tau^{\nu};\rho)\hbar\Big(\frac{2-\tau}{2}\Big)d\tau\Big(\mathfrak{m}e^{\psi\left(\frac{\varsigma_1}{\mathfrak{m}^2}\right)}+e^{\psi(\varsigma_2)}\Big)\Big]. \quad (2.13)$$

By using (1.7) and (1.6), we get

$$\hbar\Big(\frac{1}{2}\Big)\Big[\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'2^{\nu},(\frac{\varsigma_1+m\varsigma_2}{2})^+}e^{\psi}\big)(\mathfrak{m}\varsigma_2;\rho)+\mathfrak{m}^{\theta+1}\big(\varepsilon_{\nu,\theta,\jmath,w'(2\mathfrak{m})^{\nu},(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2\mathfrak{m}})^-}e^{\psi}\big)\Big(\frac{\varsigma_1}{\mathfrak{m}};\rho\Big)\Big] \quad (2.14)$$
$$\leq \hbar\Big(\frac{1}{2}\Big)\frac{(\mathfrak{m}\varsigma_2-\varsigma_1)^{\theta}}{2^{\theta}}\Big[\mathfrak{m}\Big(\mathfrak{m}e^{\psi\left(\frac{\varsigma_1}{\mathfrak{m}^2}\right)}+e^{\psi(\varsigma_2)}\Big)\int_0^1 \tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w\tau^{\nu};\rho)\hbar\Big(\frac{2-\tau}{2}\Big)d\tau$$
$$+\big[e^{\psi(\varsigma_1)}+\mathfrak{m}e^{\psi(\varsigma_2)}\big]\int_0^1 \tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w\tau^{\nu};\rho)\hbar\Big(\frac{\tau}{2}\Big)d\tau\Big].$$

From the inequality (2.14), we get the required inequality (2.9). □

Corollary 2.5. *If we put $\rho=0$ in* (2.9), *then*

$$e^{\psi\left(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2}\right)}\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'2^{\nu},(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2})^+}1\big)(\mathfrak{m}\varsigma_2)$$
$$\leq \hbar\Big(\frac{1}{2}\Big)\Big[\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'2^{\nu},(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2})^+}e^{\psi}\big)(\mathfrak{m}\varsigma_2)+\mathfrak{m}^{\theta+1}\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'(2\mathfrak{m})^{\nu},(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2\mathfrak{m}})^-}e^{\psi}\big)\Big(\frac{\varsigma_1}{\mathfrak{m}}\Big)\Big]$$
$$\leq \hbar\Big(\frac{1}{2}\Big)\frac{(\mathfrak{m}\varsigma_2-\varsigma_1)^{\theta}}{2^{\theta}}\Big[\mathfrak{m}\Big(\mathfrak{m}e^{\psi\left(\frac{\varsigma_1}{\mathfrak{m}^2}\right)}+e^{\psi(\varsigma_2)}\Big)\int_0^1 \tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w\tau^{\nu};\rho)\hbar\Big(\frac{2-\tau}{2}\Big)d\tau$$
$$+\big[e^{\psi(\varsigma_1)}+\mathfrak{m}e^{\psi(\varsigma_2)}\big]\int_0^1 \tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}(w\tau^{\nu};\rho)\hbar\Big(\frac{\tau}{2}\Big)d\tau\Big],$$

where $w'=\frac{w}{(\mathfrak{m}\varsigma_2-u)^{\nu}}$.

COROLLARY 2.6. *If we put $\hbar(\tau)=\tau, \mathfrak{m}=1$ and $\rho=0$ in* (2.9), *then we get*

$$\begin{aligned} & e^{\psi\left(\frac{\varsigma_1+\varsigma_2}{2}\right)}\left(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',\varsigma_1^+}1\right)(\varsigma_2) \\ \leq\ & \frac{1}{2}\left[\left(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',(\frac{\varsigma_1+\varsigma_2}{2})^-}e^{\psi}\right)(\varsigma_1)+\left(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',(\frac{\varsigma_1+\varsigma_2}{2})^+}e^{\psi}\right)(\varsigma_2)\right] \\ \leq\ & \frac{e^{\psi(\varsigma_1)}+e^{\psi(\varsigma_2)}}{2}\left(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w',\varsigma_2^+}1\right)(\varsigma_1). \end{aligned}$$

COROLLARY 2.7. *Letting $\hbar(\tau)=\tau, m=1$ and $\rho=w=0$ in* (2.9), *then we get*

$$\begin{aligned} e^{\psi\left(\frac{\varsigma_1+\varsigma_2}{2}\right)} &\leq \frac{2^{\theta+1}(\Gamma(\theta+1))}{(\varsigma_2-\varsigma_1)^{\theta}}\left[I^{\theta}_{(\frac{\varsigma_1+\varsigma_1}{2})^+}e^{\psi(\varsigma_2)}+I^{\theta}_{(\frac{\varsigma_1+\varsigma_2}{2})^-}e^{\psi(\varsigma_1)}\right] \\ &\leq \frac{e^{\psi(\varsigma_1)}+e^{\psi(\varsigma_2)}}{2}. \end{aligned} \tag{2.15}$$

THEOREM 2.3. *Suppose that $\mathfrak{m}\in(0,1]$, $\hbar\in L[0,1]$, $0<\varsigma_1<\varsigma_2$, $\psi:[\varsigma_1,\varsigma_2]\to\mathbb{R}$ and $g:[\varsigma_1,\varsigma_2]\to[0,\infty)$ are two real-valued functions such that $\psi,g\in L[\varsigma_1,\varsigma_2]$ and $\psi(x)=\psi(\varsigma_1+\mathfrak{m}\varsigma_2-\mathfrak{m}x)$. If ψ is an exponentially $(\hbar,\mathfrak{m})$-convex function, then we have the following inequalities for extended generalized fractional integral:*

$$\begin{aligned} & e^{\psi\left(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2}\right)}\left(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'\mathfrak{m}^{\nu},\varsigma_2^-}e^{g}\right)\left(\frac{\varsigma_1}{\mathfrak{m}};\rho\right) \qquad (2.16) \\ \leq\ & \hbar\left(\frac{1}{2}\right)(\mathfrak{m}+1)\left(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'\mathfrak{m}^{\nu},\varsigma_2^-}e^{\psi g}\right)\left(\frac{\varsigma_1}{\mathfrak{m}};\rho\right) \\ \leq\ & \hbar\left(\frac{1}{2}\right)\frac{(\mathfrak{m}\varsigma_2-\varsigma_1)^{\theta}}{\mathfrak{m}^{\theta}}\Bigg[m\left(\mathfrak{m}e^{\psi(\frac{\varsigma_1}{\mathfrak{m}^2})}+e^{\psi(\varsigma_2)}\right)\left(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w,0^+}\hbar\right)(1;\rho) \\ & +\left(\mathfrak{m}e^{\psi(\varsigma_2)}+e^{\psi(\varsigma_1)}\right)\left(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w,1^-}\hbar\right)(0;\rho)\Bigg]. \end{aligned}$$

Proof. Since ψ is an exponentially $(\hbar,\mathfrak{m})$-convex, we have

$$e^{\psi\left(\frac{x\mathfrak{m}+y}{2}\right)}\leq\hbar\left(\frac{1}{2}\right)\left[\mathfrak{m}e^{\psi(x)}+e^{\psi(y)}\right].$$

Putting in the above $x=(1-\tau)\frac{\varsigma_1}{\mathfrak{m}}+\tau\varsigma_2$ and $y=\mathfrak{m}(1-\tau)\varsigma_2+\tau\varsigma_1$, we get

$$e^{\psi\left(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2}\right)}\leq\hbar\left(\frac{1}{2}\right)\left[\mathfrak{m}e^{\psi\left((1-\tau)\frac{\varsigma_1}{\mathfrak{m}}+\tau\varsigma_2\right)}+e^{\psi\left(\mathfrak{m}(1-\tau)\varsigma_2+\tau\varsigma_1\right)}\right]. \tag{2.17}$$

Multiplying (2.17) by $\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}\big(w\tau^{\nu};\rho\big)e^{g(\tau\varsigma_2+(1-\tau)\frac{\varsigma_1}{m})}$ on both sides, then integrating over $[0,1]$, we have

$$\begin{aligned} & e^{\psi\left(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2}\right)}\int\limits_0^1\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}\big(w\tau^{\nu};\rho\big)e^{g(\tau\varsigma_2+(1-\tau)\frac{\varsigma_1}{\mathfrak{m}})}d\tau \\ \leq\ & \hbar\Big(\frac{1}{2}\Big)\Bigg[\int\limits_0^1\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}\big(w\tau^{\nu};\rho\big)e^{g(\tau\varsigma_2+(1-\tau)\frac{\varsigma_1}{\mathfrak{m}})}\mathfrak{m}e^{\psi\left((1-\tau)\frac{\varsigma_1}{\mathfrak{m}}+\tau\varsigma_2\right)}d\tau \\ & +\int\limits_0^1\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}\big(w\tau^{\nu};\rho\big)e^{g(\tau\varsigma_2+(1-\tau)\frac{\varsigma_1}{\mathfrak{m}})}e^{\psi\left(\mathfrak{m}(1-\tau)\varsigma_2+\tau\varsigma_1\right)}d\tau\Bigg]. \end{aligned}$$

Putting $x=\tau\varsigma_2+(1-\tau)\frac{\varsigma_1}{\mathfrak{m}}$ in the above, then by the assumption $\psi(x)=\psi(\varsigma_1+\mathfrak{m}\varsigma_2-\mathfrak{m}x)$, we get

$$e^{\psi\left(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2}\right)}\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'\mathfrak{m}^{\nu},\varsigma_2^-}e^{g}\big)\big(\frac{\varsigma_1}{\mathfrak{m}};\rho\big)\leq\hbar\big(\frac{1}{2}\big)(\mathfrak{m}+1)\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'\mathfrak{m}^{\nu},\varsigma_2^-}e^{\psi g}\big)\big(\frac{\varsigma_1}{\mathfrak{m}};\rho\big).$$

Again by using exponentially $(\hbar,m)$-convexity of ψ, we have

$$\begin{aligned} & e^{\psi\left(\mathfrak{m}(1-\tau)\varsigma_2+\tau\varsigma_1\right)}+\mathfrak{m}e^{\psi\left((1-\tau)\frac{\varsigma_1}{\mathfrak{m}}+\tau\varsigma_2\right)} \\ \leq\ & \hbar(\tau)\bigg(\mathfrak{m}e^{\psi(\varsigma_2)}+e^{\psi(\varsigma_1)}\bigg)+\mathfrak{m}\hbar(1-\tau)\bigg(\mathfrak{m}e^{\psi\left(\frac{\varsigma_1}{\mathfrak{m}^2}+e^{\psi(\varsigma_2)}\right)}\bigg). \end{aligned}\tag{2.18}$$

Multiplying (2.18) by $\hbar\big(\frac{1}{2}\big)\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}\big(w\tau^{\nu};\rho\big)e^{g(\tau\varsigma_2+(1-\tau)\frac{\varsigma_1}{\mathfrak{m}})}$ on both sides, then integrating over $[0,1]$, we have

$$\begin{aligned} & \hbar\Big(\frac{1}{2}\Big)\Bigg[\int\limits_0^1\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}\big(w\tau^{\nu};\rho\big)e^{g(\tau\varsigma_2+(1-\tau)\frac{\varsigma_1}{\mathfrak{m}})}e^{\psi\left((1-\tau)\mathfrak{m}\varsigma_2+\tau\varsigma_1\right)}d\tau \\ & +\mathfrak{m}\int\limits_0^1\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}\big(w\tau^{\nu};\rho\big)e^{g(\tau\varsigma_2+(1-\tau)\frac{\varsigma_1}{\mathfrak{m}})}e^{\psi\left((1-\tau)\frac{\varsigma_1}{\mathfrak{m}}+\tau\varsigma_2\right)}d\tau\Bigg] \\ \leq\ & \hbar\Big(\frac{1}{2}\Big)\Bigg[\bigg(\mathfrak{m}e^{\psi(\varsigma_2)}+e^{\psi(\varsigma_1)}\bigg)\int\limits_0^1\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}\big(w\tau^{\nu};\rho\big)\hbar(\tau)d\tau \\ & +\mathfrak{m}\bigg(\mathfrak{m}e^{\psi\left(\frac{\varsigma_1}{\mathfrak{m}^2}\right)}+e^{\psi(\varsigma_2)}\bigg)\int\limits_0^1\tau^{\theta-1}E^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath}\big(w\tau^{\nu};\rho\big)\hbar(1-\tau)d\tau\Bigg]. \end{aligned}$$

By using (1.6) and (1.7), we get

$$\begin{aligned}
&\hbar\Big(\frac{1}{2}\Big)(\mathfrak{m}+1)\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'\mathfrak{m}^{\nu},b^{-}}e^{\psi g}\big)\big(\frac{\varsigma_1}{\mathfrak{m}};\rho\big)\\
\leq\ &\hbar\Big(\frac{1}{2}\Big)\frac{(\mathfrak{m}\varsigma_2-\varsigma_1)^{\theta}}{\mathfrak{m}^{\theta}}\Bigg[m\Bigg(\mathfrak{m}e^{\psi(\frac{\varsigma_1}{\mathfrak{m}^2})}+e^{\psi(\varsigma_2)}\Bigg)\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w,0^{+}}\hbar\big)\big(1;\rho\big)\\
&+\big(\mathfrak{m}e^{\psi(\varsigma_2)}+e^{\psi(\varsigma_1)}\big)\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w,1^{-}}\hbar\big)\big(0;\rho\big)\Bigg].
\end{aligned}$$

From the above inequality and (2.18), we get the required inequality (2.16). □

COROLLARY 2.8. *If we put* $\rho=0$ *in* (2.16), *then we have*

$$\begin{aligned}
&e^{\psi\left(\frac{\varsigma_1+\mathfrak{m}\varsigma_2}{2}\right)}\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'\mathfrak{m}^{\nu},\varsigma_2^{-}}e^{g}\big)\big(\frac{\varsigma_1}{\mathfrak{m}}\big)\\
&\leq\hbar\Big(\frac{1}{2}\Big)(\mathfrak{m}+1)\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w'\mathfrak{m}^{\nu},\varsigma_2^{-}}e^{\psi g}\big)\big(\frac{\varsigma_1}{\mathfrak{m}}\big)\\
&\leq\hbar\Big(\frac{1}{2}\Big)\frac{(\mathfrak{m}\varsigma_2-\varsigma_1)^{\theta}}{\mathfrak{m}^{\theta}}\Bigg[\mathfrak{m}\Bigg(\mathfrak{m}e^{\psi(\frac{\varsigma_1}{\mathfrak{m}^2})}+e^{\psi(\varsigma_2)}\Bigg)\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w,0^{+}}\hbar\big)\big(1\big)\\
&+\big(\mathfrak{m}e^{\psi(\varsigma_2)}+e^{\psi(\varsigma_1)}\big)\big(\varepsilon^{\gamma,\eta,\zeta,c}_{\nu,\theta,\jmath,w,1^{-}}\hbar\big)\big(0\big)\Bigg].
\end{aligned}$$

By applying Theorem 2.3 similar relations can be established we leave it for the readers.

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Conflicts of Interest:

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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