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A STUDY ON THE QUASI TOPOS

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Abstract. Category *FRel* of fuzzy sets and relations does not form a topos. J. Harding, C. Walker and E. Walker [3] showed that *FRel* has a tensor product and V. Durov [1] introduced basic definitions related to the notion of vectoid endowed with a tensor product. In this paper, we show that *FRel* forms a quasi topos. Also we show that there are quasi power objects in *FRel*. And by the use of the concepts of *FRel* and quasi topos, we get the logic operators of *FRel*. Moreover, we show that *FRel* forms a vectoid.

1. Introduction

Category *FRel* of fuzzy sets and relations does not form a topos. J. Harding, C. Walker and E. Walker [3] showed that *FRel* has a tensor product and V. Durov [1] introduced basic definitions related to the notion of vectoid endowed with a tensor product. In this paper, we introduce the concepts of quasi monomorphism, quasi middle object, quasi exponential, quasi membership morphism, quasi subobject classifier, quasi topos and quasi power object. And we show that quasi middle object, equalizers, quasi exponentials and quasi subobject classifier exist in *FRel*. So *FRel* forms a quasi topos. Also we show that quasi power objects exist in *FRel*. And by the use of the concepts of *FRel* and quasi topos, we get the logic operators such as negation, conjunction,

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disjunction and implication of *FRel*. Moreover, we show that arbitrary small colimits exist in *FRel*, the bifunctor is cocontinuous, finite limits exist in *FRel*, epimorphisms are universally effective in *FRel*, all equivalence relations in *FRel* are efficient, generators exist in *FRel* and *FRel* is complete. Thus *FRel* forms a vectoid.

2. Preliminaries

In this section, we state some definitions and properties which will serve as the basic tools for the arguments used to prove our results.

Definition 2.1. An *elementary topos* is a category $\mathcal{E}$ that satisfies the following conditions:
(T1) $\mathcal{E}$ is finitely complete.
(T2) $\mathcal{E}$ has exponentials.
(T3) $\mathcal{E}$ has a subobject classifier.

Example 2.2. Category *Set* is a topos. $\{*\}$ is a terminal object, where $\{*\}$ is a singleton set, and $\Omega = \{0,1\}$ together with $\top : \{*\} \to \Omega$ defined by $\top(*) = 1$ is a subobject classifier. If we define

$$\chi_h(c) = \begin{cases} 1, & if\ c = h(d) \\ 0, & otherwise \end{cases}$$

then χ_h is the characteristic function of the monomorphism $h : D \to C$.

Category *FRel* of fuzzy sets and relations is a category whose object is (A, P_A) where A is a set and $P_A : A \to I$ is a function with $I = [0,1]$ in *Set* and morphism from (A, P_A) to (B, P_B) is a relation $r \subseteq A \times B$ satisfying $P_A(a) \leqq P_B(b)$ for all $(a, b) \in r$ (equivalently $P_A(a) \leq P_B \circ r(a)$ for all $a \in A$).

Definition 2.3. An object (M, P_M) is called a quasi middle object if for any object (A, P_A), there exists a unique morphism $r : A \to M$ such that $(a, m) \in r$ and $P_A(a) = P_M \circ r(a)$ for all $a \in A$.

Definition 2.4. A morphism $r : (X, P_X) \to (Y, P_Y)$ is called a quasi monomorphism if $(x, y) \in r$ for all $x \in X$ and $r : (X, P_X) \to (Y, P_Y)$ is a monomorphism.

Definition 2.5. A triangular norm is a function $t : I \times I \to I$, that is order preserving in both coordinates and satisfies the following conditions:

(1) $t(x, y) = t(y, x)$.
(2) $t(x, t(y, z)) = t(t(x, y), z)$.
(3) $t(1, x) = x$.

Lemma 2.6. *For any triangular norm* t *on* I*, there is a tensor product* $\otimes$ *on FRel defined as follows:*

1. $(X, P_X) \otimes (Y, P_Y) = (X \times Y, t(P_X, P_Y))$ *where*

$$t(P_X, P_Y) = t \circ (P_X \times P_Y) = min\{P_X, P_Y\}.$$

2. $r \otimes s$ *is the ordinary product relation* $r \times s$.
3. *The tensor unit is* $(\{*\}, P_{\{*\}})$ *with* $P_{\{*\}}(*) = 1$.

Proof. See [3]. □

Definition 2.7. $\mathcal{E}$ has quasi exponentials if for any objects A and B in $\mathcal{E}$ with tensor products, there exists an object B^A and a morphism $ev_A : B^A \otimes A \to B$, called a quasi evaluation morphism of A, such that for any Y and $f : Y \otimes A \to B$ in $\mathcal{E}$, there exists a unique morphism g such that the following diagram

$$\begin{tikzcd} Y \otimes A \arrow[r, "f"] \arrow[d, "g \otimes id_A"'] & B \arrow[d, "id_B"] \\ B^A \otimes A \arrow[r, "ev_A"'] & B \end{tikzcd}$$

commutes.

Definition 2.8. If $\mathcal{E}$ is a category with a quasi middle object M, then a quasi subobject classifier is an object C together with $k : M \to C$ such that for any quasi monomorphism $f : A \to D$, there exists a unique morphism $q_f : D \to C$ such that the following diagram

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, "f"'] & M \arrow[d, "k"] \\ D \arrow[r, "q_f"'] & C \end{tikzcd}$$

is a pullback.

Definition 2.9. A quasi topos is a category $\mathcal{E}$ that satisfies the following conditions:

(QT1) $\mathcal{E}$ has a quasi middle object, equalizers and finite tensor products.

(QT2) $\mathcal{E}$ has quasi exponentials.

(QT3) $\mathcal{E}$ has a quasi subobject classifier.

Definition 2.10. A category $\mathcal{E}$ with tensor products is said to have quasi power objects if for any object A there are objects $P(A)$ and ϵ_A, and a quasi monomorphism $\epsilon : \epsilon_A \to P(A) \otimes A$ such that for any object B and quasi monomorphism $r : R \to B \otimes A$ there is a unique morphism $f_r : B \to P(A)$ such that the following diagram

$$\begin{tikzcd} R \arrow[r, "f_r \otimes id_A|_R"] \arrow[d, "r"'] & \epsilon_A \arrow[d, "\epsilon"] \\ B \otimes A \arrow[r, "f_r \otimes id_A"'] & P(A) \otimes A \end{tikzcd}$$

is a pullback.

Definition 2.11. Let $\mathcal{E}$ be a category admitting arbitrary colimits. Denote by $\widehat{\mathcal{E}} = Funct(\mathcal{E}^{op}, Sets)$ the category of presheaves of sets on $\mathcal{E}$, and by $\widetilde{\mathcal{E}} \subseteq \widehat{\mathcal{E}}$ the full subcategory of $\widehat{\mathcal{E}}$ consisting of continuous presheaves $F : \mathcal{E}^{op} \to Sets$. $\mathcal{E}$ is complete if $\widetilde{\mathcal{E}} \cong \mathcal{E}$.

Definition 2.12. A vectoid is a category $\mathcal{E}$ endowed with an associative and commutative tensor product $\otimes : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$, admiting a unit and satisfying the following conditions:

(1) Arbitrary small colimits exist in $\mathcal{E}$.

(2) The bifunctor $\otimes : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is cocontinuous.

(3) Finite limits exist in $\mathcal{E}$.

(4) Epimorphisms are universally effective and all equivalence relations are efficient in $\mathcal{E}$.
(5) $\mathcal{E}$ admits a small system of generators.
(6) $\mathcal{E}$ is complete.

3. Quasi Topos

THEOREM 3.1. *Quasi middle object exists in FRel.*

Proof. Let (I, P_I) be an object with $P_I(t) = t$ for all $t \in I$. Then for any object (A, P_A), there exists a unique morphism $P_A : A \to I$ such that the following diagram

$$\begin{array}{ccc} A & \xrightarrow{P_A} & I \\ {\scriptstyle P_A}\downarrow & & \downarrow{\scriptstyle P_I} \\ I & \xrightarrow[id_I]{} & I \end{array}$$

commutes. □

THEOREM 3.2. *Equalizers exist in FRel.*

Proof. For any two objects $(A, P_A), (B, P_B)$ and two morphisms $r, s : A \rightrightarrows B$, let $E = \{a \in A \mid (a, b_i) \in r, (a, b_j) \in s \Rightarrow b_i = b_j\}$ with $P_E = P_A|_E$ and $v : E \to A$ be a morphism defined by $(a, a) \in v$. Then we get $r \circ v = s \circ v$. Also we have $P_A(a) \geq P_E(e)$ for any $(e, a) \in v$. For any $v' : E' \to A$ such that $r \circ v' = s \circ v'$, there exists a morphism $w : E' \to E$ defined by $(t, a) \in w$ where $(t, a) \in v'$ and $(a, a) \in v$. So we have $v \circ w = v'$. Since $P_A(a) \geq P_{E'}(e')$ for any $(e', a) \in v'$, $P_E = P_A|_E$ and $(e', e) \in w \cap v'$ for any $e' \in E'$, we get $P_E(e) \geq P_{E'}(e')$ for any $(e', e) \in w$. Therefore $((E, P_E), v)$ is the equalizer of r and s. □

THEOREM 3.3. *Quasi exponentials exist in FRel.*

Proof. Let (A, P_A) and (C, P_C) be two objects, then we have (H, P_H) where $H = C^A = \{h \subseteq A \times C \mid P_C(c) \geq P_A(a), (a, c) \in h\}$ with $P_H : H \to I$ defined by

$$P_H(h) = sup\{k \in I \mid min\{P_A(a), k\} \leq P_C \circ h(a), a \in A\}.$$

Also we define the quasi evaluation morphism $ev_A : H \times A \to C$ by

$$ev_A(h,a) = h(a),$$

such that for any Y and $f : Y \times A \to C$, there exists a unique morphism $g : Y \to H$ such that the following diagram

$$\begin{array}{ccc}
(Y \times A, min\{P_Y, P_A\}) & \xrightarrow{f} & (C, P_C) \\
{\scriptstyle g\times id_A}\downarrow & & \downarrow{\scriptstyle id_C} \\
(H \times A, min\{P_H, P_A\}) & \xrightarrow[ev_A]{} & (C, P_C)
\end{array}$$

commutes.

Clearly $\{(H, P_H), ev_A\}$ is the quasi exponential in *FRel.* □

THEOREM 3.4. *Quasi subobject classifier exists in FRel. That is, there exists an object (I, P_J) with $P_J(z) = 1$ for all $z \in I$ together with $i : I \to I$ defined by $i(j) = j$ for all $j \in I$. And for any quasi monomorphism $m : B \to A$, there exists a unique morphism $q_m : A \to I$ such that for all $a \in A$, $P_A(a) \geq k$ where $(a,k) \in q_m$ and the following diagram*

$$\begin{array}{ccc}
(B, P_B) & \xrightarrow{P_B} & (I, P_I) \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle i} \\
(A, P_A) & \xrightarrow[q_m]{} & (I, P_J)
\end{array}$$

is a pullback where (I, P_I) is the quasi middle object.

Proof. Let $q_m : A \to I$ be a morphism defined by

$$q_m(a) = \begin{cases} P_B(b), & if\ (b,a) \in m \\ 0, & otherwise \end{cases}$$

Then $P_A(a) \geq k$ and $q_m \circ m = i \circ P_B$. Let $n : C \to A$ be a morphism such that $P_A \circ n \geq P_C$ and $i \circ P_C = q_m \circ n$. Then there exists an element $b \in B$ such that $n(c) = m(b)$ where $(c,a) \in n$ and $(b,a) \in m$. So there exists a morphism $n' : C \to B$ defined by $n'(c) = b$ such that $m \circ n' = n$ and $P_B \circ n' = P_C$. Also $b = n'(c)$ implies $m(b) = m(n'(c))$. By $n(c) = m(n'(c))$, we get $m \circ n' = n$. Thus $P_C(c) = q_m \circ n(c) = q_m \circ m(b) = P_B(b) = P_B \circ n'(c)$. So we have $P_C = P_B \circ n'$. Hence the following diagram

$$\begin{array}{ccc} (B, P_B) & \xrightarrow{P_B} & (I, P_I) \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle i} \\ (A, P_A) & \xrightarrow[q_m]{} & (I, P_J) \end{array}$$

is a pullback. Assume that there exists another q'_m such that $q'_m \circ m = i \circ P_B = q_m \circ m$. This implies $q'_m(m(b)) = q_m(m(b))$. Also if $a \notin m(b)$, we get $q'_m(a) = q_m(a) = 0$. So $q_m = q'_m$. □

REMARK. q_m is called the quasi membership morphism.

COROLLARY 3.5. *FRel is a quasi topos.*

4. Quasi Power Objects

THEOREM 4.1. *Quasi power objects exist in FRel.*

Proof. Let $P(A) = \{k|k : A \to I,\ k(a) \leq P_A(a)\ \forall a \in A\}$ with $P_{P(A)}(k) = 1$ for all $k \in P(A)$ and $\epsilon_A = \{(k, a)|k \in P(A), k(a) \neq 0\} \subseteq P(A) \times A$ with $P_{\epsilon_A}(k, a) = k(a)$. Also we construct $s : P(A) \times A \to I$ defined by

$$s(k, a) = \begin{cases} k(a), & if(k, a) \in \epsilon_A \\ 0, & otherwise \end{cases}$$

Then s is the quasi membership morphism of $m : \epsilon_A \to P(A) \times A$ where $m(k) = k$, so the following diagram

$$\begin{array}{ccc} \epsilon_A & \xrightarrow{P_{\epsilon_A}} & I \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle i} \\ P(A) \times A & \xrightarrow[s]{} & I \end{array}$$

is a pullback, where $i : I \to I$ defined by $i(j) = j$ for all $j \in I$. Let u be the quasi membership morphism of r. Then the following diagram

$$\begin{array}{ccc} R & \xrightarrow{v} & I \\ {\scriptstyle r}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ B \times A & \xrightarrow[u]{} & I \end{array}$$

is a pullback. Also let $f_r : B \to P(A)$ be a morphism defined by

$$f_r(b)(a) = \begin{cases} v(b,a), & if (b,a) \in R \\ 0, & otherwise \end{cases}$$

Then $s \circ (f_r \times id_A) \circ r = i \circ v$ and $s \circ (f_r \times id_A) = u$. By the property of pullback, there exists a morphism $g_r = f_r \times id_A|_R : R \to \epsilon_A$ such that $P_{\epsilon_A} \circ g_r = v$ and $m \circ g_r = (f_r \times id_A) \circ r$. By the pullback lemma, the following diagram

$$\begin{array}{ccc} R & \xrightarrow{g_r} & \epsilon_A \\ {\scriptstyle r}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ B \times A & \xrightarrow[f_r \times id_A]{} & P(A) \times A \end{array}$$

is a pullback. Assume there exists another f'_r such that $m \circ g'_r = (f'_r \times id_A) \circ r$ where $g'_r = f'_r \times id_A|_R$. Then we have $i \circ P_{\epsilon_A} \circ g'_r = s \circ (f'_r \times id_A) \circ r$. Also $g'_r(b,a) = (f'_r(b), a)$ for any $(b,a) \in R$, $P_{\epsilon_A} \circ g'_r = v$ and $s \circ (f'_r \times id_A) = u$. So $u(b,a) = (f'_r(b), a) = (f_r(b), a)$ for any $(b,a) \in R$. Hence $f'_r = f_r$. Also $(f'_r(b), a) = 0 = (f_r(b), a)$ for any $(b,a) \notin R$. Therefore f_r is unique. □

5. Logic Operations of the Quasi Topos *FRel*

THEOREM 5.1. *Negation* $(\neg)$ *exists in FRel.*

Proof. Let $\perp : (I, P_I) \to (I, P_J)$ be a quasi monomorphism defined by $(u, 1-u) \in \perp$ for all $u \in I$ with $P_J(z) = 1$ for all $z \in I$ and $P_I(t) = t$ for all $t \in I$. Then $\neg : (I, P_J) \to (I, P_J)$ is the quasi membership mophism of the $\perp$. That is, the following diagram

$$\begin{array}{ccc} (I, P_I) & \xrightarrow{P_I} & (I, P_I) \\ {\scriptstyle \perp}\downarrow & & \downarrow{\scriptstyle i} \\ (I, P_J) & \xrightarrow[\neg]{} & (I, P_J) \end{array}$$

is a pullback, where $i : I \to I$ defined by $i(j) = j$ for all $j \in I$. Thus we obtain $\neg : (I, P_J) \to (I, P_J)$ defined by $(v, 1 - v) \in \neg$ for all $v \in I$. □

THEOREM 5.2. *Conjunction* $(\wedge)$ *exists in FRel.*

Proof. Let $h : I \times I \to I$ be a morphism defined by $((p, q), min\{p, q\}) \in h$ for all $p, q \in I$ and $i \times i : I \times I \to I \times I$ be a quasi monomorphism defined by $((a, b), (a, b)) \in i \times i$ for all $a, b \in I$. Then $\wedge$ is the quasi membership mophism of the $i \times i$. That is, the following diagram

$$\begin{array}{ccc} (I \times I, h) & \xrightarrow{h} & (I, P_I) \\ {\scriptstyle i\times i}\downarrow & & \downarrow{\scriptstyle i} \\ (I \times I, P_J) & \xrightarrow[\wedge]{} & (I, P_J) \end{array}$$

is a pullback, where $i : I \to I$ defined by $i(j) = j$ for all $j \in I$. Thus we obtain $\wedge : (I \times I, P_J) \to (I, P_J)$ defined by $((u, v), min\{u, v\}) \in \wedge$ for all $u, v \in I$. □

THEOREM 5.3. *Disjunction* $(\vee)$ *exists in FRel.*

Proof. Let $k : I \times I \to I$ be a morphism defined by $((p, q), max\{p, q\}) \in k$ for all $p, q \in I$ and $i \times i : I \times I \to I \times I$ be a quasi monomorphism defined by $((a, b), (a, b)) \in i \times i$ for all $a, b \in I$. Then $\vee$ is the quasi membership mophism of the $i \times i$. That is, the following diagram

$$\begin{array}{ccc} (I \times I, k) & \xrightarrow{k} & (I, P_I) \\ {\scriptstyle i\times i}\downarrow & & \downarrow{\scriptstyle i} \\ (I \times I, P_J) & \xrightarrow[\vee]{} & (I, P_J) \end{array}$$

is a pullback, where $i : I \to I$ defined by $i(j) = j$ for all $j \in I$. Thus we obtain $\vee : (I \times I, P_J) \to (I, P_J)$ defined by $((u, v), max\{u, v\}) \in \vee$ for all $u, v \in I$. □

THEOREM 5.4. *Implication* ($\Rightarrow$) *exists in FRel.*

Proof. Let $g : I \times I \to I$ be a morphism defined by $((p, q), max\{1 - p, q\}) \in g$ for all $p, q \in I$ and $i \times i : I \times I \to I \times I$ be a quasi monomorphism defined by $((a, b), (a, b)) \in i \times i$ for all $a, b \in I$. Then $\Rightarrow$ is the quasi membership mophism of the $i \times i$. That is, the following diagram

$$\begin{array}{ccc} (I \times I, g) & \xrightarrow{g} & (I, P_I) \\ {\scriptstyle i\times i}\downarrow & & \downarrow{\scriptstyle i} \\ (I \times I, P_J) & \xrightarrow[\Rightarrow]{} & (I, P_J) \end{array}$$

is a pullback, where $i : I \to I$ defined by $i(j) = j$ for all $j \in I$. Thus we obtain $\Rightarrow$: $(I \times I, P_J) \to (I, P_J)$ defined by $((u, v), max\{1 - u, v\}) \in\Rightarrow$ for all $u, v \in I$. $\square$

6. Vectoid of the Quasi Topos *FRel*

THEOREM 6.1. *Arbitrary small colimits exist in FRel.*

Proof. For any (A, P_A), there is a unique morphism $r : (\phi, P_\phi) \to (A, P_A)$ such that $P_A \circ r \geq P_\phi$. So the initial object exists in *FRel.* For any two objects (A, P_A), (B, P_B) and two morphisms $r, s : A \rightrightarrows B$, let Q be the smallest equivalence relation on B that contains all $(\{r(a)\}, \{s(a)\})$. And let $C = B/Q$ with $P_C(c) = max\{P_B(b)\}$ and q be a quotient morphism. Then (q, C) is a coequalizer of a pair r and s. By $q \circ r = q \circ s$ and for any $q' : B \to C'$ such that $q' \circ r = q' \circ s$ with

$$P_{C'}(c') = \begin{cases} max\{P_C(c)\}, & if (b, c') \in q' and\ (b, c) \in q \\ 1, & otherwise \end{cases}$$

there exists a unique morphism $u : C \to C'$ such that $u \circ q = q'$ since Q is the smallest equivalence relation on B that contains all $(\{r(a)\}, \{s(a)\})$. So the coequalizer of a pair r and s exists in *FRel* . By similar method, multiple coequalizer exists in *FRel.* A coproduct of a pair (A, P_A) and (B, P_B) is a triple$((A \sqcup B, P_{A\sqcup B}), \mu_A, \mu_B)$ where $A \sqcup B$ is the disjoint union of A and B, and two injections $\mu_A : A \to A \sqcup B$ defined by $\mu_A(a) = (a, 1)$ and $\mu_B : B \to A \sqcup B$ defined by $\mu_B(b) = (b, 2)$ with

$$P_{A\sqcup B}(a, 1) = P_A(a), a \in A$$

$$P_{A\sqcup B}(b, 2) = P_B(b), b \in B.$$

So the coproduct of a pair exists in *FRel*. By similar method, *FRel* has coproducts. By the coequalizer and the coproduct, the pushout of s along r where $s : A \to C$ and $r : A \to B$ exists in *FRel*. By similar method, *FRel* has multiple pushouts. □

THEOREM 6.2. *The bifunctor* $\otimes : FRel \times FRel \to FRel$ *is cocontinuous.*

Proof. Given (X, P_X) and for any (A, P_A), there is a unique morphism $r : \phi \to X \times A$ such that $P_{X\times A} \circ r \geq P_\phi$. Also we have $X \times \phi = \phi$. So it preserves the initial object. For any two objects $X \times A$, $X \times B$ and two morphisms $id_X \times r, id_X \times s : X \times A \rightrightarrows X \times B$, let Q' be the smallest equivalence relation on $X \times B$ that contains all $((x, \{r(a)\}), (x, \{s(a)\}))$. And let $C' = (X \times B)/Q'$ with $P_{C'}(c') = max\{P_{X\times B}(b)\}$ and q' be a quotient morphism. Then (q', C') is the coequalizer of a pair $id_X \times r$ and $id_X \times s$. And we have $X \times B/Q \cong (X \times B)/Q'$ where Q is the smallest equivalence relation on B that contains all $(\{r(a)\}, \{s(a)\})$. So it preserves a coequalizer of a pair. By similar method, it preserves multiple coequalizers. A coproduct of a pair $X \times A$ and $X \times B$ is a triple$(X \times A \sqcup X \times B, \mu_{X\times A}, \mu_{X\times B})$ with two injections $\mu_{X\times A} : X \times A \to X \times A \sqcup X \times B$ and $\mu_{X\times B} : X \times B \to X \times A \sqcup X \times B$. And we have $X \times (A \sqcup B) \cong (X \times A) \sqcup (X \times B)$. So it preserves a coproduct of a pair. By similar method, it preserves coproducts. By the coequalizer and the coproduct, the pushout of $id_X \times s$ along $id_X \times r$ where $id_X \times s : X \times A \to X \times C$ and $id_X \times r : X \times A \to X \times B$ exists in *FRel*. So it preserves pushouts. By similar method, it preserves multiple pushouts. □

THEOREM 6.3. *Finite limits exist in FRel.*

Proof. For any (A, P_A), there is a unique morphism $r : (A, P_A) \to (\phi, P_\phi)$ such that $P_\phi \circ r \geq P_A$. So the terminal object exists in *FRel*. For any two objects (A, P_A), (B, P_B) and two morphisms $r, s : A \rightrightarrows B$, let $E = \{a \in A \mid (a, b) \in r, (a, c) \in s \Rightarrow b = c\}$ with $P_E(a) = P_A(a)$ and $q : E \to A$ defined by $(a, a) \in q$ for all $a \in E$. Then $r \circ q = s \circ q$. For any $q' : E' \to A$ such that $r \circ q' = s \circ q'$, since E is the largest subobject with $r \circ q = s \circ q$, there is a unique morphism $v : E' \to E$ such that $q \circ v = q'$. So the equalizer of a pair r and s exists in *FRel*. A product of a pair (A, P_A) and (B, P_B) is a triple$((A \sqcup B, P_{A\sqcup B}), \pi_A, \pi_B)$ where $A \sqcup B$ is the disjoint union of A and B, and two projections $\pi_A : A \sqcup B \to A$ defined by $\pi_A(a,1) = a$ and $\pi_B : A \sqcup B \to B$ defined by $\pi_B(b,2) = b$ with

$$P_{A \sqcup B}(a,1) = P_A(a), a \in A$$

$$P_{A \sqcup B}(b,2) = P_B(b), b \in B.$$

So the product of a pair exists in *FRel*. By the equalizer and the product, the pullback of s along r where $s : Y \to C$ and $r : X \to C$ exists in *FRel*. □

THEOREM 6.4. *Epimorphisms are universally effective and all equivalence relations are efficient in FRel.*

Proof. For any epimorphism $e : (X, P_X) \to (Y, P_Y)$, there is a kernel pair (p, q) of e where $B = \{(a,b) | (a,y), (b,y) \in e\}$ and $p, q : B \rightrightarrows X$ defined by $((a,b),a) \in p$ and $((a,b),b) \in q$. That is, the following diagram

$$\begin{array}{ccc} B & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle e} \\ X & \xrightarrow[e]{} & Y \end{array}$$

is a pullback. If there is a morphism $s : X \to Z$ such that $s \circ p = s \circ q$, since Y is the largest object with $e \circ p = e \circ q$, there is a unique morphism $k : Y \to Z$ such that $k \circ e = s$. So the epimorphism $e : (X, P_X) \to (Y, P_Y)$ is universally effective.

For any (X, P_X) and an equivalence relation $r \subseteq X \times X$, there is a fibered product $X \times_{X/r} X$ such that the following diagram

$$\begin{array}{ccc} X \times_{X/r} X & \xrightarrow{\pi_1} & X \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[h]{} & X/r \end{array}$$

is a pullback. So for two morphisms $\pi_1', \pi_2' : r \rightrightarrows X$ such that $h \circ \pi_1' = h \circ \pi_2'$, there is a morphism $\varphi : r \to X \times_{X/r} X$ such that $\pi_1 \circ \varphi = \pi_1'$ and $\pi_2 \circ \varphi = \pi_2'$ where $\varphi(a,b) = (\varphi_1(a), \varphi_2(b))$. Thus $\pi_1 \circ \varphi(a,b) = \pi_1(\varphi_1(a), \varphi_2(b)) = \varphi_1(a)$ and $\pi_1'(a,b) = a$. So we get $\varphi_1(a) = a$ and $\varphi_2(b) = b$. Therefore $r = X \times_{X/r} X$. □

THEOREM 6.5. *FRel admits a small system of generators.*

Proof. For any (A, P_A) and (B, P_B), let $r \neq s : A \rightrightarrows B$. Then there is an element $a \in A$ such that $(a, b) \in r$ and $(a, c) \in s$. We construct $X = (\{*\}, P_{\{*\}})$ with $P_{\{*\}}(*) = min\{P_A(a)\}$ for all $a \in A$ and $q : X \to A$ defined by $(*, a) \in q$. Then $r \circ q \neq s \circ q$. □

THEOREM 6.6. *FRel is complete.*

Proof. For any $G \in ob(\widetilde{FRel})$, since $FRel$ is cocomplete and G is representable, G has a left adjoint F. So for any $K \in ob(FRel)$, there is an object $F \circ G(K)$ such that $F \circ G(K) \in ob(FRel)$. Since $FRel \subseteq \widetilde{FRel}$, we have $FRel \cong \widetilde{FRel}$. □

COROLLARY 6.7. *FRel is a vectoid.*

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