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EVALUATIONS OF THE CUBIC CONTINUED FRACTION BY SOME THETA FUNCTION IDENTITIES

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Abstract. In this paper, we use some theta function identities involving two parameters $h_{n,k}$ and $h'_{n,k}$ for the theta function φ to establish new evaluations of Ramanujan's cubic continued fraction.

1. Introduction

Ramanujan's cubic continued fraction $G(q)$, for $|q| < 1$, is defined by

$$G(q) = \frac{q^{1/3}}{1} \, \underset{+}{} \, \frac{q+q^2}{1} \, \underset{+}{} \, \frac{q^2+q^4}{1} \, \underset{+}{} \, \frac{q^3+q^6}{1} \, \underset{+}{} \, \cdots \, .$$

It has been known that evaluating values of $G(e^{-\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$ for some specific positive rational numbers n is quite difficult in general. Ramanathan [11] evaluated

$$G(e^{-\pi\sqrt{10}}) = \frac{\sqrt{9+3\sqrt{6}} - \sqrt{7+3\sqrt{6}}}{(1+\sqrt{5}\,)\sqrt{\sqrt{5}+\sqrt{6}}}$$

by using Kronecker's limit formula. Andrews and Berndt [3] have also given a proof of the evaluation of $G(e^{-\pi\sqrt{10}})$. Berndt, Chan, and Zhang

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[6] evaluated $G(e^{-\pi\sqrt{n}})$ for $n = 2, 10, 22, 58$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 1$, 5, 13, 37 by using Ramanujan's class invariants. Chan [7] explicitly evaluated $G(e^{-\pi\sqrt{n}})$ for $n = 1, 2, 4, \frac{2}{9}$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 1, 5$ by applying some reciprocity theorems for the cubic continued fraction.

Adiga, Vasuki, and Mahadeva Naika [2] found the numerical values of $G(e^{-2\pi})$ and $G(-e^{-\pi\sqrt{n}})$ for $n = \frac{1}{3}, \frac{25}{3}, \frac{49}{3}, \frac{1}{75}, \frac{1}{147}$ by using some modular equations. Adiga, Kim, Mahadeva Naika, and Madhusudhan [1] found values of $G(-e^{-\pi\sqrt{n}})$ for $n = 1, 3, 5, \frac{1}{3}, \frac{1}{5}, \frac{1}{9}, \frac{1}{27}$. Moreover, Yi [12] found explicit values of $G(e^{-\pi\sqrt{n}})$ for $n = 3, 6, 7, 8, 12, 16, 28, \frac{1}{2}, \frac{1}{3}, \frac{4}{3}, \frac{1}{4}, \frac{1}{9}, \frac{4}{9}$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 2, 3, 4, 7, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{9}$ by using some eta-function identities. In [14] the values of $G(e^{-\pi\sqrt{n}})$ for $n = 1, 4, 9, \frac{1}{3}$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 4, 9$ were evaluated by employing modular equations of degrees 3 and 9.

Recently, Paek and Yi [8, 9, 10] exploited some theta function identities related to modular equations of degrees 3 and 9 to obtain explicit values of $G(e^{-\pi\sqrt{n}})$ for $n = 1, 8, 16, 32, 36, 64, 81, 128, 144, 256, 324, \frac{1}{2}, \frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \frac{32}{3}, \frac{64}{3}, \frac{128}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{12}, \frac{1}{16}, \frac{1}{24}, \frac{1}{32}, \frac{1}{48}, \frac{1}{96}, \frac{1}{128}, \frac{1}{192}, \frac{1}{384}$ and also $G(-e^{-\pi\sqrt{n}})$ for $n = 8, 16, 32, 36, 64, 81, \frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \frac{32}{3}, \frac{1}{4}, \frac{1}{8}, \frac{1}{12}, \frac{1}{16}, \frac{1}{24}, \frac{1}{32}, \frac{1}{48}, \frac{1}{96}, \frac{1}{128}, \frac{1}{192}, \frac{1}{384}$.

In this paper, we employ some theta function identities involving two parameters $h_{n,k}$ and $h'_{n,k}$ for the theta function φ to establish 36 new explicit values of $G(e^{-\pi\sqrt{n}})$ for $n = 5, 20, 27, 45, 48, 80, 108, 180, 432, 720, \frac{1}{5}, \frac{4}{5}, \frac{9}{5}, \frac{16}{5}, \frac{36}{5}, \frac{144}{5}, \frac{5}{9}, \frac{20}{9}, \frac{80}{9}, \frac{1}{27}, \frac{4}{27}, \frac{16}{27}, \frac{1}{45}, \frac{4}{45}, \frac{16}{45}$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 20, 27, 45, 180, \frac{4}{5}, \frac{9}{5}, \frac{36}{5}, \frac{5}{9}, \frac{20}{9}, \frac{1}{45}, \frac{4}{45}$.

Ramanujan's theta function $\varphi(q)$, for $|q| < 1$, is defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

Recall two parameters $h_{k,n}$ and $h'_{k,n}$ for the theta-function φ from [13]. For any positive real numbers k and n, define $h_{k,n}$ by

$$h_{k,n} = \frac{\varphi(q)}{k^{1/4}\varphi(q^k)},$$

where $q = e^{-\pi\sqrt{n/k}}$ and define $h'_{k,n}$ by

$$h'_{k,n} = \frac{\varphi(-q)}{k^{1/4}\varphi(-q^k)},$$

where $q = e^{-2\pi\sqrt{n/k}}$.

Yi [13] has established some useful properties of $h_{k,n}$:

$$h_{k,\frac{1}{n}} = h_{k,n}^{-1}, \tag{1.1}$$

and

$$h_{k,n} = h_{n,k}. \tag{1.2}$$

Note that general formulas for $G(e^{-2\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$ in terms of $h'_{9,n}$ and $h_{9,n}$ were given in [15, Theorem 6.2(iii) and (iv)] as follows:

$$G(e^{-2\pi\sqrt{n}}) = \frac{1-\sqrt{3}\,h'_{9,n}}{2} \tag{1.3}$$

and

$$G(-e^{-\pi\sqrt{n}}) = \frac{1-\sqrt{3}\,h_{9,n}}{2}. \tag{1.4}$$

In view of (1.3) and (1.4), in order to compute $G(e^{-2\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$, it suffices to evaluate $h'_{9,n}$ and $h_{9,n}$, respectively. Throughout this paper, we assume the subscript n in $h'_{9,n}$ and $h_{9,n}$ to be a positive real number.

2. Evaluations of $h_{9,n}$ and $h'_{9,n}$

We begin this section by establishing the evaluations of $h_{9,3}$ and $h_{9,5}$ which will play key roles in finding $h_{9,n}$ for some n. We first need the following theta function identities involving $h_{3,n}$ and $h_{3,\frac{n}{9}}$.

Lemma 2.1. *For any n, we have*

$$(\sqrt{3}\,h_{3,n}h_{3,\frac{n}{9}} - 1)^3 = 3\,h_{3,n}^4 - 1. \tag{2.1}$$

Proof. By [4, Entry 1(iii), p. 345], we have

$$\left(\frac{\varphi(q^{1/3})}{\varphi(q^3)} - 1\right)^3 = \frac{\varphi^4(q)}{\varphi^4(q^3)} - 1,$$

or equivalently

$$\left(\frac{\varphi(q^{1/3})}{\varphi(q)}\cdot\frac{\varphi(q)}{\varphi(q^3)} - 1\right)^3 = \frac{\varphi^4(q)}{\varphi^4(q^3)} - 1.$$

Rewrite the last equality in terms of $h_{3,n}$ to complete the proof. $\square$

We next need the following theta function identities involving $h_{5,n}$ and $h_{5,9n}$ that follows from a modular equation in [5, Entry 67, p. 235] such as $PQ+\dfrac{5}{PQ}=\left(\dfrac{Q}{P}\right)^2+3\dfrac{Q}{P}+3\dfrac{P}{Q}-\left(\dfrac{P}{Q}\right)^2$, where $P=\frac{\varphi(q)}{\varphi(q^5)}$ and $Q=\frac{\varphi(q^3)}{\varphi(q^{15})}$.

Lemma 2.2 ([13], Theorem 4.14(i)). *For any n, we have*

(2.2)

$$\sqrt{5}\,h_{5,n}h_{5,9n}+\frac{\sqrt{5}}{h_{5,n}h_{5,9n}}=\left(\frac{h_{5,9n}}{h_{5,n}}\right)^2+3\left(\frac{h_{5,9n}}{h_{5,n}}+\frac{h_{5,n}}{h_{5,9n}}\right)-\left(\frac{h_{5,n}}{h_{5,9n}}\right)^2.$$

We need another theta function identity involving $h_{9,n}$ and $h_{9,9n}$ to establish some further evaluations of $h_{9,n}$ that follows from a modular equation $\left(P-3+\dfrac{3}{P}\right)\left(Q-3+\dfrac{3}{Q}\right)=\left(\dfrac{Q}{P}\right)^2$, where $P=\frac{\varphi(q)}{\varphi(q^9)}$ and $Q=\frac{\varphi(q^3)}{\varphi(q^{27})}$ in [14, Theorem 3.5].

Lemma 2.3 ([14], Corollary 3.6). *For any n, we have*

(2.3) $$3h_{9,n}(h_{9,n}^2-\sqrt{3}\,h_{9,n}+1)(h_{9,9n}^2-\sqrt{3}\,h_{9,9n}+1)=h_{9,9n}^3.$$

We begin with the evaluations of $h_{9,n}$ for $n=3$, 27, $\frac{1}{3}$, and $\frac{1}{27}$.

Theorem 2.4. *We have*

(i) $h_{9,3}=\dfrac{\sqrt{3}}{1+\sqrt[3]{2}}$,

(ii) $h_{9,27}=\dfrac{3^{5/6}}{\sqrt[3]{2}+\sqrt[3]{3}+\sqrt[3]{4}}$,

(iii) $h_{9,\frac{1}{3}}=\dfrac{1+\sqrt[3]{2}}{\sqrt{3}}$,

(iv) $h_{9,\frac{1}{27}}=\dfrac{\sqrt[3]{2}+\sqrt[3]{3}+\sqrt[3]{4}}{3^{5/6}}$.

Proof. For (i), it is sufficient to find the value of $h_{3,9}$ by (1.2). Let $n=9$ in (2.1) and put $h_{3,1}=1$, then it follows that

$$h_{3,9}^3-\sqrt{3}\,h_{3,9}^2+3h_{3,9}-\sqrt{3}=0.$$

Employing *Mathematica* to solve the above equation for $h_{3,9}$ and then using $h_{3,9}>0$, we find that $h_{3,9}=\dfrac{\sqrt{3}}{1+\sqrt[3]{2}}$. Thus (i) has been established.

For (ii), let $n = 3$ in (2.3) and put the value of $h_{9,3}$ obtained from (i), then it follows that

$$(1 + \sqrt[3]{2}\,)h_{9,27}^3 + 3\sqrt{3}\,(1 - \sqrt[3]{2}\,)(h_{9,27}^2 - \sqrt{3}\,h_{9,27} + 1) = 0.$$

Employing *Mathematica* again to solve the last equation for $h_{9,27}$ and then using $h_{9,27} > 0$, we complete the proof. The proofs of (iii) and (iv) are clear by (1.1). □

We next evaluate $h_{9,n}$ for $n = 5$, 45, $\frac{1}{5}$, $\frac{9}{5}$, $\frac{5}{9}$, and $\frac{1}{45}$.

Theorem 2.5. *We have*

(i) $h_{9,5} = \dfrac{1+\sqrt{3}}{\sqrt{3}+\sqrt{5}}$,

(ii) $h_{9,45} = \dfrac{3 - 3\sqrt[3]{2\sqrt{3}+2\sqrt{5}} + 6\sqrt[3]{4+\sqrt{15}}}{6+\sqrt{3}+2\sqrt{15}}$,

(iii) $h_{9,\frac{5}{9}} = \dfrac{1+\sqrt[3]{-2\sqrt{3}+2\sqrt{5}}}{\sqrt{3}}$,

(iv) $h_{9,\frac{1}{5}} = \dfrac{\sqrt{3}+\sqrt{5}}{1+\sqrt{3}}$,

(v) $h_{9,\frac{1}{45}} = \dfrac{6+\sqrt{3}+2\sqrt{15}}{3 - 3\sqrt[3]{2\sqrt{3}+2\sqrt{5}} + 6\sqrt[3]{4+\sqrt{15}}}$,

(vi) $h_{9,\frac{9}{5}} = \dfrac{\sqrt{3}}{1+\sqrt[3]{-2\sqrt{3}+2\sqrt{5}}}$.

Proof. For (i), it is enough to find the value of $h_{5,9}$ by (1.2). Letting $n = 1$ in (2.2) and then putting $h_{5,1} = 1$, we find that

$$h_{5,9}^4 + (3 - \sqrt{5}\,)h_{5,9}^3 + (3 - \sqrt{5}\,)h_{5,9} - 1 = 0.$$

Solving the above equation for $h_{5,9}$ and then using $h_{5,9} > 0$, we deduce that $h_{5,9} = \dfrac{1+\sqrt{3}}{\sqrt{3}+\sqrt{5}}$, where we utilized *Mathematica*. Hence we complete the proof of (i).

For (ii), setting $n = 5$ in (2.3) and putting the value of $h_{9,5}$ obtained from (i), we find that

$$(1 + 2\sqrt{3} + 2\sqrt{5}\,)h_{9,45}^3 - 3\sqrt{3}\,h_{9,45}^2 + 9\,h_{9,45} - 3\sqrt{3} = 0.$$

Utilizing *Mathematica* to solve the last equation for $h_{9,45}$ and then using $h_{9,45} > 0$, we have completed the proof of (ii).

The proof of (iii) is similar to that of (ii) and the proofs of (iv)–(vi) are clear by (1.1). $\square$

See [13, Theorem 4.15(i)] for an alternative proof of Theorem 2.5(i). We now turn to the evaluations of $h'_{9,n}$ for $n = 3$, 5, 27, 45, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{9}{5}$, $\frac{5}{9}$, $\frac{1}{27}$, and $\frac{1}{45}$. But first we need the following theta function identity involving $h_{9,n}$ and $h'_{9,n}$ that comes from a modular equation $\dfrac{P}{Q} + \dfrac{Q}{P} + 2 = Q + \dfrac{3}{Q}$, where $P = \frac{\varphi(q)}{\varphi(q^9)}$ and $Q = \frac{\varphi(-q^2)}{\varphi(-q^{18})}$ in [14, Theorem 3.3].

Lemma 2.6 ([14], Corollary 3.4). *For any n, we have*

$$\sqrt{3}\left(h'_{9,n} + \frac{1}{h'_{9,n}}\right) = \frac{h_{9,n}}{h'_{9,n}} + \frac{h'_{9,n}}{h_{9,n}} + 2\,. \tag{2.4}$$

We first evaluate $h'_{9,n}$ for $n = 3$, 27, $\frac{1}{3}$, and $\frac{1}{27}$.

Theorem 2.7. *We have*

(i) $h'_{9,3} = \dfrac{(2+\sqrt[3]{4})(1-\sqrt{3}+\sqrt[3]{2})}{2\sqrt{3}}$,

(ii) $h'_{9,27} = \dfrac{-2-\sqrt[3]{2}+\sqrt[3]{4}+\sqrt[3]{36}-3^{5/6}+\sqrt[3]{9(-1+\sqrt[3]{2})^2}}{\sqrt[3]{2}-2\sqrt[3]{3}+\sqrt[3]{4}}$,

(iii) $h'_{9,\frac{1}{3}} = \dfrac{1-\sqrt{3}+\sqrt[3]{2}}{\sqrt{3}\sqrt[3]{2}}$,

(iv) $h'_{9,\frac{1}{27}} = \dfrac{1}{\sqrt{3}} + \dfrac{1-\sqrt{3+2\sqrt[3]{2}+2\sqrt[3]{4}}}{3^{1/6}(\sqrt[3]{2}+\sqrt[3]{4})}$.

Proof. For (i), letting $n = 3$ in (2.4) and then putting the value of $h_{9,3}$ in Theorem 2.4(i), we find that

$$(1+\sqrt[3]{2})(2-\sqrt[3]{2})h'^2_{9,3} - 2\sqrt{3}\,(1+\sqrt[3]{2})h'_{9,3} + 3\sqrt[3]{2} = 0.$$

Using *Mathematica* to solve the above equation for $h'_{9,3}$ and then using $h'_{9,3} < 1$, we complete the proof of (i). The proofs of (ii)–(iv) are similar to that of (i). $\square$

We next evaluate $h'_{9,n}$ for $n = 5$, 45, $\frac{1}{5}$, $\frac{9}{5}$, $\frac{5}{9}$, and $\frac{1}{45}$.

Theorem 2.8. *We have*

(i) $h'_{9,5} = \dfrac{1}{4}(1+\sqrt{3})\left(3+\sqrt{5}-\sqrt{6+6\sqrt{5}}\right)$,

(ii) $h'_{9,45} = \dfrac{\sqrt{3}\sqrt[3]{2\sqrt{3}+2\sqrt{5}}}{1+\sqrt{1+2\sqrt{3}+2\sqrt{5}}+\sqrt[3]{2\sqrt{3}+2\sqrt{5}}}$,

(iii) $h'_{9,\frac{5}{9}} = \dfrac{1}{\sqrt{3}}\left(1+\dfrac{1-\sqrt{1-2\sqrt{3}+2\sqrt{5}}}{\sqrt[3]{-2\sqrt{3}+2\sqrt{5}}}\right)$,

(iv) $h'_{9,\frac{1}{5}} = \dfrac{1+\sqrt{3}+(3-\sqrt{15}\,)\sqrt{-1+2\sqrt{3}+2\sqrt{5}}}{(2+\sqrt{3}\,)(3-\sqrt{5}\,)}$,

(v) $h'_{9,\frac{1}{45}} = \dfrac{1}{\sqrt{3}}\left(1+\dfrac{1-\sqrt{1+2\sqrt{3}+2\sqrt{5}}}{\sqrt[3]{2\sqrt{3}+2\sqrt{5}}}\right)$,

(vi) $h'_{9,\frac{9}{5}} = \dfrac{\sqrt{3}\sqrt[3]{-2\sqrt{3}+2\sqrt{5}}}{1+\sqrt{1-2\sqrt{3}+2\sqrt{5}}+\sqrt[3]{-2\sqrt{3}+2\sqrt{5}}}$.

Proof. For (i), let $n = 5$ in (2.4) and then put the value of $h_{9,5}$ in Theorem 2.5(i), then we deduce that

$$(3-\sqrt{5}\,)h'^{2}_{9,5} - 2(1+\sqrt{3}\,)h'_{9,5} + (2+\sqrt{3}\,)(3-\sqrt{5}\,) = 0.$$

Employing *Mathematica* to solve the above equation for $h'_{9,5}$ and then using $h'_{9,3} < 1$, we complete the proof of (i). The proofs of (ii)–(iv) are similar to that of (i). □

We evaluate some more values of $h_{9,n}$ and $h'_{9,n}$ by utilizing the following two theta function identities involving $h'_{9,n}$, $h'_{9,\frac{n}{4}}$, and $h_{9,n}$.

LEMMA 2.9 ([9], Corollary 3.2). *For any n, we have*

$$\sqrt{3}\left(h'_{9,n}+\frac{1}{h'_{9,n}}\right) = \frac{h'_{9,n/4}}{h'_{9,n}} + \frac{h'_{9,n}}{h'_{9,n/4}} + 2. \tag{2.5}$$

Note that (2.5) follows from a modular equation $\dfrac{P}{Q}+\dfrac{Q}{P}+2 = Q+\dfrac{3}{Q}$, where $P = \frac{\varphi(-q)}{\varphi(-q^9)}$ and $Q = \frac{\varphi(-q^2)}{\varphi(-q^{18})}$ in [9, Theorem 3.1].

LEMMA 2.10 ([14], Corollary 3.2). *For any n, we have*

$$\sqrt{3\,h_{9,n}h'_{9,n/4}} + \sqrt{\frac{3}{h_{9,n}h'_{9,n/4}}} = \sqrt{\frac{h_{9,n}}{h'_{9,n/4}}} + \sqrt{\frac{h'_{9,n/4}}{h_{9,n}}} + 2\,. \tag{2.6}$$

Note that (2.6) follows from the modular equation in [14, Theorem 3.1] such as $\sqrt{PQ}+\dfrac{3}{\sqrt{PQ}} = \sqrt{\dfrac{Q}{P}}+\sqrt{\dfrac{P}{Q}}+2$, where $P = \frac{\varphi(q)}{\varphi(q^9)}$ and

$Q = \frac{\varphi(-q)}{\varphi(-q^9)}$. We establish the evaluations of $h'_{9,n}$ for $n = 12$, 108, $\frac{4}{3}$, $\frac{3}{4}$, $\frac{27}{4}$, $\frac{1}{12}$, $\frac{4}{27}$, and $\frac{1}{108}$.

Theorem 2.11. *We have*

(i) $$h'_{9,12} = \frac{\sqrt{3}\left(1-\sqrt{\sqrt[3]{2}+\sqrt[3]{4}-\sqrt{3}}\right)}{2-\sqrt[3]{5+3\sqrt{3}}},$$

(ii) $$h'_{9,\frac{3}{4}} = -4-3\sqrt[3]{2}-2\sqrt[3]{4}+\frac{7+5\sqrt[3]{2}+4\sqrt[3]{4}}{\sqrt{3}},$$

(iii) $$h'_{9,108} = \frac{a-\sqrt{\sqrt{3}\,a(1-\sqrt{3}\,a+a^2)}}{-1+\sqrt{3}\,a},$$

(iv) $$h'_{9,\frac{27}{4}} = \frac{\sqrt{3}\left(4+2\sqrt[3]{2}+\sqrt[3]{4}+2\sqrt[3]{9}-2\sqrt[3]{9}\sqrt{3+2\sqrt[3]{2}+2\sqrt[3]{4}}\right)}{(\sqrt[3]{2}+\sqrt[3]{4}-2\sqrt[3]{3})^2},$$

(v) $$h'_{9,\frac{4}{3}} = \frac{1}{2^{1/6}}+\frac{1}{\sqrt{3}}-\frac{1+\sqrt{3}}{\sqrt{3}\sqrt[3]{4}},$$

(vi) $$h'_{9,\frac{1}{12}} = -\sqrt[3]{2}+\frac{1+\sqrt[3]{2}}{\sqrt{3}},$$

(vii) $$h'_{9,\frac{4}{27}} = \frac{1}{\sqrt{3}}+\frac{(\sqrt[3]{2}+\sqrt[3]{4})\left(1-\sqrt{2+2\sqrt[3]{2}-\sqrt{3+6\sqrt[3]{2}+6\sqrt[3]{4}}}\right)}{3^{5/6}\left(1-\sqrt{3+2\sqrt[3]{2}+2\sqrt[3]{4}}\right)},$$

(viii) $$h'_{9,\frac{1}{108}} = \frac{4(1-\sqrt[3]{2}+\sqrt[3]{3})-2(\sqrt[3]{4}+\sqrt[3]{6})+\sqrt[3]{9}}{3(3^{1/6}+\sqrt[3]{6})+3^{5/6}(2-\sqrt[3]{2})},$$

where

$$a = \frac{-2-\sqrt[3]{2}+\sqrt[3]{4}+\sqrt[3]{36}-3^{5/6}+\sqrt[3]{9(-1+\sqrt[3]{2})^2}}{\sqrt[3]{2}+\sqrt[3]{4}-2\sqrt[3]{3}}.$$

Proof. For (i), let $n = 12$ in (2.5) and put the value $h'_{9,3}$ in Theorem 2.7(i), then we find that

$$2\left(-2+\sqrt[3]{5+3\sqrt{3}}\right)h'^2_{9,12}+4\sqrt{3}\,h'_{9,12}+(2+\sqrt[3]{4})(1-\sqrt{3}+\sqrt[3]{2})-6=0.$$

Using *Mathematica* to solve the above equation for $h'_{9,12}$ and then using $h'_{9,12} > 0$, we complete the proof of (i).

For (ii), let $n = 3$ in (2.6) and then put the value $h_{9,3}$ in Theorem 2.4(i), then we find that

$$(2-\sqrt[3]{2})h'_{9,\frac{3}{4}} - 2\sqrt{\sqrt{3}\,(1+\sqrt[3]{2})h'_{9,\frac{3}{4}}} + \sqrt{3}\sqrt[3]{2} = 0.$$

Employing *Mathematica* to solve the above equation for $h'_{9,\frac{3}{4}}$ and then using $h'_{9,\frac{3}{4}} < 1$, we see that we complete the proof of (ii). For the proofs of (iii)–(viii), either apply (2.5) and Theorem 2.7 or apply (2.6) and Theorem 2.4, and repeat the same arguments as in the proofs of (i) and (ii). □

We end this section by evaluating $h'_{9,n}$ for $n = 20$, 180, $\frac{5}{4}$, $\frac{45}{4}$, $\frac{4}{5}$, $\frac{36}{5}$, $\frac{20}{9}$, $\frac{1}{20}$, $\frac{9}{20}$, $\frac{5}{36}$, $\frac{4}{45}$, and $\frac{1}{180}$.

THEOREM 2.12. *We have*

(i) $$h'_{9,20} = \frac{(1+\sqrt{3}\,)\left(2-\sqrt{6+6\sqrt{5}-6\sqrt{2+2\sqrt{5}}}\right)}{3+2\sqrt{3}-\sqrt{5}-\sqrt{6+6\sqrt{5}}},$$

(ii) $$h'_{9,\frac{5}{4}} = \frac{(1+\sqrt{3}\,)(5+3\sqrt{5}\,)}{-2(\sqrt{3}+\sqrt{5}\,)} - \sqrt{\frac{-6+12\sqrt{3}+12\sqrt{5}}{47-21\sqrt{5}}},$$

(iii) $$h'_{9,180} = \frac{-\sqrt{3}\,b}{1-2b+\sqrt{1+b^3}}\left(1-\sqrt{\frac{2+2b^2+b^3+2\sqrt{1+b^3}}{b(1+b+\sqrt{1+b^3})}-1}\right),$$

(iv) $$h'_{9,\frac{45}{4}} = \frac{6+6b^2+3b^3+6\sqrt{1+b^3}-6\sqrt{(2-b^3)\sqrt{1+b^3}+\sqrt{5}\,b^3}}{2\sqrt{3}(1+b+\sqrt{1+b^3}\,)^2},$$

(v) $$h'_{9,\frac{20}{9}} = \frac{1}{\sqrt{3}} + \frac{2-2\sqrt{4+\sqrt{3}+\sqrt{5}-\sqrt{17+4\sqrt{3}+8\sqrt{5}+2\sqrt{15}}}}{\sqrt{3}\sqrt[3]{2\sqrt{3}+2\sqrt{5}}\left(1-\sqrt{1-2\sqrt{3}+2\sqrt{5}}\right)},$$

(vi) $$h'_{9,\frac{5}{36}} = \frac{1+\sqrt[3]{4+\sqrt{15}}\,\left(1-\sqrt{1-2\sqrt{3}+2\sqrt{5}}\right)}{\sqrt{3}},$$

(vii) $$h'_{9,\frac{4}{5}} = \frac{-4+2\sqrt{6+6\sqrt{5}+6\sqrt{2+2\sqrt{5}}}}{3-\sqrt{3}+\sqrt{5}+\sqrt{15}+2\sqrt{3(2+\sqrt{3}\,)(1+\sqrt{5}\,)}},$$

(viii) $$h'_{9,\frac{1}{20}} = \frac{(-1+\sqrt{3}\,)(5+3\sqrt{5}\,)}{2\sqrt{3}+2\sqrt{5}} - \sqrt{\frac{-6-12\sqrt{3}+12\sqrt{5}}{47-21\sqrt{5}}},$$

(ix) $h'_{9,\frac{4}{45}} = \dfrac{1}{\sqrt{3}} + \dfrac{b^{3/2} - \sqrt{4 + 4b^3 - (4 + b^3)\sqrt{1 + b^3}}}{\sqrt{3b}\,(1 - \sqrt{1 + b^3}\,)}$,

(x) $h'_{9,\frac{1}{180}} = \dfrac{1}{\sqrt{3}} + \dfrac{1 - \sqrt{1 + 2\sqrt{3} + 2\sqrt{5}}}{\sqrt{3}\sqrt[3]{4 + \sqrt{15}}}$,

(xi) $h'_{9,\frac{36}{5}} = \dfrac{2\sqrt{3}\,b}{4b - b^2 - \sqrt{8b + b^4}} \left(1 - \sqrt{1 - \dfrac{8b - 2b^2 - 2\sqrt{8b + b^4}}{4b - b^3 + b\sqrt{8b + b^4}}}\right)$,

(xii) $h'_{9,\frac{9}{20}} = \dfrac{\sqrt{3}\left(2 - 2\sqrt{1 - 2\sqrt{3} + 2\sqrt{5}} + \sqrt[3]{4 - \sqrt{15}}\right)}{\left(2 - \sqrt[3]{-2\sqrt{3} + 2\sqrt{5}}\right)^2}$,

where

$$b = \sqrt[3]{2\sqrt{3} + 2\sqrt{5}}.$$

Proof. For (i), let $n = 20$ in (2.5) and put the value of $h'_{9,5}$ in Theorem 2.8(i), then we deduce that

$$h'^2_{9,20} - \frac{3 + 2\sqrt{3} - \sqrt{5} - \sqrt{6 + 6\sqrt{5}}}{2 + \sqrt{3} - \sqrt{15}}\, h'_{9,20} + \frac{2 + 2\sqrt{-6 + 3\sqrt{5}}}{1 + \sqrt{3} - 3\sqrt{5} + \sqrt{15}} = 0.$$

Employing *Mathematica* to solve the above equation for $h'_{9,20}$ and then using $h'_{9,12} > 0$, we complete the proof of (i).

For (ii)–(xii), repeat the same argument as in the proof of (i) by either applying (2.5) and Theorem 2.8 or applying (2.6) and Theorem 2.5. □

3. Evaluations of $G(q)$

In this section, we establish the explicit evaluations of the cubic continued fraction. We explicitly evaluate 25 new values of $G(e^{-\pi\sqrt{n}})$ and 11 new values of $G(-e^{-\pi\sqrt{n}})$ for some n. We first evaluate $G(e^{-\pi\sqrt{n}})$ for $n = 12$, 108, $\frac{4}{3}$, and $\frac{4}{27}$.

Theorem 3.1. *We have*

(i) $G(e^{-2\sqrt{3}\pi}) = \dfrac{1}{2} - \dfrac{1}{4}(2 + \sqrt[3]{4}\,)(1 - \sqrt{3} + \sqrt[3]{2}\,)$,

(ii) $G(e^{-6\sqrt{3}\pi})$

$$= \frac{1}{2} + \frac{\sqrt{3}\left(2 + \sqrt[3]{2} - \sqrt[3]{4} - \sqrt[3]{36} + 3^{5/6} - \sqrt[3]{9(\sqrt[3]{2} - 1)^2}\right)}{2(\sqrt[3]{2} + \sqrt[3]{4} - 2\sqrt[3]{3}\,)},$$

(iii) $G(e^{-2\pi/\sqrt{3}}) = \dfrac{\sqrt[3]{-5+3\sqrt{3}}}{2}$,

(iv) $G(e^{-2\pi/3\sqrt{3}}) = \dfrac{\sqrt[3]{3}\left(-1+\sqrt{3+2\sqrt[3]{2}+2\sqrt[3]{4}}\right)}{2(\sqrt[3]{2}+\sqrt[3]{4}\,)}$,

Proof. The proofs are clear by (1.3) and Theorem 2.7. □

See [12, Theorem 6.3.7(ii)] and [8, Theorem 5.1(i)] for alternative proofs of Theorem 3.1(i) and (iii), respectively.

We now evaluate $G(e^{-\pi\sqrt{n}})$ for $n = 20$, 180, $\frac{4}{5}$, $\frac{36}{5}$, $\frac{20}{9}$, and $\frac{4}{45}$.

Theorem 3.2. *We have*

(i) $G(e^{-2\sqrt{5}\pi}) = \dfrac{\sqrt{3}+\sqrt{5}-3(1+\sqrt{3}\,)\sqrt{-2+\sqrt{5}}}{2(-3+\sqrt{5}\,)}$,

(ii) $G(e^{-6\sqrt{5}\pi}) = \dfrac{1}{2} - \dfrac{3\sqrt[3]{2\sqrt{3}+2\sqrt{5}}}{2\left(1+\sqrt{1+2\sqrt{3}+2\sqrt{5}}+\sqrt[3]{2\sqrt{3}+2\sqrt{5}}\right)}$,

(iii) $G(e^{-2\sqrt{5}\pi/3}) = \dfrac{-1+\sqrt{1-2\sqrt{3}+2\sqrt{5}}}{2\sqrt[3]{-2\sqrt{3}+2\sqrt{5}}}$,

(iv) $G(e^{-2\pi/\sqrt{5}}) = \dfrac{1+\sqrt{9+6\sqrt{5}-6\sqrt{6+3\sqrt{5}}}}{2(1+\sqrt{3}\,)}$,

(v) $G(e^{-2\pi/3\sqrt{5}}) = \dfrac{-1+\sqrt{1+2\sqrt{3}+2\sqrt{5}}}{2\sqrt[3]{2\sqrt{3}+2\sqrt{5}}}$,

(vi) $G(e^{-6\pi/\sqrt{5}}) = \dfrac{1}{2} - \dfrac{3\sqrt[3]{-2\sqrt{3}+2\sqrt{5}}}{2\left(1+\sqrt{1-2\sqrt{3}+2\sqrt{5}}+\sqrt[3]{-2\sqrt{3}+2\sqrt{5}}\right)}$.

Proof. The results follow from (1.3) and Theorem 2.8. □

We now find $G(e^{-\pi\sqrt{n}})$ for $n = 3$, 27, 48, 80, 432, $\frac{1}{3}$, $\frac{16}{3}$, $\frac{1}{27}$, and $\frac{16}{27}$.

Theorem 3.3. *We have*

(i) $G(e^{-\sqrt{3}\pi}) = \dfrac{1}{2}\left(-6-5\sqrt[3]{2}-4\sqrt[3]{4}+\sqrt{3}\,(4+3\sqrt[3]{2}+2\sqrt[3]{4}\,)\right)$,

(ii) $G(e^{-3\sqrt{3}\pi})$

$$= \frac{1}{2} - \frac{3\left(4+2\sqrt[3]{2}+\sqrt[3]{4}+2\sqrt[3]{9}-2\sqrt[3]{9}\sqrt{3+2\sqrt[3]{2}+2\sqrt[3]{4}}\right)}{2(\sqrt[3]{2}+\sqrt[3]{4}-2\sqrt[3]{3}\,)^2},$$

(iii) $G(e^{-4\sqrt{3}\pi}) = \dfrac{1}{2} - \dfrac{3\left(1-\sqrt{\sqrt[3]{2}+\sqrt[3]{4}-\sqrt{3}}\right)}{2\left(2-\sqrt[3]{5+3\sqrt{3}}\right)}$,

(iv) $G(e^{-12\sqrt{3}\pi}) = \dfrac{1-\sqrt{3\sqrt{3}\,a(1-\sqrt{3}\,a+a^2)}}{2-2\sqrt{3}\,a}$,

(v) $G(e^{-\pi/\sqrt{3}}) = \dfrac{-1+\sqrt{3}}{\sqrt[3]{4}}$,

(vi) $G(e^{-4\pi/\sqrt{3}}) = \dfrac{1+\sqrt{3}-\sqrt{6}}{2\sqrt[3]{4}}$,

(vii) $G(e^{-\pi/3\sqrt{3}}) = -\dfrac{1}{\sqrt[3]{3}} + \sqrt[3]{\dfrac{5+3\sqrt{3}}{6}}$,

(viii) $G(e^{-4\pi/3\sqrt{3}}) = \dfrac{(1+\sqrt[3]{2})\left(1-\sqrt{2+2\sqrt[3]{2}-\sqrt{3+6\sqrt[3]{2}+6\sqrt[3]{4}}}\right)}{\sqrt[3]{12}\left(-1+\sqrt{3+2\sqrt[3]{2}+2\sqrt[3]{4}}\right)}$,

where

$$a = \frac{-2-\sqrt[3]{2}+\sqrt[3]{4}+\sqrt[3]{36}-3^{5/6}+\sqrt[3]{9(-1+\sqrt[3]{2})^2}}{\sqrt[3]{2}+\sqrt[3]{4}-2\sqrt[3]{3}}.$$

Proof. The results are immediate consequences of (1.3) and Theorem 2.11. □

See [12, Theorem 6.3.3(i)], [12, Theorem 6.3.3(vi)], and [8, Theorem 5.1(ii)] for alternative proofs of Theorem 3.3(i), (v), and (vi), respectively. We further evaluate $G(e^{-\pi\sqrt{n}})$ for $n = 5$, 45, 80, 720, $\frac{1}{5}$, $\frac{9}{5}$, $\frac{16}{5}$, $\frac{144}{5}$, $\frac{5}{9}$, $\frac{80}{9}$, $\frac{1}{45}$, and $\frac{16}{45}$.

Theorem 3.4. *We have*

(i) $G(e^{-\sqrt{5}\pi}) = \dfrac{1}{2} + \dfrac{(3+\sqrt{3})(5+3\sqrt{5})}{4(\sqrt{3}-\sqrt{5})} + 3\sqrt{\dfrac{-1+2\sqrt{3}+2\sqrt{5}}{94-42\sqrt{5}}}$,

(ii) $G(e^{-3\sqrt{5}\pi})$

$$= \frac{1}{2} - \frac{3\left(2+2b^2+b^3+2\sqrt{1+b^3}\right)}{4(1+b+\sqrt{1+b^3})^2} + \frac{3\sqrt{(2-b^3)\sqrt{1+b^3}+\sqrt{5}\,b^3}}{2(1+b+\sqrt{1+b^3})^2},$$

(iii) $G(e^{-4\sqrt{5}\pi}) = \dfrac{1}{2} - \dfrac{(3+\sqrt{3})\left(2-\sqrt{6+6\sqrt{5}-6\sqrt{2+2\sqrt{5}}}\right)}{2\left(3+2\sqrt{3}-\sqrt{5}-\sqrt{6+6\sqrt{5}}\right)}$,

(iv) $G(e^{-12\sqrt{5}\pi})$

$$= \frac{1}{2}+\frac{3b}{2-4b+2\sqrt{1+b^3}}\left(1-\sqrt{\frac{2+2b^2+b^3+2\sqrt{1+b^3}}{b(1+b+\sqrt{1+b^3})}-1}\right),$$

(v) $G(e^{-\pi/\sqrt{5}}) = \dfrac{1}{2} - \dfrac{(3-\sqrt{3})(5+3\sqrt{5})}{4(\sqrt{3}+\sqrt{5})} + \dfrac{3}{2}\sqrt{\dfrac{-2-4\sqrt{3}+4\sqrt{5}}{47-21\sqrt{5}}}$,

(vi) $G(e^{-3\pi/\sqrt{5}}) = \dfrac{1}{2} - \dfrac{3\left(1-\sqrt{1-2\sqrt{3}+2\sqrt{5}}+\sqrt[3]{4-\sqrt{15}}\right)}{\left(2-\sqrt[3]{-2\sqrt{3}+2\sqrt{5}}\right)^2}$,

(vii) $G(e^{-4\pi/\sqrt{5}}) = \dfrac{1}{2} + \dfrac{2\sqrt{3}-3\sqrt{2+2\sqrt{5}+2\sqrt{2+2\sqrt{5}}}}{3-\sqrt{3}+\sqrt{5}+\sqrt{15}+2\sqrt{3(2+\sqrt{3})(1+\sqrt{5})}}$,

(viii) $G(e^{-12\pi/\sqrt{5}})$

$$= \frac{1}{2} - \frac{3b}{4b-b^2-\sqrt{8b+b^4}}\left(1-\sqrt{1-\frac{8b-2b^2-2\sqrt{8b+b^4}}{4b-b^3+b\sqrt{8b+b^4}}}\right),$$

(ix) $G(e^{-\sqrt{5}\pi/3}) = \dfrac{\sqrt[3]{4+\sqrt{15}}}{2}\left(\sqrt{1-2\sqrt{3}+2\sqrt{5}}-1\right)$,

(x) $G(e^{-4\sqrt{5}\pi/3}) = \dfrac{\sqrt{4+\sqrt{3}+\sqrt{5}-\sqrt{17+4\sqrt{3}+8\sqrt{5}+2\sqrt{15}}}-1}{\left(\sqrt{-2\sqrt{3}+2\sqrt{5}}\right)\sqrt[3]{2\sqrt{3}+2\sqrt{5}}}$,

(xi) $G(e^{-\pi/3\sqrt{5}}) = \dfrac{\sqrt{1+2\sqrt{3}+2\sqrt{5}}-1}{2\sqrt[3]{4+\sqrt{15}}}$,

(xii) $G(e^{-4\pi/3\sqrt{5}}) = \dfrac{\sqrt{4b+4b^4-(4b+b^4)\sqrt{1+b^3}}-b^2}{2b(1-\sqrt{1+b^3})}$,

where

$$b = \sqrt[3]{2\sqrt{3}+2\sqrt{5}}.$$

Proof. The proofs are straightforward by (1.3) and Theorem 2.12. □

We turn to the evaluations of $G(-e^{-\pi\sqrt{n}})$ for $n = 3$, 27, $\frac{1}{3}$, and $\frac{1}{27}$.

Theorem 3.5. *We have*

(i) $G(-e^{-\sqrt{3}\pi})=\dfrac{1}{2}-\dfrac{3}{2(1+\sqrt[3]{2}\,)}$,

(ii) $G(-e^{-3\sqrt{3}\pi})=\dfrac{1}{2}-\dfrac{3\sqrt[3]{3}}{2(\sqrt[3]{2}+\sqrt[3]{3}+\sqrt[3]{4}\,)}$,

(iii) $G(-e^{-\pi/\sqrt{3}})=-\dfrac{1}{\sqrt[3]{4}}$,

(iv) $G(-e^{-\pi/3\sqrt{3}})=-\dfrac{1+\sqrt[3]{2}}{\sqrt[3]{12}}$.

Proof. The results follow directly from (1.4) and Theorem 2.4. □

See [12, Theorem 6.3.5(ii) and (vi)] and [1, Theorem 5.6(iv) and (iii)] for different proofs of Theorem 3.5(i) and (iii), respectively. See also [1, Theorem 5.6(v)] for a different proof of Theorem 3.5(iv). We now evaluate $G(-e^{-\pi\sqrt{n}})$ for $n=5$, 45, $\frac{1}{5}$, $\frac{9}{5}$, $\frac{5}{9}$, and $\frac{1}{45}$.

Theorem 3.6. *We have*

(i) $G(-e^{-\sqrt{5}\pi})=\dfrac{-3+\sqrt{5}}{2(\sqrt{3}+\sqrt{5}\,)}$,

(ii) $G(-e^{-3\sqrt{5}\pi})=\dfrac{1}{2}-\dfrac{3\left(1-\sqrt[3]{2\sqrt{3}+2\sqrt{5}}+2\sqrt[3]{4+\sqrt{15}}\right)}{2(1+2\sqrt{3}+2\sqrt{5}\,)}$,

(iii) $G(-e^{-\pi/\sqrt{5}})=\dfrac{-2+\sqrt{3}-\sqrt{15}}{2(1+\sqrt{3}\,)}$,

(iv) $G(-e^{-3\pi/\sqrt{5}})=\dfrac{1}{2}-\dfrac{3}{2\left(1+\sqrt[3]{-2\sqrt{3}+2\sqrt{5}}\right)}$,

(v) $G(-e^{-\sqrt{5}\pi/3})=-\dfrac{\sqrt[3]{-\sqrt{3}+\sqrt{5}}}{\sqrt[3]{4}}$,

(vi) $G(-e^{-\pi/3\sqrt{5}})=-\dfrac{\sqrt[3]{\sqrt{3}+\sqrt{5}}}{\sqrt[3]{4}}$.

Proof. The results follow from (1.4) and Theorem 2.5. □

See [1, Theorem 5.6(i) and (ii)] for different proofs of Theorem 3.6(i) and (iii). We close this section by evaluating $G(-e^{-\pi\sqrt{n}})$ for $n=20$, 180, $\frac{4}{5}$, $\frac{36}{5}$, $\frac{20}{9}$, and $\frac{4}{45}$.

Theorem 3.7. *We have*

(i) $$G(-e^{-2\sqrt{5}\pi}) = \frac{3-\sqrt{5}}{\sqrt{3}+\sqrt{5}-3(1+\sqrt{3}\,)\sqrt{-2+\sqrt{5}}} \times \left(1 - \frac{(3+\sqrt{3}\,)\left(2-\sqrt{6+6\sqrt{5}-6\sqrt{2+2\sqrt{5}}}\right)}{3+2\sqrt{3}-\sqrt{5}-\sqrt{6+6\sqrt{5}}}\right),$$

(ii) $$G(-e^{-6\sqrt{5}\pi}) = -\frac{1+b+\sqrt{1+b^3}}{(1-2b+\sqrt{1+b^3}\,)^2} \times \left(1+b+\sqrt{1+b^3}-3b\sqrt{-1+\frac{2+2b^2+b^3+2\sqrt{1+b^3}}{b(1+b+\sqrt{1+b^3})}}\right),$$

(iii) $$G(-e^{-2\pi/\sqrt{5}}) = -\frac{1+\sqrt{3}}{1+\sqrt{9+6\sqrt{5}-6\sqrt{6+2\sqrt{5}}}} \times \left(1+\frac{4\sqrt{3}-6\sqrt{2+2\sqrt{5}+2\sqrt{2+2\sqrt{5}}}}{3-\sqrt{3}+\sqrt{5}+\sqrt{15}+2\sqrt{3(2+\sqrt{3}\,)(1+\sqrt{5}\,)}}\right),$$

(iv) $$G(-e^{-6\pi/\sqrt{5}}) = \frac{2b+b^2+\sqrt{8b+b^4}}{4b-b^2-\sqrt{8b+b^4}} \times \left(1-\frac{6b}{4b-b^2-\sqrt{8b+b^4}}\left(1-\sqrt{1-\frac{8b-2b^2-2\sqrt{8b+b^4}}{4b-b^3+b\sqrt{8b+b^4}}}\right)\right),$$

(v) $$G(-e^{-2\sqrt{5}\pi/3}) = \frac{2\sqrt{4+\sqrt{3}+\sqrt{5}-\sqrt{17+4\sqrt{3}+8\sqrt{5}+2\sqrt{15}}}-2}{\sqrt[3]{4+\sqrt{15}}\left(1-\sqrt{-2\sqrt{3}+2\sqrt{5}}\right)^2},$$

(vi) $$G(-e^{-2\pi/3\sqrt{5}}) = \frac{\sqrt{4b+4b^4-(4b+b^4)\sqrt{1+b^3}\,)}-b^2}{(1-\sqrt{1+b^3}\,)^2},$$

where

$$b = \sqrt[3]{2\sqrt{3}+2\sqrt{5}}.$$

Proof. For (i), let $n = 20$ in $G(e^{-2\pi\sqrt{n}}) = -G(e^{-\pi\sqrt{n}})G(-e^{-\pi\sqrt{n}})$ in [12, Lemma 6.3.6] and put the values of $G(e^{-4\sqrt{5}\pi})$ and $G(e^{-2\sqrt{5}\pi})$ in Theorems 3.4(iii) and 3.2(i), respectively, then we have the desired result. The proofs of (ii)–(vi) are similar to that of (i). □

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