Korean J. Math. **27** (2019), No. 4, pp. 1061–1075
https://doi.org/10.11568/kjm.2019.27.4.1061

SASAKIAN STRUCTURES ON PRODUCTS OF REAL LINE AND KÄHLERIAN MANIFOLD

Gherici Beldjilali, Ahmed Mohammed Cherif, and Kaddour Zaga

Abstract. In this paper, we construct a Sasakian manifold by the product of real line and Kählerian manifold with exact Kähler form. This result demonstrates the close relation between Sasakian and Kählerian manifold with exact Kähler form. We present an example and an open problem.

1. Introduction

A $(2n+1)$-dimensional Riemannian manifold (M,g) is said to be an almost contact metric manifold if there exist a $(1,1)$-tensor field φ on M, a vector field $\xi \in \Gamma(TM)$ and a 1-form $\eta \in \Gamma(T^*M)$, such that:

$$\eta(\xi) = 1, \quad \varphi^2 X = -X + \eta(X)\xi$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for all $X, Y \in \Gamma(TM)$. In particular, in an almost contact metric manifold we also have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. It can be proved that an almost contact metric manifold is Sasakian if and only if (see [1], [2], [6]):

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X, \tag{1.1}$$

Received Autust 13, 2019. Revised October 30, 2019. Accepted November 6, 2019.

2010 Mathematics Subject Classification: 53C15, 53C25, 53C55.

Key words and phrases: Product manifolds, Sasakian manifolds, Kählerian manifolds.

for any $X, Y \in \Gamma(TM)$. For a Sasakian manifold the following equation holds:

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{1.2}$$

$$\nabla_X \xi = -\varphi X, \quad (\nabla_X \eta)Y = -g(\varphi X, Y). \tag{1.3}$$

The sectional curvature of the plane section spanned by the unit tangent vector field X orthogonal to ξ and φX is called a φ-sectional curvature. If any Sasakian manifold M has a constant φ-sectional curvature c, then M is called a Sasakian space form and denoted by $M^{2n+1}(c)$. The Riemannian curvature tensor of Sasakian space form is given by the following formula:

$$R(X,Y) = X\wedge Y + \frac{c-1}{4}\big(\varphi^2 X\wedge\varphi^2 Y + \varphi X\wedge\varphi Y + 2g(X,\varphi Y)\varphi\big), \tag{1.4}$$

where $(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y$ for all $X, Y, Z \in \Gamma(TM)$.

Let N be an $2n$-dimensional differentiable manifold. An almost Hermitian structure on N is by definition a pair (J, h) of an almost complex structure J and a Riemannian metric g satisfying:

$$J^2X = -X, \quad h(JX, JY) = h(X,Y), \tag{1.5}$$

for all $X, Y \in \Gamma(TN)$. An almost complex stucture J is integrable, and hence the manifold is a complex manifold, if and only if its Nijenhuis tensor N_J vanishes, with:

$$N_J(X,Y) = [JX, JY] - [X,Y] - J[X, JY] - J[JX, Y]. \tag{1.6}$$

For an almost Hermitian manifold (N, J, h), we define the Kähler form Ω as $\Omega(X,Y) = h(X, JY)$. (N, J, h) is then called almost Kähler if Ω is closed i.e. $d\Omega = 0$. It can be shown that this condition for (N, J, h) to be almost Kähler is equivalent to:

$$h((\nabla_X J)Y, Z) + h((\nabla_Y J)Z, X) + h((\nabla_Z J)X, Y) = 0. \tag{1.7}$$

An almost Kähler manifold with integrable J is called a Kähler manifold, and thus is characterized by the conditions $d\Omega = 0$ and $N_J = 0$. One can prove that both of these conditions combined are equivalent with the single condition (see [6], [3]):

$$\nabla_X JY = J\nabla_X Y, \quad X, Y \in \Gamma(TN). \tag{1.8}$$

In the next section, we construct a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ by the product of real line $\mathbb{R}$ and Kählerian manifold (N, J, h) with exact Kähler form, that is $\Omega = d\theta$, where $\theta \in \Gamma(T^*N)$ (Theorem 2.1).

2. Sasakian structures on products of $\mathbb{R}$ and Kählerian manifold

Let (N, J, h) be a $2n$-dimensional almost Hermitian manifold with exact Kähler form, such that $\Omega = d\theta$, where $\theta \in \Gamma(T^*N)$, and let $M = \mathbb{R} \times N$ be the product manifold of a real line $\mathbb{R}$ and N equipped with the following Riemannian metric $g = h + \eta \otimes \eta$ with $\eta = dr + \theta$, where r is the standard coordinate with respect to the frame ∂_r on $\mathbb{R}$. For all $X, Y \in \Gamma(TN)$, we have:

$$g(X,Y) = h(X,Y) + \theta(X)\theta(Y), \quad g(X, \partial_r) = \theta(X), \quad g(\partial_r, \partial_r) = 1.$$

PROPOSITION 2.1. *We set:*

$$\xi = \partial_r, \quad \varphi\partial_r = 0, \quad \varphi X = JX - \theta(JX)\xi, \quad \forall X \in \Gamma(TN). \tag{2.1}$$

Then, (φ, ξ, η, g) defines an almost contact metric structure on M.

Proof. We have $\eta = dr + \theta$ and $\xi = \partial_r$, so $\eta(\xi) = 1$. As $\varphi\partial_r = 0$, we get $\varphi^2\partial_r = 0$, on the other hand, $-\partial_r + \eta(\partial_r)\partial_r = 0$. Let $X \in \Gamma(TN)$, we compute:

$$\begin{aligned}
\varphi^2 X &= \varphi(JX - \theta(JX)\xi) \\
&= \varphi(JX) - \theta(JX)\varphi\xi \\
&= J^2X - \theta(J^2X)\xi \\
&= -X + \theta(X)\xi \\
&= -X + \eta(X)\xi.
\end{aligned}$$

Let $X, Y \in \Gamma(TN)$, we have

$$\begin{aligned}
g(\varphi X, \varphi Y) &= g(JX - \theta(JX)\xi, JY - \theta(JY)\xi) \\
&= g(JX, JY) - \theta(JY)g(JX, \xi) - \theta(JX)g(\xi, JY) \\
&\quad + \theta(JX)\theta(JY)g(\xi, \xi),
\end{aligned}$$

by the definition of the metric g with $\eta = dr + \theta$, we obtain

$$\begin{aligned}
g(\varphi X, \varphi Y) &= h(JX, JY) + \theta(JX)\theta(JY) - \theta(JX)\theta(JY) \\
&\quad - \theta(JX)\theta(JY) + \theta(JX)\theta(JY) \\
&= h(JX, JY),
\end{aligned}$$

as $h(JX,JY)=h(X,Y)$ and $g(X,Y)=h(X,Y)+\theta(X)\theta(Y)$, we conclude that

$$\begin{aligned} g(\varphi X,\varphi Y) &= g(X,Y)-\theta(X)\theta(Y) \\ &= g(X,Y)-\eta(X)\eta(Y). \end{aligned}$$

As $\varphi\xi=0$, $g(X,\xi)=\theta(X)=\eta(X)$, $\eta(\xi)=1$ and $g(\xi,\xi)=1$, we get

$$g(\varphi X,\varphi\xi)=g(X,\xi)-\eta(X)\eta(\xi)=0,$$

$$g(\varphi\xi,\varphi\xi)=g(\xi,\xi)-\eta(\xi)\eta(\xi)=0.$$

□

Now, we denote by ∇^N (resp. ∇^M) the covariant derivative with respect to the metric h on N (resp. g on M). From the koszul formula (see [3], [4]), we have the following:

PROPOSITION 2.2. *For all $X,Y,Z\in\Gamma(TN)$, we have:*

1): $g(\nabla^M_\xi\xi,\xi)=g(\nabla^M_\xi\xi,X)=0$;
2): $g(\nabla^M_\xi X,\xi)=g(\nabla^M_X\xi,\xi)=0$;
3): $g(\nabla^M_\xi X,Y)=g(\nabla^M_X\xi,Y)=h(X,JY)$;
4): $g(\nabla^M_X Y,\xi)=\frac{1}{2}\big[X\theta(Y)+Y\theta(X)+\theta([X,Y])\big]$;
5): $g(\nabla^M_X Y,Z)=h(\nabla^N_X Y,Z)+\frac{1}{2}\big[X(\theta(Y))+Y(\theta(X))+\theta([X,Y])\big]\theta(Z)$
: $+h(X,JZ)\theta(Y)+h(Y,JZ)\theta(X)$.

Proof. **5)** First we shall use the Koszul formula for a Riemann metric g and the Levi-Civita connection ∇^M:

$$\begin{aligned} 2g(\nabla^M_X Y,Z) &= Xg(Y,Z)+Yg(Z,X)-Zg(X,Y) \\ &\quad +g(Z,[X,Y])+g(Y,[Z,X])-g(X,[Y,Z]), \end{aligned}$$

by the definition of the metric g, we have:

$$\begin{aligned} 2g(\nabla^M_X Y,Z) &= Xh(Y,Z)+X(\theta(Y)\theta(Z))+Yh(Z,X)+Y(\theta(Z)\theta(X)) \\ &\quad -Zh(X,Y)-Z(\theta(X)\theta(Y))+h(Z,[X,Y])+\theta(Z)\theta([X,Y]) \\ &\quad +h(Y,[Z,X])+\theta(Y)\theta([Z,X])-h(X,[Y,Z])-\theta(X)\theta([Y,Z]), \end{aligned}$$

by the Koszul formula for h and ∇^N, we get:

$$\begin{aligned} 2g(\nabla^M_X Y,Z) &= 2h(\nabla^N_X Y,Z)+X(\theta(Y))\theta(Z)+\theta(Y)X(\theta(Z)) \\ &\quad +Y(\theta(Z))\theta(X)+\theta(Z)Y(\theta(X))-Z(\theta(X))\theta(Y) \\ &\quad -\theta(X)Z(\theta(Y))+\theta(Z)\theta([X,Y])+\theta(Y)\theta([Z,X]) \\ &\quad -\theta(X)\theta([Y,Z]), \end{aligned}$$

from the formula:

$$2d\theta(X,Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y]),$$

(see [3]), we obtain:

$$\begin{aligned} 2g(\nabla^M_X Y, Z) &= 2h(\nabla^N_X Y, Z) + [X(\theta(Y)) + Y(\theta(X)) + \theta([X,Y])]\theta(Z) \\ &\quad + 2\theta(Y)d\theta(X,Z) + 2\theta(X)d\theta(Y,Z), \end{aligned}$$

finally, by the condition $\Omega = d\theta$ and the definition of Ω, we obtain:

$$\begin{aligned} g(\nabla^M_X Y, Z) &= h(\nabla^N_X Y, Z) + \theta(X)h(Y,JZ) + \theta(Y)h(X,JZ) \\ &\quad + \frac{1}{2}[X(\theta(Y)) + Y(\theta(X)) + \theta([X,Y])]\theta(Z). \end{aligned}$$

For **1)**, we have:

$$g(\nabla^M_\xi \xi, \xi) = \frac{1}{2}\xi g(\xi,\xi) = 0,$$

because $g(\xi,\xi) = 1$. We compute

$$\begin{aligned} 2g(\nabla^M_\xi \xi, X) &= \xi g(\xi, X) + \xi g(X,\xi) - Xg(\xi,\xi) \\ &\quad + g(X,[\xi,\xi]) + g(\xi,[X,\xi]) - g(\xi,[\xi,X]), \end{aligned}$$

as $g(\xi,X) = \eta(X)$, $g(\xi,\xi) = 1$, $[\xi,\xi] = 0$ and $[X,\xi] = [\xi,X] = 0$, we obtain

$$2g(\nabla^M_\xi \xi, X) = 2\xi\eta(X) = 0,$$

because $\eta(X) \in C^\infty(N)$ does not depend on r. **2)** we have:

$$g(\nabla^M_\xi X, \xi) = \xi g(X,\xi) - g(X, \nabla^M_\xi \xi) = 0,$$

from **1)** we obtain:

$$\begin{aligned} g(\nabla^M_\xi X, \xi) &= \xi g(X,\xi) - g(X, \nabla^M_\xi \xi) \\ &= \xi\eta(X) = 0. \end{aligned}$$

3) By the Koszul formula and the definition of g, we have:

$$\begin{aligned} 2g(\nabla^M_\xi X, Y) &= \xi g(X,Y) + Xg(Y,\xi) - Yg(\xi,X) \\ &\quad + g(Y,[\xi,X]) + g(X,[Y,\xi]) - g(\xi,[X,Y]) \\ &= \xi h(X,Y) + \xi(\theta(X)\theta(Y)) + X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y]) \\ &= X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y]), \end{aligned}$$

here $[\xi,X] = [Y,\xi] = 0$ and $\xi h(X,Y) = \xi(\theta(X)\theta(Y)) = 0$. We conclude that:

$$g(\nabla^M_\xi X, Y) = d\theta(X,Y) = h(X,JY).$$

We compute:

$$\begin{aligned}
2g(\nabla^M_X \xi, Y) &= Xg(\xi, Y) + \xi g(Y, X) - Yg(X, \xi) \\
&\quad + g(Y, [X, \xi]) + g(\xi, [Y, X]) - g(X, [\xi, Y]) \\
&= X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y]) \\
&= 2d\theta(X, Y),
\end{aligned}$$

that is, $g(\nabla^M_X \xi, Y) = h(X, JY)$. **4)** We have:

$$\begin{aligned}
g(\nabla^M_X Y, \xi) &= Xg(Y, \xi) - g(Y, \nabla^M_X \xi) \\
&= X(\theta(Y)) - h(X, JY) \\
&= X(\theta(Y)) - d\theta(X, Y) \\
&= X(\theta(Y)) - \frac{1}{2}\big[X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y])\big] \\
&= \frac{1}{2}\big[X(\theta(Y)) + Y(\theta(X)) + \theta([X, Y])\big].
\end{aligned}$$

□

Lemma 2.1. *We choose an orthonormal basis $\{e_1, ..., e_{2n}\}$ of the tangent space T_xN at each point $x \in N$, then*

$$\{\xi, e_1 - \theta(e_1)\xi, ..., e_{2n} - \theta(e_{2n})\xi\}$$

is an orthonormal basis of $T_{(r,x)}M$, where $r \in \mathbb{R}$.

Proof. For any $i, j \in \{1, ..., 2n\}$, we have:

$$\begin{aligned}
g(e_i - \theta(e_i)\xi, e_j - \theta(e_j)\xi) &= g(e_i, e_j) - \theta(e_j)g(e_i, \xi) - \theta(e_i)g(\xi, e_j) \\
&\quad + \theta(e_i)\theta(e_j)g(\xi, \xi) \\
&= h(e_i, e_j) + \theta(e_i)\theta(e_j) - \theta(e_i)\theta(e_j) - \theta(e_i)\theta(e_j) \\
&\quad + \theta(e_i)\theta(e_j) \\
&= h(e_i, e_j) = \delta_{ij}.
\end{aligned}$$

We compute:

$$\begin{aligned}
g(e_i - \theta(e_i)\xi, \xi) &= g(e_i, \xi) - \theta(e_i)g(\xi, \xi) \\
&= \theta(e_i) - \theta(e_i) = 0,
\end{aligned}$$

with $g(\xi, \xi) = 1$. □

So that, for all vector $v \in T_{(r,x)}M$, there exist constants $a, b_1, ..., b_{2n}$ such that:

$$v = a\xi + \sum_{i=1}^{2n} b_i\big(e_i - \theta(e_i)\xi\big). \tag{2.2}$$

Note that, $a = g(v,\xi)$ and $b_i = g\big(v, e_i - \theta(e_i)\xi\big)$ for all $i = 1, .., 2n$. From the Proposition 2.2, and the Lemma 2.1, we get the following:

PROPOSITION 2.3. *For all $X, Y \in \Gamma(TN)$, we have:*

1): $\nabla^M_\xi \xi = 0$;
2): $\nabla^M_\xi X = \nabla^M_X \xi = -\varphi X$;
3): $\nabla^M_X Y = \nabla^N_X Y - \theta(Y)\varphi X - \theta(X)\varphi Y + \frac{1}{2}\big[(\nabla^N_X\theta)Y + (\nabla^N_Y\theta)X\big]\xi$.

Proof. Let $\{e_1, ..., e_{2n}\}$ be an orthonormal frame on N. From the Proposition 2.2, we have:
1)

$$g(\nabla^M_\xi \xi, \xi) = 0,$$

$$g(\nabla^M_\xi \xi, e_i - \theta(e_i)\xi) = g(\nabla^M_\xi \xi, e_i) - \theta(e_i) g(\nabla^M_\xi \xi, \xi) = 0.$$

2)

$$g(\nabla^M_\xi X, \xi) = 0,$$

$$\begin{aligned} g(\nabla^M_\xi X, e_i - \theta(e_i)\xi) &= g(\nabla^M_\xi X, e_i) - \theta(e_i) g(\nabla^M_\xi X, \xi) \\ &= h(X, Je_i) = -h(JX, e_i), \end{aligned}$$

so that:

$$\begin{aligned} \nabla^M_\xi X &= -h(JX, e_i)e_i + h(JX, e_i)\theta(e_i)\xi \\ &= -JX + \theta(JX)\xi = -\varphi X, \end{aligned}$$

with the same method we find that $\nabla^M_X \xi = -\varphi X$.
3) We compute:

$$g(\nabla^M_X Y, \xi) = \; = \frac{1}{2}\big[X\theta(Y) + Y\theta(X) + \theta([X,Y])\big],$$

$$\begin{aligned}
g(\nabla^M_X Y, e_i-\theta(e_i)\xi) &= g(\nabla^M_X Y, e_i)-\theta(e_i)g(\nabla^M_X Y,\xi)\\
&= h(\nabla^N_X Y, e_i)+\frac{1}{2}\big[X(\theta(Y))+Y(\theta(X))+\theta([X,Y])\big]\theta(e_i)\\
&\quad +h(X,Je_i)\theta(Y)+h(Y,Je_i)\theta(X)\\
&\quad -\frac{1}{2}\theta(e_i)\big[X\theta(Y)+Y\theta(X)+\theta([X,Y])\big]\\
&= h(\nabla^N_X Y, e_i)-h(JX,e_i)\theta(Y)-h(JY,e_i)\theta(X),
\end{aligned}$$

we conclude that:

$$\begin{aligned}
\nabla^M_X Y &= \frac{1}{2}\big[X\theta(Y)+Y\theta(X)+\theta([X,Y])\big]\xi\\
&\quad +h(\nabla^N_X Y,e_i)e_i-h(\nabla^N_X Y,e_i)\theta(e_i)\xi\\
&\quad -h(JX,e_i)\theta(Y)e_i+h(JX,e_i)\theta(Y)\theta(e_i)\xi\\
&\quad -h(JY,e_i)\theta(X)e_i-h(JY,e_i)\theta(X)\theta(e_i)\xi\\
&= \frac{1}{2}\big[X\theta(Y)+Y\theta(X)+\theta([X,Y])\big]\xi\\
&\quad +\nabla^N_X Y-\theta(\nabla^N_X Y)\xi\\
&\quad -\theta(Y)JX+\theta(Y)\theta(JX)\xi\\
&\quad -\theta(X)JY-\theta(X)\theta(JY)\xi,
\end{aligned}$$

note that:

$$-\theta(Y)JX+\theta(Y)\theta(JX)\xi=-\theta(Y)\varphi X,\quad -\theta(X)JY-\theta(X)\theta(JY)\xi=-\theta(X)\varphi Y,$$

$$\frac{1}{2}\big[X\theta(Y)+Y\theta(X)+\theta([X,Y])\big]\xi-\theta(\nabla^N_X Y)\xi=\frac{1}{2}\big[(\nabla^N_X\theta)Y+(\nabla^N_Y\theta)X\big]\xi.$$

□

From conditions (1.1), (1.8), and the Proposition 2.3, we deduce:

THEOREM 2.1. *The manifold (M,φ,ξ,η,g) is Sasakian if and only if (N,J,h) is Kählerian manifold.*

Proof. The manifold (M,φ,ξ,η,g) is Sasakian if it satisfies:

$$(\nabla^M_\xi\varphi)\xi = g(\xi,\xi)\xi-\eta(\xi)\xi \tag{2.3}$$

$$(\nabla^M_\xi\varphi)X = g(\xi,X)\xi-\eta(X)\xi \tag{2.4}$$

$$(\nabla^M_X\varphi)\xi = g(X,\xi)\xi-\eta(\xi)X \tag{2.5}$$

$$(\nabla^M_X\varphi)Y = g(X,Y)\xi-\eta(Y)X, \tag{2.6}$$

for all $X, Y \in \Gamma(TN)$. It is easy to prove that conditions (2.3), (2.4), and (2.5) are satisfied, and we have:

$$(\nabla^M_\xi \varphi)\xi = \nabla^M_\xi \varphi\xi - \varphi(\nabla^M_\xi \xi) = g(\xi,\xi)\xi - \eta(\xi)\xi = 0,$$

$$\begin{aligned}(\nabla^M_\xi \varphi)X &= \nabla^M_\xi \varphi X - \varphi(\nabla^M_\xi X) \\ &= \nabla^M_\xi JX - \nabla^M_\xi(\theta(JX)\xi) + \varphi^2(X) \\ &= -\varphi(JX) - \theta(JX)\nabla^M_\xi \xi - X + \theta(X)\xi \\ &= -(J^2X - \theta(J^2X)\xi) - X + \theta(X)\xi \\ &= X + \theta(X)\xi - X + \theta(X)\xi = 0,\end{aligned}$$

so that, $(\nabla^M_\xi \varphi)X = g(\xi, X)\xi - \eta(X)\xi = 0$. We compute:

$$\begin{aligned}(\nabla^M_X \varphi)\xi &= \nabla^M_X \varphi\xi - \varphi(\nabla^M_X \xi) \\ &= \varphi^2 X = -X + \theta(X)\xi,\end{aligned}$$

so, $(\nabla^M_X \varphi)\xi = g(X,\xi)\xi - \eta(\xi)X = -X + \theta(X)\xi$. For the condition (2.6), we compute:

$$(\nabla^M_X \varphi)Y = \nabla^M_X \varphi Y - \varphi(\nabla^M_X Y), \tag{2.7}$$

the first term of (2.7), is given by:

$$\begin{aligned}\nabla^M_X \varphi Y &= \nabla^M_X (JY - \theta(JY)\xi) \\ &= \nabla^M_X JY - X(\theta(JY))\xi - \theta(JY)\nabla^M_X \xi,\end{aligned}$$

using the Proposition 2.2, we have

$$\begin{aligned}\nabla^M_X \varphi Y &= \nabla^N_X JY - \theta(JY)\varphi X - \theta(X)\varphi JY + \frac{1}{2}\big[(\nabla^N_X \theta)JY + (\nabla^N_{JY}\theta)X\big]\xi \\ &\quad -X(\theta(JY))\xi + \theta(JY)\varphi X \\ &= \nabla^N_X JY + \theta(X)Y - \theta(X)\theta(Y)\xi - \frac{1}{2}X(\theta(JY))\xi - \frac{1}{2}\theta(\nabla^N_X JY)\xi \\ &\quad +\frac{1}{2}(JY)(\theta(X))\xi - \frac{1}{2}\theta(\nabla^N_{JY} X)\xi, \qquad (2.8)\end{aligned}$$

the second term of (2.7), is given by:

$$\begin{aligned}-\varphi(\nabla^M_X Y) &= -\varphi(\nabla^N_X Y) + \theta(Y)\varphi^2 X + \theta(X)\varphi^2 Y \\ &= -J\nabla^N_X Y + \theta(J\nabla^N_X Y)\xi - \theta(Y)X + 2\theta(X)\theta(Y)\xi - \theta(X)Y,\end{aligned} \tag{2.9}$$

Substituting the formulas (2.8) and (2.9) in (2.7), we obtain:

$$\begin{aligned}
(\nabla^M_X \varphi)Y &= (\nabla^N_X J)Y + \frac{1}{2}\big[(JY)(\theta(X)) - X(\theta(JY)) - \theta([JY, X])\big]\xi \\
&\quad -\theta(\nabla^N_X JY)\xi - \theta(Y)X + \theta(X)\theta(Y)\xi + \theta(J\nabla^N_X Y)\xi \\
&= (\nabla^N_X J)Y + d\theta(JY, X)\xi - \theta((\nabla^N_X J)Y)\xi - \theta(Y)X + \theta(X)\theta(Y)\xi \\
&= (\nabla^N_X J)Y + h(X, Y)\xi - \eta((\nabla^N_X J)Y)\xi - \eta(Y)X + \eta(X)\eta(Y)\xi \\
&= (\nabla^N_X J)Y - \eta((\nabla^N_X J)Y)\xi + g(X, Y)\xi - \eta(Y)X,
\end{aligned}$$

that is:

$$(\nabla^M_X \varphi)Y = -\varphi^2(\nabla^N_X J)Y + g(X, Y)\xi - \eta(Y)X, \tag{2.10}$$

the Theorem 2.1 follows from equation (2.10). □

REMARK 2.1.

1. From the Proposition 2.3, and the Theorem 2.1, the structure $(M, \varphi, \xi, \eta, g)$ is K-contact (i.e. $\nabla^M_X \xi = -\varphi X$) and not necessarily Sasakian.
2. Using Theorem 2.1, we can construct many examples for Sasakian manifolds.

EXAMPLE 2.1. For this example we use the product of the Kählerian manifold $(\mathbb{R}^2, J, h)$ by the real line $\mathbb{R}$, with $h = dx^2 + dy^2$ and $J\partial_x = \partial_y$, $J\partial_y = -\partial_x$. The Kähler form of $(\mathbb{R}^2, J, h)$ is given by $\Omega = -2dx \wedge dy$, we set $\theta = -2xdy$, we get $\Omega = d\theta$. Then by using (2.1) with $g = h + \eta \otimes \eta$ and $\eta = dr + \theta$, we have:

$$\xi = \partial_r, \quad \eta = dr - 2xdy$$

$$g = \begin{pmatrix} 1 & 0 & -2x \\ 0 & 1 & 0 \\ -2x & 0 & 1 + 4x^2 \end{pmatrix}, \quad \varphi = \begin{pmatrix} 0 & 2x & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We can easily verify that $(\mathbb{R} \times \mathbb{R}^2, \varphi, \xi, \eta, g)$ is a Sasakian manifold.

REMARK 2.2. By considering the above Theorem, it is proved that there are Kähler and Sasakian manifolds of any dimensions. We start with $\mathbb{R}^2$ and its natural Kähler structure. There is a 3-dimension Sasakian manifold $M = \mathbb{R} \times \mathbb{R}^2$. Then, by the method of Oubin$\tilde{a}$ [5] and consequently we have a Kählerian manifold of dimension 4. Continuing the current method, we produce Kähler and Kenmotsu manifolds of any dimensions $n \geq 2$.

Open Problem. Is it possible to construct a Sasakian manifold by the product of real line and Kählerian manifold without exact Kähler form?

2.1. Curvature formulas and main result. Suppose that the manifold (N, J, h) is Kählerian, so the manifold $(M, \varphi, \xi, \eta, g)$ defined in (2.1) is Sasakian. By a direct computation using the proposition (2.3), we get the following:

PROPOSITION 2.4. *For all $X, Y \in \Gamma(TN)$, we have:*

1): $R^M(X,Y)\xi = (X \wedge_g Y)\xi$;
2): $R^M(X,Y)Z = R^N(X,Y)Z + (X \wedge_g Y)Z$
: $\qquad -\Big(\varphi^2 X \wedge_g \varphi^2 Y + \varphi X \wedge_g \varphi Y + 2g(X,\varphi Y)\varphi\Big)Z$;
3): $R^M(\xi,X)Y = (\xi \wedge_g X)Y$;
4): $R^M(\xi,X)\xi = (\xi \wedge_g X)\xi$.

where R^M (resp. R^N) are the curvature tensors for g (resp. h).

Proof. Let $X, Y, Z \in \Gamma(TN)$. **1)** From equation (1.2), we have:

$$\begin{aligned} R^M(X,Y)\xi &= \eta(Y)X - \eta(X)Y \\ &= g(Y,\xi)X - g(X,\xi)Y \\ &= (X \wedge_g Y)\xi. \end{aligned}$$

2) Suppose that, at $x_0 \in N$, $\nabla^N_{U_i} U_j = 0$, $\forall i, j = 1,2,3$ with $U_1 = X$, $U_2 = Y$ and $U_3 = Z$. We compute:

$$\nabla^M_X \nabla^M_Y Z = \nabla^M_X \Big\{ \nabla^N_Y Z - \theta(Z)\varphi Y - \theta(Y)\varphi Z + \frac{1}{2}\big[(\nabla^N_Y \theta)Z + (\nabla^N_Z \theta)Y\big]\xi \Big\}$$

by using equation (1.3), we get:

$$\begin{aligned} (\nabla^N_Y \theta)Z + (\nabla^N_Z \theta)Y &= (\nabla^N_Y \eta)Z + (\nabla^N_Z \eta)Y \\ &= g(Y,\varphi Z) + g(Z,\varphi Y) = 0. \end{aligned}$$

So that:

$$\begin{aligned}
\nabla^M_X\nabla^M_Y Z &= \nabla^M_X\nabla^N_Y Z - \nabla^M_X(\theta(Z)\varphi Y) - \nabla^M_X(\theta(Y)\varphi Z)\\
&= \nabla^N_X\nabla^N_Y Z - \theta(\nabla^N_Y Z)\varphi X - \theta(X)\varphi(\nabla^N_Y Z)\\
&\quad -X(\theta(Z))\varphi Y - \theta(Z)\nabla^M_X\varphi Y - X(\theta(Y))\varphi Z - \theta(Y)\nabla^M_X\varphi Z\\
&= \nabla^N_X\nabla^N_Y Z - \eta(\nabla^N_Y Z)\varphi X - \eta(X)\varphi(\nabla^N_Y Z)\\
&\quad -X(\eta(Z))\varphi Y - \eta(Z)\nabla^M_X\varphi Y - X(\eta(Y))\varphi Z - \eta(Y)\nabla^M_X\varphi Z.
\end{aligned}$$

At x_0, we have:

$$\begin{aligned}
\nabla^M_X\nabla^M_Y Z &= \nabla^N_X\nabla^N_Y Z - X(\eta(Z))\varphi Y - \eta(Z)[(\nabla^M_X\varphi)Y + \varphi(\nabla^M_X Y)]\\
&\quad -X(\eta(Y))\varphi Z - \eta(Y)[(\nabla^M_X\varphi)Z + \varphi(\nabla^M_X Z)].
\end{aligned}$$

By equations (1.2) and (1.3), we conclude that:

$$\begin{aligned}
\nabla^M_X\nabla^M_Y Z &= \nabla^N_X\nabla^N_Y Z - [(\nabla^N_X\eta)Z]\varphi Y - \eta(Z)[g(X,Y)\xi - \eta(Y)X\\
&\quad +\varphi(\nabla^N_X Y) - \eta(Y)\varphi^2 X - \eta(X)\varphi^2 Y)] - [(\nabla^N_X\eta)Y]\varphi Z\\
&\quad -\eta(Y)[g(X,Z)\xi - \eta(Z)X + \varphi(\nabla^N_X Z)\\
&\quad -\eta(Z)\varphi^2 X - \eta(X)\varphi^2 Z)]\\
&= \nabla^N_X\nabla^N_Y Z - g(X,\varphi Z)\varphi Y - \eta(Z)[g(X,Y)\xi - \eta(Y)X\\
&\quad -\eta(Y)\varphi^2 X - \eta(X)\varphi^2 Y)] - g(X,\varphi Y)\varphi Z\\
&\quad -\eta(Y)[g(X,Z)\xi - \eta(Z)X - \eta(Z)\varphi^2 X - \eta(X)\varphi^2 Z)],
\end{aligned}$$

with the same method we find that:

$$\begin{aligned}
\nabla^M_Y\nabla^M_X Z &= \nabla^N_Y\nabla^N_X Z - g(Y,\varphi Z)\varphi X - \eta(Z)[g(Y,X)\xi - \eta(X)Y\\
&\quad -\eta(X)\varphi^2 Y - \eta(Y)\varphi^2 X)] - g(Y,\varphi X)\varphi Z\\
&\quad -\eta(X)[g(Y,Z)\xi - \eta(Z)Y - \eta(Z)\varphi^2 Y - \eta(Y)\varphi^2 Z)],
\end{aligned}$$

from the definition of curvature tensor R^M of M,

$$R^M(X,Y)Z = \nabla^M_X\nabla^M_Y Z - \nabla^M_Y\nabla^M_X Z - \nabla^M_{[X,Y]}Z$$

with at x_0, $[X,Y]=0$, we get:

$$\begin{aligned}
R^M(X,Y)Z &= R^N(X,Y)Z - 2g(X,\varphi Y)\varphi Z + g(\varphi X,Z)\varphi Y\\
&\quad -g(\varphi Y,Z)\varphi X + \eta(Y)\eta(Z)\varphi^2 X - \eta(X)\eta(Z)\varphi^2 Y\\
&\quad +2\eta(Y)\eta(Z)X - 2\eta(X)\eta(Z)Y + \eta(X)g(Y,Z)\xi\\
&\quad -\eta(Y)g(X,Z)\xi,
\end{aligned}$$

note that:

$$-(\varphi X \wedge_g \varphi Y)Z = g(\varphi X, Z)\varphi Y - g(\varphi Y, Z)\varphi X,$$

$$\begin{aligned}
&(X \wedge_g Y)Z - (\varphi^2 X \wedge_g \varphi^2 Y)Z \\
&\quad = g(Y,Z)X - g(X,Z)Y \\
&\qquad -g(\varphi^2 Y, Z)\varphi^2 X + g(\varphi^2 X, Z)\varphi^2 Y \\
&\quad = g(Y,Z)X - g(X,Z)Y \\
&\qquad +g(\varphi Y, \varphi Z)\varphi^2 X - g(\varphi X, \varphi Z)\varphi^2 Y \\
&\quad = g(Y,Z)X - g(X,Z)Y + g(Y,Z)\varphi^2 X \\
&\qquad -\eta(Y)\eta(Z)\varphi^2 X - g(X,Z)\varphi^2 Y + \eta(X)\eta(Z)\varphi^2 Y \\
&\quad = g(Y,Z)X - g(X,Z)Y - g(Y,Z)X \\
&\qquad +g(Y,Z)\eta(X)\xi - \eta(Y)\eta(Z)\varphi^2 X + g(X,Z)Y \\
&\qquad -g(X,Z)\eta(Y)\xi + \eta(X)\eta(Z)\varphi^2 Y \\
&\quad = g(Y,Z)\eta(X)\xi - \eta(Y)\eta(Z)\varphi^2 X \\
&\qquad -g(X,Z)\eta(Y)\xi + \eta(X)\eta(Z)\varphi^2 Y \\
&\quad = g(Y,Z)\eta(X)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Y)\eta(Z)\xi \\
&\qquad -g(X,Z)\eta(Y)\xi - \eta(X)\eta(Z)Y + \eta(X)\eta(Y)\eta(Z)\xi,
\end{aligned}$$

that is:

$$\begin{aligned}
(X \wedge_g Y)Z - (\varphi^2 X \wedge_g \varphi^2 Y)Z &= g(Y,Z)\eta(X)\xi + \eta(Y)\eta(Z)X \\
&\quad -g(X,Z)\eta(Y)\xi - \eta(X)\eta(Z)Y,
\end{aligned}$$

by the formula $\varphi^2 X + X = \eta(X)\xi$, we obtain:

$$\begin{aligned}
\eta(Y)\eta(Z)\varphi^2 X &- \eta(X)\eta(Z)\varphi^2 Y + \eta(Y)\eta(Z)Y - \eta(X)\eta(Z)Y \\
&= \eta(Y)\eta(Z)\eta(X)\xi - \eta(X)\eta(Z)\eta(Y)\xi \\
&= 0.
\end{aligned}$$

so that:

$$\begin{aligned}
R^M(X,Y)Z &= R^N(X,Y)Z - 2g(X,\varphi Y)\varphi Z - (\varphi X \wedge_g \varphi Y)Z \\
&\quad +(X \wedge_g Y)Z - (\varphi^2 X \wedge_g \varphi^2 Y)Z.
\end{aligned}$$

3) Let $U \in \Gamma(TM)$, by equation (1.2), we have:

$$\begin{aligned}
g(R^M(\xi,X)Y,U) &= g(R^M(Y,U)\xi, X) \\
&= \eta(U)g(Y,X) - \eta(Y)g(U,X) \\
&= g(U,\xi)g(Y,X) - g(Y,\xi)g(U,X),
\end{aligned}$$

so that:

$$R^M(\xi, X)Y = g(Y, X)\xi - g(Y, \xi)X = (\xi \wedge_g X)Y.$$

4) From equation (1.2), we have:

$$\begin{aligned} R^M(\xi, X)\xi &= \eta(X)\xi - \eta(\xi)X \\ &= g(X, \xi)\xi - g(\xi, \xi)X \\ &= (\xi \wedge_g X)\xi. \end{aligned}$$

□

Remark 2.3. From the Proposition 2.4, we can write:

$$\begin{aligned} R^M(U, V)W =& R^N(X, Y)Z + (U \wedge_g V)W \\ & - \Big(\varphi^2 U \wedge_g \varphi^2 V + \varphi U \wedge_g \varphi V + 2g(U, \varphi V)\varphi\Big)W \end{aligned} \tag{2.11}$$

where $U = \xi + X$, $V = \xi + Y$ and $W = \xi + Z$ and $X, Y, Z \in \Gamma(TN)$.

From the above remark and equation (1.4), we get the following:

Proposition 2.5. *The manifold $(M, \varphi, \xi, \eta, g)$ defined in (2.1) is Sasakian space form with constant φ-sectional curvature $c = -3$ if and only if (N, J, h) is flat Kählerian manifold.*

References

[1] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics Vol. **203**, Birhauser, Boston, (2002).

[2] C.P. Boyer, K. Galicki, *Sasakian Geometry*, Oxford Univ. Press, Oxford, 2007.

[3] S. Kobayashi, and K. Nomizu, *Foundations of Differential Geometry I, II*, Interscience Tract., 1963, 1969.

[4] O'Neil, *Semi- Riemannian Geometry*, Academic Press, New York, 1983.

[5] J. A. Oubiña, *New classes of almost contact metric structures.* Publ. Math. Debrecen, **32** (1985), 187–193.

[6] K. Yano, M. Kon, *Structures on Manifolds*, Series in Pure Math., Vol MAT, World Sci.,(1984).

Gherici Beldjilali
Département de Mathématiques, Faculté des sciences exactes
Université de Mascara, Mascara, Algeria
E-mail: gherici.beldjilali@univ-mascara.dz

Ahmed Mohammed Cherif
Département de Mathématiques, Faculté des sciences exactes
Université de Mascara, Mascara, Algeria
E-mail: med_cherif_ahmed@yahoo.fr

Kaddour Zaga
Département de Mathématiques, Faculté des sciences exactes
Université de Mascara, Mascara, Algeria
E-mail: zegga.kaddour@yahoo.fr