Korean J. Math. **27** (2019), No. 4, pp. 1109–1118
https://doi.org/10.11568/kjm.2019.27.4.1109

FUZZY CONNECTIONS ON ADJOINT TRIPLES

Jung Mi Ko and Yong Chan Kim*

Abstract. In this paper, we introduce the notion of residuated and Galois connections on adjoint triples and investigate their properties. Using the properties of residuated and Galois connections, we solve fuzzy relation equations and give their examples.

1. Introduction

Ward et al.[19] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool as algebraic structures for many valued logics [2,6-10,18]. Pawlak [11,12] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. For an extension of classical rough sets, many researchers [2,9,10] developed L-lower and L-upper approximation operators in complete residuated lattices.

Abdel-Hamid [1] introduced the notion of adjoint triples. By using this concepts, Medina et al.[3-5] developed information systems and decision rules. Sanchez [15] introduced the theory of fuzzy relation equations with various types of composition: max-min, min-max, min-α. Fuzzy

Received Autust 28, 2019. Revised September 26, 2019. Accepted December 3, 2019.

2010 Mathematics Subject Classification: 03E72, 03G10, 06A15, 54F05.

Key words and phrases: Adjoint triples, residuated connections, Galois connections, fuzzy relational erosion(dilation), fuzzy closure (interior) operator.

This work was supported by the Research Institute of Natural Science of Gangneung-Wonju National University.

∗ Corresponding author.

relation equations with new types of composition(continuous t-norm [16], residuated lattice [13,14]) is developed.

In this paper, we show that there exists the residuated connection between fuzzy relational erosion and fuzzy relational dilation on adjoint triples. Moreover, we study residuated and Galois connections on adjoint triples and investigate their properties. Using the properties of residuated and Galois connections, we solve fuzzy relation equations and give their examples.

2. Preliminaries

Definition 2.1. [2,9,17] Let X and Y be sets. Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets. Let $\delta, \gamma : P_1^X \to P_2^Y$ and $\epsilon, \rho : P_2^Y \to P_1^X$.

(1) $(P_1^X, \delta, \epsilon, P_2^Y)$ is called a *residuated connection* if $\delta(f) \leq_2 g$ iff $f \leq_1 \epsilon(g)$ for all $f \in P_1^X, g \in P_2^Y$.

(2) $(P_1^X, \gamma, \rho, P_2^Y)$ is called a *Galois connection* if $g \leq_2 \gamma(f)$ iff $f \leq_1 \rho(g)$ for all $f \in P_1^X, g \in P_2^Y$.

Definition 2.2.[2] Let X be a set and $(P, \leq)$ be a poset. An operator $C : P^X \to P^X$ is called a *fuzzy closure operator* on X if it satisfies the following conditions:

(C1) $f \leq C(f)$ and $C(C(f)) \leq C(f)$, for all $f \in P^X$.

(C2) If $f \leq g$, then $C(f) \leq C(g)$ for all $f, g \in P^X$.

An operator $I : P^X \to P^X$ is called a *fuzzy interior operator* on X if it satisfies the conditions

(I1) $I(f) \leq f$ and $I(f) \leq I(I(f))$ for all $f \in P^X$,

(I2) If $f \leq g$, then $I(f) \leq I(g)$ for all $f, g \in P^X$.

Definition 2.3. [3-5] Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets. We say that the mappings $\& : P_1 \times P_2 \to P_3$, $\to : P_2 \times P_3 \to P_1$ and $\Rightarrow : P_1 \times P_3 \to P_2$ is called an *adjoint triple* if it satisfies the following conditions:

$x \leq_1 (y \to z)$ iff $x \& y \leq_3 z$ iff $y \leq_2 (x \Rightarrow z)$ for $x \in P_1, y \in P_2, z \in P_3$.

Example 2.4. Let $[0,1]_m$ be a regular partition of $[0,1]$ in m pieces with $[0,1]_m = \{0, \frac{1}{m}, \frac{2}{m}, ..., \frac{m-1}{m}, 1\}$. A discretization of a t-norm T :

$[0,1]\times[0,1]\to[0,1]$ is the operator $T^0:[0,1]_m\times[0,1]_n\to[0,1]_k$ defined as

$$T^0(x,y)=\frac{[kT(x,y)]}{k}$$

where $[x]=\bigwedge\{n\in Z\mid x\le n\}$ is the ceiling function. For this operator, the corresponding implication operators $\to^0$: $[0,1]_n\times[0,1]_k\to[0,1]_m$ and $\Rightarrow^0$: $[0,1]_m\times[0,1]_k\to[0,1]_n$ defined as

$$y\to^0 z=\frac{<m(y\to z)>}{m},\quad x\Rightarrow^0 z=\frac{<n(x\to z)>}{n}$$

where $<x>=\bigvee\{n\in Z\mid n\le x\}$ is the floor function.

Let $x\le y\to^0 z=\frac{<m(y\to z)>}{m}$. Since $x-1\le<x>\le x$, $x\le\frac{<m(y\to z)>}{m}\le\frac{m(y\to z)}{m}=y\to z$. Hence $T(x,y)\le z$. Since $x\le[x]<x+1$,

$$T^0(x,y)=\frac{[kT(x,y)]}{k}<\frac{kT(x,y)+1}{k}\le\frac{z+1}{k}.$$

Hence $T^0(x,y)=\frac{[kT(x,y)]}{k}\le z$.

Let $T^0(x,y)=\frac{[kT(x,y)]}{k}\le z$. Since $kT(x,y)\le[kT(x,y)]$, $T(x,y)\le z$ iff $y\le x\to z$. Hence

$$x\Rightarrow^0 z=\frac{<n(x\to z)>}{n}>\frac{<n(x\to z)>}{n}\ge y.$$

Other cases are similarly proved.

3. Fuzzy connections on adjoint triples

LEMMA 3.1. *Let P_i be complete lattices for $i\in\{1,2,3\}$. Let $(\&,\to,\Rightarrow)$ be an adjoint triple with respect to $(P_1,\le_1)$, $(P_2,\le_2)$, $(P_3,\le_3)$. Then the following properties hold.*

(1) *If $x_1\le_1 x_2$, then $x_1\&y\le_3 x_2\&y$.*

(2) *If $y_1\le_2 y_2$, then $x\&y_1\le_3 x\&y_2$.*

(3) *$\to,\Rightarrow$ are order-preserving on the second argument and order-reversing on the first argument.*

(4) *$y\le_2(x\Rightarrow(x\&y))$, $x\le_1(y\to(x\&y))$.*

(5) *$x\&(x\Rightarrow z)\le_3 z$, $(y\to z)\&y\le_3 z$.*

(6) *$y\le_2((y\to z)\Rightarrow z)$, $x\le_1((x\Rightarrow z)\to z)$.*

(7) *$(\bigvee_i x_i)\&y=\bigvee_i(x\&y_i)$ and $x\&(\bigvee_i y_i)=\bigvee_i(x\&y_i)$.*

(8) *$x\Rightarrow(\bigwedge_i z_i)=\bigwedge_i(x\to z_i)$ and $(\bigvee_i x_i)\Rightarrow z=\bigwedge_i(x_i\Rightarrow z)$.*

(9) $y \to (\bigwedge_i z_i) = \bigwedge_i (y \to z_i)$ *and* $(\bigvee_i y_i) \to z = \bigwedge_i (y_i \to z)$.

Proof. (1) Since $x_1 \leq_1 x_2 \leq_1 (y \to x_2 \& y)$, $x_1 \& y \leq_3 x_2 \& y$.

(2) Since $y_1 \leq_2 y_2 \leq_2 (y \Rightarrow x \& y_2)$, $x \& y_1 \leq_3 x \& y_2$.

(3) Let $x_1 \leq_1 x_2$. Since $x_1 \& (x_2 \Rightarrow z) \leq_3 x_2 \& (x_2 \Rightarrow z) \leq_3 z$, $(x_2 \Rightarrow z) \leq_2 (x_1 \Rightarrow z)$.

Let $z_1 \leq_3 z_2$. Since $(x \to z_1) \& x \leq_3 z_1 \leq_3 z_2$, $(x \to z_1) \leq_1 (x \to z_2)$.

Other cases are similarly proved.

(4) It follows $x \& y \leq_3 x \& y$.

(5) It follows $(x \Rightarrow z) \leq_2 (x \Rightarrow z)$ iff $x \& (x \Rightarrow z) \leq_3 z$. Moreover, $(y \to z) \leq_1 (y \to z)$ iff $(y \to z) \& y \leq_3 z$.

(6) By (5), $(y \to z) \& y \leq_3 z$ iff $y \leq_2 ((y \to z) \Rightarrow z)$. Moreover,$x \& (x \Rightarrow z) \leq_3 z$ iff $x \leq_1 ((x \Rightarrow z) \to z)$.

(7) By (1), $(\bigvee_i x_i) \& y \geq_3 \bigvee_i (x \& y_i)$. Since $x_i \leq_2 (y \to x_i \& y) \leq_2 (y \to \bigvee_i (x_i \& y))$, $\bigvee_i x_i \leq_2 (y \to \bigvee_i (x_i \& y))$ iff $(\bigvee_i x_i) \& y \leq_3 \bigvee_i (x \& y_i)$.

(8) By (3), $x \Rightarrow (\bigwedge_i z_i) \leq_2 \bigwedge_i (x \Rightarrow z_i)$. Since $x \& (\bigwedge_i (x \Rightarrow z_i)) \leq_3 x \& (x \Rightarrow z_i) \leq_3 z_i$, then $x \& (\bigwedge_i (x \Rightarrow z_i)) \leq_3 \bigwedge_i (x \& (x \Rightarrow z_i)) \leq_3 \bigwedge_i z_i$. Thus $\bigwedge_i (x \Rightarrow z_i) \leq_2 x \Rightarrow (\bigwedge_i z_i)$.

Moreover, by (3), $(\bigvee_i x_i) \Rightarrow z \leq_2 \bigwedge_i (x_i \Rightarrow z)$. Since $(\bigvee_i x_i) \& \bigwedge_i (x_i \Rightarrow z) \leq_3 (\bigvee_i x_i) \& (x_i \Rightarrow z) = \bigvee_i (x_i \& (x_i \Rightarrow z)) \leq_3 z$, then $\bigwedge_i (x_i \Rightarrow z) \leq_2 (\bigvee_i x_i) \Rightarrow z$. □

Definition 3.2. [5] Let X, Y be sets and P_i be complete lattices. Let $(\&, \to, \Rightarrow)$ be an adjoint triple with respect to $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$.

(1) The *fuzzy relational erosion with respect to* $R \in P_1^{X \times Y}$, $\epsilon_R : P_3^X \to P_2^Y$ is defined as

$$\epsilon_R(f)(y) = \bigwedge_{x \in X} (R(x,y) \Rightarrow f(x)).$$

(2) The *fuzzy relational dilation with respect to* R, $\delta_R : P_2^Y \to P_3^X$ is defined as

$$\delta_R(g)(x) = \bigvee_{y \in Y} (R(x,y) \& g(y)).$$

(3) The *fuzzy relational property-oriented erosion with respect to* R, $\epsilon_{R_p} : P_3^Y \to P_1^X$ is defined as

$$\epsilon_{R_p}(g)(x) = \bigwedge_{y \in Y} (R(x,y) \to g(y)).$$

(4) The *fuzzy relational property-oriented dilation with respect to* R, $\delta_{R_p} : P_1^X \to P_3^Y$ is defined as

$$\delta_{R_p}(f)(y) = \bigvee_{x\in X}(f(x)\& R(x,y)).$$

THEOREM 3.3. *Let* X, Y *be sets and* P_i *be complete lattices. Let* $(\&, \to, \Rightarrow)$ *be an adjoint triple with respect to* $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$. *Then the following properties hold.*

(1) $(P_2^Y, \delta_R, \epsilon_R, P_2^Y)$ *is a residuated connection.*

(2) $\delta_R(\epsilon_R(f)) \leq_3 f$ *iff* $g \leq_2 \epsilon_R(\delta_R(g))$ *for all* $f \in P_3^X, g \in P_2^Y$.

(3) *If* $f_1 \leq_3 f_2$ *and* $g_1 \leq_2 g_2$, *for all* $f_1, f_2 \in P_3^X, g_1, g_2 \in P_2^Y$,

$$\epsilon_R(f_1) \leq_2 \epsilon_R(f_2),\ \delta_R(g_1) \leq_3 \delta_R(g_2).$$

(4) $\bigvee_{i\in\Gamma} \delta_R(g_i) = \delta_R(\bigvee_{i\in\Gamma} g_i)$ *for all* $g_i \in P_2^Y$.

(5) $\bigwedge_{i\in\Gamma} \epsilon_R(f_i) = \epsilon_R(\bigwedge_{i\in\Gamma} f_i)$ *for all* $f_i \in P_3^X$.

(6) $\delta_R(g) = \delta_R(\epsilon_R(\delta_R(g)))$ *for all* $g \in P_2^Y$. *If* $g = g_0$ *is a solution of* $\delta_R(g) = f$, *then* $g_1 = \epsilon_R(f)$ *is a solution of* $\delta_R(g) = f$ *such that* $g_0 \leq \epsilon_R(f)$.

(7) $\epsilon_R(\delta_R(\epsilon_R(f))) = \epsilon_R(f)$ *for all* $f \in P_3^X$. *If* $f = f_1$ *is a solution of* $\epsilon_R(f) = g$, *then* $f_2 = \delta_R(g)$ *is a solution of* $\epsilon_R(f) = g$ *such that* $\delta_R(g) \leq f_1$.

(8) $\delta_R \circ \epsilon_R : P_3^X \to P_3^X$ *is a fuzzy interior operator.*

(9) $\epsilon_R \circ \delta_R : P_2^Y \to P_2^Y$ *is a fuzzy closure operator.*

Proof. (1) We show that $\delta_R(g) \leq_3 f$ iff $g \leq_2 \epsilon_R(f)$ for all $f \in P_3^X, g \in P_2^Y$. For $f \in P_3^X, g \in P_2^Y$, $\delta_R(g)(x) \leq_3 f(x)$ iff $\bigvee_{y\in Y}(R(x,y)\& g(y)) \leq_3 f(x)$ iff $g(y) \leq_2 \bigwedge_{y\in Y}(R(x,y) \Rightarrow f(x))$ iff $g(y) \leq_2 \epsilon_R(f)(y)$.

(2) For $f \in P_3^X, g \in P_2^Y$, $\delta_R(g)(x) \leq_3 \delta_R(g)(x)$ iff $g(y) \leq_2 \epsilon_R(\delta_R(g))(y)$ and $\epsilon_R(f)(y) \leq_2 \epsilon_R(f)(x)$ iff $\delta_R(\epsilon_R(f))(x) \leq_3 f(x)$.

(3) Since $g_1(y) \leq_2 g_2(y) \leq_2 \epsilon_R(\delta_R(g_2))(y)$, then $\delta_R(g_1)(x) \leq \delta_R(g_2)(x)$. Moreover, since $\delta_R(\epsilon_R(f_1))(x) \leq_3 f_2(x)$, $\epsilon_R(f_1)(y), \epsilon_R(f_2)(y)$.

(4) From (3), $\bigvee_{i\in\Gamma} \delta_R(g_i) \leq_3 \delta_R(\bigvee_{i\in\Gamma} g_i)$.

Since $g_i \leq_2 \epsilon_R(\delta_R(g_i))$ and $\bigvee_{i\in\Gamma} g_i \leq_2 \epsilon_R(\bigvee_{i\in\Gamma} \delta_R(g_i))$, $\delta_R(\bigvee_{i\in\Gamma} g_i) \leq_3 \bigvee_{i\in\Gamma} \delta_R(g_i)$.

(5) From (3), $\epsilon_R(\bigwedge_{i\in\Gamma} f_i) \leq_2 \bigwedge_{i\in\Gamma} \epsilon_R(f_i)$.

Since $\delta_R(\epsilon_R(f_i)) \leq_3 f_i$ and $\delta_R(\bigwedge_{i\in\Gamma} \epsilon_R(f_i)) \leq_3 \bigwedge_{i\in\Gamma} \delta_R(\epsilon_R(f_i)) \leq_3 \bigwedge_{i\in\Gamma} f_i$, $\bigwedge_{i\in\Gamma} \epsilon_R(f_i) \leq_2 \epsilon_R(\bigwedge_{i\in\Gamma} f_i)$.

(6) By (2), $\delta_R(g) = \delta_R(\epsilon_R(\delta_R(g)))$ for all $g \in P_2^Y$. If $\delta_R(g_0) = f$, Then $\delta_R(\epsilon_R(\delta_R(g_0))) = \delta_R(\epsilon_R(f)) = \delta_R(g_0) = f$. Moreover, $g_0 \leq_2 \epsilon_R(\delta_R(g_0)) = \epsilon_R(f)$.

(7) It is similarly proved as (6).

(8) For each $f, f_1, f_2 \in L^X$, $\delta_R \circ \epsilon_R(f) \leq_3 f$ and $(\delta_R \circ \epsilon_R)(\delta_R \circ \epsilon_R)(f) = \delta_R \circ \epsilon_R(f)$, if $f_1 \leq_3 f_2$,

$$(\delta_R \circ \epsilon_R)(f_1) \leq_3 (\delta_R \circ \epsilon_R)(f_2)$$

(9) It is similarly proved as (8). □

Corollary 3.4. *Let X, Y be sets and P_i be complete lattices. Let $(\&, \rightarrow, \Rightarrow)$ be an adjoint triple with respect to $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$. Then the following properties hold.*

(1) *$(P_1^X, \delta_{R_p}, \epsilon_{R_p}, P_3^Y)$ is a residuated connection.*

(2) *$\delta_{R_p}(\epsilon_{R_p}(g) \leq_3 g$ iff $f \leq_1 \epsilon_{R_p}(\delta_{R_p}(f))$ for all $f \in P_1^X, g \in P_3^Y$.*

(3) *If $f_1 \leq_3 f_2$ and $g_1 \leq_2 g_2$, for all $f_1, f_2 \in P_1^X, g_1, g_2 \in P_3^Y$,*

$$\epsilon_{R_p}(g_1) \leq_1 \epsilon_{R_p}(g_2),\ \delta_{R_p}(f_1) \leq_3 \delta_{R_p}(f_2).$$

(4) *$\bigvee_{i\in\Gamma} \delta_{R_p}(f_i) = \delta_{R_p}(\bigvee_{i\in\Gamma} f_i)$ for all $f_i \in P_1^X$.*

(5) *$\bigwedge_{i\in\Gamma} \epsilon_{R_p}(g_i) = \epsilon_{R_p}(\bigwedge_{i\in\Gamma} g_i)$ for all $g_i \in P_3^Y$.*

(6) *$\delta_{R_p}(f) = \delta_{R_p}(\epsilon_{R_p}(\delta_{R_p}(f)))$ for all $f \in P_1^Y$. If $f = f_0$ is a solution of $\delta_{R_p}(f) = g$, then $f_1 = \epsilon_{R_p}(g)$ is a solution of $\delta_{R_p}(f) = g$ such that $f_0 \leq_1 \epsilon_{R_p}(g)$.*

(7) *$\epsilon_{R_p}(\delta_{R_p}(\epsilon_{R_p}(g))) = \epsilon_{R_p}(g)$ for all $g \in P_3^Y$. If $g = g_1$ is a solution of $\epsilon_{R_p}(g) = f$, then $g_2 = \delta_{R_p}(f)$ is a solution of $\epsilon_{R_p}(g) = f$ such that $\delta_{R_p}(f) \leq_3 g_1$.*

(8) *$\delta_{R_p} \circ \epsilon_{R_p} : P_3^Y \rightarrow P_3^Y$ is a fuzzy interior operator.*

(9) *$\epsilon_{R_p} \circ \delta_{R_p} : P_1^X \rightarrow P_1^X$ is a fuzzy closure operator.*

Theorem 3.5. *Let X, Y be sets and P_i be complete lattices. Let $(\&, \rightarrow, \Rightarrow)$ be an adjoint triple with respect to $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$. An operation $\gamma_R : P_1^X \rightarrow P_2^Y$ is defined as*

$$\gamma_R(f)(y) = \bigwedge_{x\in X} (f(x) \Rightarrow R(x, y)).$$

An operation $\rho_R : P_2^Y \rightarrow P_1^X$ is defined as

$$\rho_R(g)(x) = \bigwedge_{y\in Y} (g(y) \rightarrow R(x, y)).$$

Then following properties hold.

(1) $(P_1^X, \gamma_R, \rho_R, P_2^Y)$ *is a Galois connection.*

(2) $f \leq_1 \rho_R(\gamma_R(f))$ *and* $g \leq_2 \gamma_R(\rho_R(g))$ *for all* $f \in P_1^X$ *and* $g \in P_2^Y$.

(3) *If* $f_1 \leq_1 f_2$ *for all* $f_1, f_2 \in P_1^X$*, then* $\gamma_R(f_2)(y) \leq_2 \gamma_R(f_1)(y)$.

(4) *If* $g_1 \leq_2 g_2$ *for all* $g_1, g_2 \in P_2^X$*, then* $\rho_R(g_2)(x) \leq_1 \rho_R(g_1)(y)$.

(5) $\bigwedge_{i\in\Gamma} \gamma_R(f_i) = \gamma_R(\bigvee_{i\in\Gamma} f_i)$ *for all* $f_i \in P_1^X$.

(6) $\bigwedge_{i\in\Gamma} \rho_R(g_i) = \rho_R(\bigvee_{i\in\Gamma} g_i)$ *for all* $g_i \in P_2^Y$.

(7) $\gamma_R(f) = \gamma_R(\rho_R(\gamma_R(f)))$ *for all* $A \in L^X$. *If* $f = f_1$ *is a solution of* $\gamma_R(f) = g$*, then* $f = \rho_R(g)$ *is a solution of* $\gamma_R(f) = g$ *such that* $f_1 \leq_1 \rho_R(g)$.

(8) $\rho_R(g) = \rho_R(\gamma_R(\rho_R(g)))$ *for all* $g \in P_2^Y$. *If* $g = g_1$ *is a solution of* $\rho_R(g) = f$*, then* $g = \gamma_R(f)$ *is a solution of* $\rho_R(g) = f$ *such that* $g_1 \leq \gamma_R(f)$.

(9) $\rho_R \circ \gamma_R : P_1^X \to P_1^X$ *and* $\gamma_R \circ \rho_R : P_2^Y \to P_2^Y$ *are fuzzy closure operators.*

Proof. (1) We show that $g \leq_2 \gamma_R(f)$ iff $f \leq_1 \rho_R(g)$ for all $f \in P_1^X, g \in P_2^Y$.

For $f \in P_1^X, g \in P_2^Y$, $g(y) \leq_2 \gamma_R(f)(y) = \bigwedge_{x\in X}(f(x) \Rightarrow R(x,y))$ iff $f(x) \leq_1 \bigwedge_{y\in Y}(g(y) \to R(x,y))$ iff $f(x) \leq_1 \rho_R(g)(x)$.

(2) It follows from $\rho_R(g) \leq_1 \rho_R(g)$ iff $g \leq_2 \gamma_R(\rho_R(g))$ and $\gamma_R(f) \leq_2 \gamma_R(f)$ iff $f \leq_1 \rho_R(\gamma_R(f))$.

(3) Since $f_1 \leq_1 f_2 \leq_1 \rho_R(\gamma_R(f_2))$, $\gamma_R(f_2) \leq_2 \gamma_R(f_1)$.

(4) Since $g_1 \leq_2 g_2 \leq_2 \gamma_R(\rho_R(g_2))$, $\rho_R(g_2) \leq_1 \rho_R(g_1)$.

(5) By (3), since $f_1 \leq_1 f_2$, then $\gamma_R(f_2) \leq_2 \gamma_R(f_1)$. Hence $\gamma_R(\bigvee_{i\in\Gamma} f_i) \leq_2 \bigwedge_{i\in\Gamma} \gamma_R(f_i)$.

Since $f_i \leq_1 \rho_R(\gamma_R(f_i))$ and $\bigvee_{i\in\Gamma} f_i \leq_1 \bigvee_{i\in\Gamma} \rho_R(\gamma_R(f_i)) \leq_1 \rho_R(\bigwedge_{i\in\Gamma} \gamma_R(f_i))$, Thus $\gamma_R(\bigvee_{i\in\Gamma} f_i) \leq_2 \bigwedge_{i\in\Gamma} \gamma_R(f_i)$,

(6) It is similarly proved as (5).

(7) By (2), $\gamma_R(f) = \gamma_R(\rho_R(\gamma_R(f)))$ for all $f \in P_1^X$. If $\gamma_R(f_1) = g$, then $\gamma_R(\rho_R(\gamma_R(f_1))) = \gamma_R(\rho_R(g)) = \gamma_R(f_1) = g$. Moreover, $f_1 \leq_1 \rho_R(\gamma_R(f_1)) = \rho_R(g)$.

(8) It is similarly proved as (7).

(9) For each $g, h \in P_2^Y$, $g \leq_2 \gamma_R \circ \rho_R(g)$ and $(\gamma_R \circ \rho_R) \circ (\gamma_R \circ \rho_R(g)) = \gamma_R \circ \rho_R(g)$. If $g \leq_2 h$, then $\rho_R(h) \leq_1 \rho_R(g)$. Moreover $(\gamma_R \circ \rho_R)(g) \leq_2 (\gamma_R \circ \rho_R)(h)$. Hence $\gamma_R \circ \rho_R : P_2^Y \to P_2^Y$ is a fuzzy closure operator. Similarly, $\rho_R \circ \gamma_R$ is a fuzzy closure operator.

□

EXAMPLE 3.6. Let $X = \{x, y, z\}$ be a set of cars and $Y = \{a, b\}$ be a set of attributes. Let $([0,1], \odot, \to, 0, 1)$ be a t-norm (ref.[2,6-8]) as

$$x \odot y = \max\{0, x + y - 1\},\ x \to y = \min\{1 - x + y, 1\}.$$

Let $[0,1]_m$ be a regular partition of $[0,1]$ in m pieces with $[0,1]_m = \{0, \frac{1}{m}, \frac{2}{m}, ..., \frac{m-1}{m}, 1\}$.

Let $\&^0 : [0,1]_3 \times [0,1]_4 \to [0,1]_2$, $\Rightarrow^0$: $[0,1]_3 \times [0,1]_2 \to [0,1]_4$, $\to^0$: $[0,1]_4 \times [0,1]_2 \to [0,1]_3$ defined as

$$x\&^0 y = \frac{[2(x \odot y)]}{2},\ x \Rightarrow^0 y = \frac{< 4(x \to y) >}{4}\ x \to^0 y = \frac{< 3(x \to y) >}{3}$$

$\&^0$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
0	0	0	0	0	0
$\frac{1}{3}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{2}{3}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
1	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1

$\Rightarrow^0$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{3}$	$\frac{1}{2}$	1	1
$\frac{2}{3}$	$\frac{1}{4}$	$\frac{3}{4}$	1
1	0	$\frac{1}{2}$	1

$\to^0$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{4}$	$\frac{2}{3}$	1	1
$\frac{1}{2}$	$\frac{1}{3}$	1	1
$\frac{3}{4}$	0	$\frac{2}{3}$	1
1	0	$\frac{1}{3}$	1

where $[x] = \bigwedge\{n \in Z \mid x \leq n\}, < x >= \bigvee\{n \in Z \mid n \leq x\}$.

(1) Define $R : X \times Y \to [0,1]_3$ as

$$R(x,a) = \tfrac{1}{3}, R(y,a) = 1, R(z,a) = \tfrac{2}{3}$$
$$R(x,b) = 0, R(y,b) = \tfrac{2}{3}, R(z,b) = 1.$$

For $f_1 \in [0,1]_2^X$ with $f_1 = (\frac{1}{2}, 0, \frac{1}{2})$. Then $\epsilon_R(f_1) = (0, \frac{1}{4})$. $f_1 = (\frac{1}{2}, 0, \frac{1}{2}) \in [0,1]_2^X$ is a solution of $\epsilon_R(f_1) = (0, \frac{1}{4})$. Also, $\delta_R(0, \frac{1}{4}) = (0, 0, \frac{1}{2})$ is a solution of $\epsilon_R(f_1) = (0, \frac{1}{4})$ such that $\delta_R(0, \frac{1}{4}) = (0, 0, \frac{1}{2}) \leq_2 f_1 = (\frac{1}{2}, 0, \frac{1}{2})$.

For $g_0 \in [0,1]_4^X$ with $g_0 = (\frac{3}{4}, \frac{1}{4})$. Then $\delta_R(g_0) = (\frac{1}{2}, 1, \frac{1}{2})$. $g_0 = (\frac{3}{4}, \frac{1}{4}) \in [0,1]_4^X$ is a solution of $\delta_R(g) = (\frac{1}{2}, 1, \frac{1}{2})$. Also, $\delta_R(\frac{1}{2}, 1, \frac{1}{2}) = (\frac{3}{4}, \frac{1}{2})$ is a solution of $\delta_R(g) = (\frac{1}{2}, 1, \frac{1}{2})$ such that $g_0 = (\frac{3}{4}, \frac{1}{4}) \leq_2 \delta_R(\frac{1}{2}, 1, \frac{1}{2}) = (\frac{3}{4}, \frac{1}{2})$.

Then $([0,1]_4^Y, \delta_R, \epsilon_R, [0,1]_2^X)$ is a residuated connection.

(2) Define $R : X \to [0,1]_4$ for $c \in \{a, b\}$ as

$$R(x,a) = \tfrac{3}{4}, R(y,a) = \tfrac{1}{4}, R(z,a) = 1$$
$$R(x,b) = \tfrac{1}{2}, R(y,b) = 1, R(z,b) = \tfrac{1}{4}.$$

For $f_1 \in [0,1]_3^X$ with $f_1 = (\frac{1}{3}, \frac{2}{3}, 1)$. Then $\delta_{R_p}(f_1) = (\frac{1}{2}, 1)$. $f_1 = (\frac{1}{3}, \frac{2}{3}, \frac{1}{3}) \in [0,1]_3^X$ is a solution of $\delta_{R_p}(f_1) = (\frac{1}{2}, 1)$. Also, $\epsilon_{R_p}(\frac{1}{2}, 1) = (\frac{2}{3}, 1, \frac{1}{3})$ is a solution of $\delta_{R_p}(f) = (\frac{1}{2}, 1)$ such that $f_1 = (\frac{1}{3}, \frac{2}{3}, \frac{1}{3}) \leq_1 \epsilon_{R_p}(\frac{1}{2}, 1) = (\frac{2}{3}, 1, \frac{1}{3})$.

For $g_0 \in [0,1]_2^X$ with $g_0 = (\frac{1}{2}, \frac{1}{2})$. Then $\epsilon_{R_p}(g_0) = (\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$. $g_0 = (\frac{1}{2}, \frac{1}{2}) \in [0,1]_2^X$ is a solution of $\epsilon_{R_p}(g) = (\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$. Also, $\delta_{R_p}(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) = (\frac{1}{2}, \frac{1}{2})$ is a solution of $\epsilon_{R_p}(g) = (\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$ such that $g_0 = (\frac{1}{2}, \frac{1}{2}) = \delta_{R_p}(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) = (\frac{1}{2}, \frac{1}{2})$.

Then $([0,1]_3^Y, \delta_{R_p}, \epsilon_{R_p}, [0,1]_2^X)$ is a residuated connection.

(3) Define $R : X \times Y \to [0,1]_2$ as

$$\begin{array}{l} R(x,a) = 0, R(y,a) = \frac{1}{2}, R(z,a) = \frac{1}{2} \\ R(x,b) = \frac{1}{2}, R(y,b) = 1, R(z,b) = 0. \end{array}$$

Since $\gamma_R(f)(a) = \bigwedge_{x \in X}(f(x) \Rightarrow R(x,a))$, $f_1 = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \in [0,1]_3^X$ is a solution of $\gamma_R(f) = (\frac{1}{2}, \frac{1}{4})$. Also, $\rho_R(\frac{1}{2}, \frac{1}{4}) = (\frac{1}{3}, 1, \frac{2}{3})$ is a solution of $\gamma_R(f) = (\frac{1}{2}, \frac{1}{4})$ such that $f_1 = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \leq_1 \rho_R(\frac{1}{2}, \frac{1}{4}) = (\frac{1}{3}, 1, \frac{2}{3})$.

Since $\rho_R(g)(x) = \bigwedge_{x \in X}(g(a) \to R(x,a))$, $g = g_1 = (\frac{1}{4}, \frac{1}{2}) \in [0,1]_4^Y$ is a solution of $\rho_R(g) = (\frac{2}{3}, 1, \frac{1}{3})$. Also, $g = \gamma_R(\frac{2}{3}, 1, \frac{1}{3}) = (\frac{1}{4}, \frac{1}{2})$ is a solution of $\rho_R(g) = (\frac{2}{3}, 1, \frac{1}{3})$ such that $g_1 = (\frac{1}{4}, \frac{1}{2}) = \gamma_R(\frac{2}{3}, 1, \frac{1}{3})$.

Then $([0,1]_3^X, \gamma_R, \rho_R, [0,1]_4^Y)$ is a Galois connection.

References

[1] A.A. Abdel-Hamid, N.N. Morsi, *Associatively tied implications*, Fuzzy Sets and Systems, **136** (3) (2003), 291–311.

[2] R. Bělohlávek, *Fuzzy Relational Systems*, Kluwer Academic Publishers, New York, 2002.

[3] M.E. Cornejo, J. Medina, E. Ramírez, *A comparative study of adjoint triples*, Fuzzy Sets and Systems, **211** (2013), 1–14.

[4] M.E. Cornejo, J. Medina and E. Ramírez, *Multi-adjoint algebras versus non-commutative residuated structures*, International Journal of Approximate Reasoning **66** (2015), 119-138.

[5] N. Madrid, M. Ojeda-Aciego, J. Medina and I. Perfilieva, *L*-fuzzy relational mathematical morphology based on adjoint triples, Information Sciences **474** (2019), 75–89.

[6] P. Hájek, *Metamathematices of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, 1998.

[7] U. Höhle, E.P. Klement, *Non-classical logic and their applications to fuzzy subsets*, Kluwer Academic Publishers, Boston, 1995.

[8] U. Höhle, S.E. Rodabaugh, *Mathematics of Fuzzy Sets, Logic, Topology and Measure Theory*, The Handbooks of Fuzzy Sets Series, Kluwer Academic Publishers, Dordrecht, 1999.

[9] Y.C. Kim, *Join-meet preserving maps and Alexandrov fuzzy topologies*, Journal of Intelligent and Fuzzy Systems **28** (2015), 457–467.

[10] M. Kryszkiewicz, *Rough set approach to incomplete information systems*, Information Sciences **112** (1998), 39–49.
[11] Z. Pawlak, *Rough sets*, Internat. J. Comput. Inform. Sci., **11** (1982), 341–356.
[12] Z. Pawlak, *Rough sets: Theoretical Aspects of Reasoning about Data, System Theory, Knowledge Engineering and Problem Solving*, Kluwer Academic Publishers, Dordrecht, The Netherlands (1991)
[13] I. Perfilieva, *Finitary solvability conditions for systems of fuzzy relation equations*, Information Sciences, **234** (2013), 29–43.
[14] I. Perfilieva and L. Noskova, *System of fuzzy relation equations with inf- composition: Commplete set of solutions*, Fuzzy Sets and Systems **159** (2008), 2256–2271.
[15] E. Sanchez, *Resolution of composite fuzzy relation equations*, Inform. and Control **30** (1976), 38–48.
[16] B.S. Shieh, Solutions of fuzzy relation equations based on continuous t-norms, Information Sciences, **177** (2007), 4208-4215.
[17] P. Sussner, *Lattice fuzzy transforms from the perspective of mathematical morphology*, Fuzzy Sets and Systems, **288** (2016), 115–128.
[18] S. P. Tiwari, I. Perfilieva and A.P. Singh, *Generalized residuated lattices based F-transformation*, Iranian Journal of Fuzzy Systems **15** (2) (2018), 165–182.
[19] M. Ward, R.P. Dilworth, *Residuated lattices*, Trans. Amer. Math. Soc. **45** (1939), 335–354,

Jung Mi Ko
Department of Mathematics
Gangneung-Wonju National
Gangneung 25457, Korea
E-mail: jmko@gwnu.ac.kr

Yong Chan Kim
Department of Mathematics
Gangneung-Wonju National
Gangneung 25457, Korea
E-mail: yck@gwnu.ac.kr