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C^*-ALGEBRA VALUED SYMMETRIC SPACES AND FIXED POINT RESULTS WITH AN APPLICATION

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Abstract. In this paper, we firstly introduce the class of C^*-algebra valued symmetric spaces and utilize the same to prove our fixed point results. We furnish an example to highlight the utility of our main result. Finally, we apply our result to examine the existence and uniqueness of a solution for a system of Fredholm integral equations.

1. Introduction

In the context of fixed point theory, Banach [6] proved an effective and powerful tool in nonlinear analysis known as the Banach contraction principle. This principle has been extended and generalized in numerous different directions (see [1–4, 7, 10, 13, 16]). Most recently, researchers of the fixed point theory come across situations wherein all the metric conditions are not needed (see [9, 11, 12, 17]). Inspired by this fact, many researchers established fixed point results in semi-metric spaces [19] (or symmetric spaces). A mapping d on $X \neq \emptyset$ is called symmetric if $d(a,b) = d(b,a)$ and $d(a,b) = 0$ if and only if $a = b$, for all $a, b \in X$. The pair (X, d) is called symmetric space. Due to the absence of triangular inequality, the function 'd' is not continuous in general so the uniqueness of the limit of the sequence is not guaranteed. To overcome the earlier

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mentioned difficulties, Wilson [19] suggested several related weaker conditions. Such weaker conditions are adopting our setting which will be stated in preliminaries soon.

Very recently, Asim et al. [5] enlarged the class of symmetric spaces and partial metric spaces by introducing the class of partial symmetric spaces and utilized the same to prove fixed point results for singlevalued mappings as well as multivalued mappings. On the other hand, in 2014, Ma et al. [14] established the notion of C^*-algebra valued metric spaces (in short C^*-avMS) by replacing the range set $\mathbb{R}$ with a unital C^*-algebra which is more general class than the class of metric spaces and gave some fixed point results in such spaces. In 2015, Ma et al. [15], introduced the notion of C^*-algebra valued b-metric spaces as a generalization of C^*-avMS and proved some fixed point results also used their results as an application for an integral type operator. In the recent past, Chandok [8], generalized the class of C^*-avMS by introducing the class of C^*-algebra valued partial metric spaces and utilized the same to prove some fixed point theorems. Later on, many researchers worked on C^*-av metric and proved numerous fixed point theorems also used their results as applications for integral type operators.

Inspired by the ideas of symmetric spaces and C^*-avMS, we introduce the notion of C^*-av symmetric spaces and prove some fixed point results. We also furnish some examples which demonstrate the utility of our main result. Moreover, we apply one of our main results to examine the existence and uniqueness of a solution for the system of integral type operators.

2. Preliminaries

Throughout the paper, we denote $\mathcal{A}$ by a unital ($i.e.$, unity element I) C^*-algebra with linear involution $*$ such that for all $a, b \in \mathcal{A}$, $(ab)^* = b^*a^*$ and $a^{**} = a$. A positive element $a \in \mathcal{A}$ is denoted by $0_{\mathcal{A}} \preccurlyeq a$, if $a = a^*$ and $\sigma(a) = \{\lambda \in \mathbb{R} : \lambda I - a \text{ is non-invertible } \} \subseteq [0, \infty)$, where $0_{\mathcal{A}}$ is a zero element in $\mathcal{A}$. Also, $\mathcal{A}_+ = \{a \in \mathcal{A}; a \succcurlyeq 0_{\mathbb{A}}\}$. The partial ordering on $\mathcal{A}$ can be defined as follows: $a \preccurlyeq b$ if and only if $0_{\mathcal{A}} \preccurlyeq b - a$. The pair $(\mathcal{A}, *)$ is said to be an unital $*$-algebra, if it contains the unity element I. A unital $*$-algebra $(\mathcal{A}, *)$ is called a Banach $*$-algebra, if it satisfies $\|a^*\| = \|a\|$ along with a complete sub-multiplicative norm. A Banach

C^*-algebra satisfying $\|a^*a\| = \|a\|^2$, for all $a \in \mathcal{A}$ is called a C^*-algebra.

The following definition is introduced by Ma et al. [14]:

Definition 2.1. Let $A \neq \emptyset$. The mapping $d : A \times A \to \mathcal{A}$ is called a C^*-av metric on A, if it satisfies the following (for all $a, b, c \in A$):

1. $d(a, b) \succcurlyeq 0_{\mathcal{A}}$ and $d(a, b) = 0_{\mathcal{A}}$ iff $a = b$;
2. $d(a, b) = d(b, a)$;
3. $d(a, b) \preccurlyeq d(a, c) + d(c, b)$.

The triplet $(A, \mathcal{A}, d)$ is called a C^*-avMS.

In 2015, again Ma et al. [15] introduced the notion of C^*-av b-metric space as follows:

Definition 2.2. Suppose A is a non-empty set and $s \in \mathcal{A}$ such that $s \succcurlyeq I$. The mapping $d : A \times A \to \mathcal{A}$ is called a C^*-av b-metric on A, if it satisfies the following for all $a, b, c \in A$:

(i) $d(a, b) \succcurlyeq 0_{\mathcal{A}}$ and $d(a, b) = 0_{\mathcal{A}}$ iff $a = b$;
(ii) $d(a, b) = d(b, a)$;
(iii) $d(a, b) \preccurlyeq s[d(a, c) + d(c, b)]$.

The triplet $(A, \mathcal{A}, d)$ is called a C^*-avbMS.

Remark 2.3. Clearly, if $s = I$ then a C^*-avbMS reduces to a C^*-avMS.

Now, we recall the definition of C^*-algebra valued partial metric space introduced by Chandok et al. [8].

Definition 2.4. Let $A \neq \emptyset$. The mapping $d : A \times A \to \mathcal{A}$ is called a C^*-av partial metric on A, if it satisfies the following for all $a, b, c \in A$:

(i) $d(a, b) \succcurlyeq 0_{\mathcal{A}}$ and $d(a, a) = d(b, b) = d(a, b)$ iff $a = b$;
(ii) $d(a, a) \preccurlyeq d(a, b)$;
(iii) $d(a, b) = d(b, a)$;
(iv) $d(a, b) \preccurlyeq d(a, c) + d(c, b) - d(c, c)$.

The triplet $(A, \mathcal{A}, d)$ is called a C^*-avPMS.

Remark 2.5. Obviously, if $d(a, a) = 0_{\mathcal{A}}$ for all $a \in \mathcal{A}$, then $(\mathcal{A}, d)$ is a C^*-avMS.

Now, we introduce the C^*-algebra valued symmetric space as follows:

Definition 2.6. Suppose A is a non-empty set. The mapping $d : A \times A \to \mathcal{A}$ is called a C^*-av symmetric on A, if it satisfies the following for all $a, b \in A$:

(i) $d(a,b) \succcurlyeq 0_{\mathcal{A}}$ and $d(a,b) = 0_{\mathcal{A}}$ iff $a = b$;
(ii) $d(a,b) = d(b,a)$.

The triplet $(A, \mathcal{A}, d)$ is called a C^*-av symmetric space.

A C^*-av symmetric space $(A, \mathcal{P})$ reduces to a symmetric space if $\mathcal{A} = \mathbb{R}$. Obviously, every symmetric space is a C^*-av symmetric space but not conversely.

Definition 2.7. A sequence $\{a_n\}$ in $(A, \mathcal{A}, d)$ is called convergent to $a \in \mathcal{A}$ (with respect to $\mathcal{A}$), if

$$\lim_{n\to\infty} d(a_n, a) = 0_{\mathcal{A}}.$$

Definition 2.8. A sequence $\{a_n\}$ in $(A, \mathcal{A}, d)$ is called Cauchy sequence (with respect to $\mathcal{A}$), if

$$\lim_{n,m\to\infty} d(a_n, a_m) = 0_{\mathcal{A}}.$$

Definition 2.9. The triplet $(A, \mathcal{A}, d)$ is called complete C^*-av symmetric space if every Cauchy sequence in A is convergent to some point $a \in A$.

Now, we furnish some examples in support of our newly introduced C^*-av metric space as follows:

Example 2.10. Let E be a Lebesgue measurable set, $H = L^2(E)$ a Hilbert space, $L(H)$ the set of all bounded and linear operators on H. Obviously, $L(H)$ is a C^*-algebra with the usual norm. Let $A = L^\infty(E)$ and define $d : A \times A \to L(H)$ by (for all $a, b \in A$ and $p, q \geq 1$):

$$d(a,b) = \pi_{|a-b|^p} + \pi_{|a-b|^q},$$

where $\pi_u : H \to H$ is the multiplicative operator defined by:

$$\pi_u(\psi) = u.\psi, \text{ for all } \psi \in H.$$

Then $(A, L(H), d)$ is C^*-av symmetric space. For completeness, we take a Cauchy sequence $\{f_n\}$ in A, that is, for each $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that (for all $n, m \geq N$)

$$\begin{aligned}\|d(f_n, f_m)\| &= \|\pi_{|f_n - f_m|^p} + \pi_{|f_n - f_m|^q}\| \\ &\leq \|\pi_{|f_n - f_m|^p}\| + \|\pi_{|f_n - f_m|^q}\| \\ &= \|f_n - f_m\|_\infty^p + \|f_n - f_m\|_\infty^q \\ &< \epsilon.\end{aligned}$$

Thus, there exists $f \in A$ and $N \in \mathbb{N}$ such that $\|f_n - f\|_\infty^p < \epsilon$. Then we have

$$\begin{aligned}\|d(f_n, f)\| &= \|\pi_{|f_n - f|^p} + \pi_{|f_n - f|^q}\| \\ &\leq \|\pi_{|f_n - f|^p}\| + \|\pi_{|f_n - f|^q}\| \\ &= \|f_n - f\|_\infty^p + \|f_n - f\|_\infty^q \\ &< \epsilon.\end{aligned}$$

Thus, the sequence $\{f_n\}$ converges to f in X. $(A, L(H), d)$ is complete C^*-av symmetric space.

Example 2.11. Let $A = \mathbb{R}$ and $\mathcal{A} = M_3(\mathbb{R})$. Define $d : A \times A \to \mathcal{A}$ by (for all $a, b \in A$ and $p, q \geq 1$):

$$d(a,b) = \begin{bmatrix} |a-b|^p & 0 & 0 \\ 0 & |a-b|^p & 0 \\ 0 & 0 & k|a-b|^p \end{bmatrix} + \begin{bmatrix} |a-b|^q & 0 & 0 \\ 0 & |a-b|^q & 0 \\ 0 & 0 & k|a-b|^q \end{bmatrix}$$

where, $k \geq 0$. Observe that, d is C^*-av symmetric and $(A, \mathcal{A}, d)$ is a complete C^*-av symmetric space with the coefficient $s = 2^{p-1}$.

Let $(A, \mathcal{A}, d)$ be a C^*-av symmetric space. Then, the open ball of center $a \in A$ and radius $0_{\mathcal{A}} \prec \epsilon \in \mathcal{A}$ is defined by:

$$B_d(a, \epsilon) = \{b \in A : d(a, b) \prec \epsilon\}.$$

Similarly, the closed ball with center $a \in A$ and radius $\epsilon \succ 0_{\mathcal{A}}$ is defined by:

$$B_d[a, \epsilon] = \{b \in A : d(a, b) \preccurlyeq \epsilon\}.$$

The family of the open balls (for all $a \in A$ and $\epsilon \succ 0_{\mathcal{A}}$)

$$\mathcal{U}_d = \{B_d(a, \epsilon) : a \in A, \epsilon \succ 0_{\mathcal{A}}\},$$

forms a basis of some topology τ_d on A.

Lemma 2.12. *Let (A, τ_d) be a topological space and $f : A \to A$. If f is continuous then every sequence $\{a_n\} \subseteq A$ such that $a_n \to a$ implies $fa_n \to fa$. The converse holds if A is metrizable.*

Next, we adopt some definitions from symmetric spaces to the setting of C^*-av symmetric space:

Definition 2.13. Let $(A, \mathcal{A}, d)$ be a C^*-av symmetric space. Then

$(AC1)$ $\lim_{n\to\infty} d(a_n, a) = 0_{\mathcal{A}}$ and $\lim_{n\to\infty} d(a_n, b) = 0_{\mathcal{A}}$ imply $a = b$, where $\{a_n\}$ a sequence in X and $a, b \in A$.

$(AC2)$ d is said to be 1-continuous if $\lim_{n\to\infty} d(a_n, a) = 0_{\mathcal{A}}$ implies that $\lim_{n\to\infty} d(a_n, b) = d(a, b)$, where $\{a_n\}$ a sequence in X and $a, b \in A$.

$(AC3)$ d is said to be continuous if $\lim_{n\to\infty} d(a_n, a) = 0_{\mathcal{A}}$ and $\lim_{n\to\infty} d(a_n, b) = d(a, b)$ imply that $\lim_{n\to\infty} d(a_n, b_n) = d(a, b)$, where $\{a_n\}$ and $\{b_n\}$ are sequences in A and $a, b \in A$.

$(AC4)$ $\lim_{n\to\infty} d(a_n, a) = 0_{\mathcal{A}}$ and $\lim_{n\to\infty} d(a_n, b_n) = 0_{\mathcal{A}}$ imply $\lim_{n\to\infty} d(b_n, a) = 0_{\mathcal{A}}$, for sequences $\{a_n\}$, $\{b_n\}$ in A and $a \in A$.

$(AC5)$ $\lim_{n\to\infty} d(a_n, b_n) = 0_{\mathcal{A}}$ and $\lim_{n\to\infty} d(b_n, c_n) = 0_{\mathcal{A}}$ imply $\lim_{n\to\infty} d(a_n, c_n) = 0_{\mathcal{A}}$, for sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ in A.

Remark 2.14. From Definition 2.13, it is observed that $(AC3) \Rightarrow (AC2)$, $(AC4) \Rightarrow (AC1)$ and $(AC2) \Rightarrow (AC1)$ but in general the converse implications are not true.

3. Fixed point results

The following definition is used in our subsequent discussions.

Definition 3.1. The max function on $\mathcal{A}$ (C^*-algebra) with the partial order relation '$\preccurlyeq$' is defined by (for all $a, b \in \mathcal{A}_+$):

$$\max\{a, b\} = b \Leftrightarrow \; a \preccurlyeq b \text{ and } \|a\| \leq \|b\|.$$

Now, we recall the definition of Kannan-Ćirić C^*-contraction condition [18]:

Definition 3.2. Let $(A, \mathcal{A}, d)$ be a C^*-av symmetric space. A mapping $f : A \to A$ is said to be Kannan-Ćirić type C^*-contraction if there exists $\rho \in \mathcal{A}$ with $\|\rho\| < 1$ such that (for all $a, b \in A$)

$$d(fa, fb) \preccurlyeq \rho^* \max\{d(a, fa), d(b, fb)\}\rho. \tag{1}$$

Next, we prove a fixed point result via C^*-av Kannan-Ćirić type contraction in the setting of C^*-av symmetric space:

THEOREM 3.3. *Let $(A, \mathcal{A}, d)$ be a complete C^*-av symmetric space and $f : A \to A$. Suppose the following conditions hold:*

(i) f satisfies condition (1),
(ii) f is continuous.

Then f has a unique fixed point $a \in A$.

Proof. Take $a_0 \in A$ and construct an iterative sequence $\{a_n\}$ by:

$$a_1 = fa_0,\ a_2 = f^2a_0,\ a_3 = f^3a_0, \cdots, a_n = f^na_0, \cdots .$$

Now, we assert that $\lim_{n\to\infty} d(a_n, a_{n+1}) = 0_{\mathcal{A}}$. On setting $a = a_n$ and $b = a_{n+1}$ in (1), we get

$$\begin{aligned} d(a_n, a_{n+1}) &= d(fa_{n-1}, fa_n) \\ &\preccurlyeq \rho^* \max\{d(a_{n-1}, fa_{n-1}), d(a_n, fa_n)\}\rho \\ &= \rho^* \max\{d(a_{n-1}, a_n), d(a_n, a_{n+1})\}\rho. \end{aligned} \tag{2}$$

Assume that $\max\{d(a_{n-1}, a_n), d(a_n, a_{n+1})\} = d(a_n, a_{n+1})$, then from (2), we have

$$d(a_n, a_{n+1}) \preccurlyeq \rho^* d(a_n, a_{n+1})\rho,$$

yielding thereby

$$\begin{aligned} d(a_n, a_{n+1}) &\preccurlyeq \rho^* d(a_n, a_{n+1})\rho \\ &= \rho^*(d(a_n, a_{n+1}))^{\frac{1}{2}}(d(a_n, a_{n+1}))^{\frac{1}{2}}\rho \\ &= \Big((d(a_n, a_{n+1})\rho)^{\frac{1}{2}}\Big)^*\Big((d(a_n, a_{n+1})\rho)^{\frac{1}{2}}\Big) \\ &= \Big\|\big(d(a_n, a_{n+1})\rho\big)^{\frac{1}{2}}\Big\|^2 I \\ &\preccurlyeq \|\rho\|^n\|d(a_n, a_{n+1})\|I \end{aligned}$$

a contradiction (since $\|\rho\|^2 < 1$). Thus, $\max\{d(a_{n-1}, a_n), d(a_n, a_{n+1})\} = d(a_{n-1}, a_n)$. Therefore, (2) gives rise

$$d(a_n, a_{n+1}) \preccurlyeq \rho^* d(a_{n-1}, a_n)\rho\ n \in \mathbb{N}.$$

Thus, inductively we have

$$\begin{aligned} d(a_n, a_{n+1}) &\preccurlyeq (\rho^*)^n d(a_0, a_1)\rho^n \\ &= (\rho^*)^n (d(a_0, a_1))^{\frac{1}{2}} (d(a_0, a_1))^{\frac{1}{2}} \rho^n \\ &= \left((d(a_0, a_1)\rho^n)^{\frac{1}{2}}\right)^* \left((d(a_0, a_1)\rho^n)^{\frac{1}{2}}\right) \\ &= \left\| (d(a_0, a_1)\rho^n)^{\frac{1}{2}} \right\|^2 I \\ &\preccurlyeq \|\rho\|^n \|d(a_0, a_1)\| I. \end{aligned}$$

On making limit as $n \to \infty$, we get

$$\lim_{n\to\infty} d(a_n, a_{n+1}) = 0_{\mathcal{A}}. \tag{3}$$

Now, we assert that $\{a_n\}$ is a Cauchy sequence. Form (1), we have (for $n, m \in \mathbb{N}$)

$$\begin{aligned} d(a_n, a_m) &= d(fa_{n-1}, fa_{m-1}) \\ &\leq \rho^* \max\{d(a_{n-1}, fa_{n-1}), d(a_{m-1}, fa_{m-1})\}\rho \\ &\leq \rho^* \max\{d(a_{n-1}, a_n), d(a_{m-1}, a_m)\}\rho. \end{aligned}$$

By using (3), we have

$$\lim_{n,m\to\infty} d(a_n, a_m) = 0_{\mathcal{A}}. \tag{4}$$

Hence, $\{a_n\}$ is a Cauchy sequence. Since $(A, \mathcal{A}, d)$ is complete, there exists $a \in A$, such that $\{a_n\}$ converges to a. Now, we will show that $a \in A$ is a fixed point of f. By condition (ii), we have

$$a = \lim_{n\to\infty} a_{n+1} = f(\lim_{n\to\infty} a_n) = fa.$$

Therefore, a is a fixed point of f. To prove the uniqueness of fixed point, suppose that $a, b \in A$ are such that $fa = a$ and $fb = b$. Then from (1), we have

$$\begin{aligned} d(a, b) &= d(fa, fb) \\ &\preccurlyeq \rho^* \max\{d(a, fa), d(b, fb)\}\rho, \\ &= \rho^* \max\{d(a, a), d(b, b)\}\rho \\ &= 0_{\mathcal{A}}. \end{aligned}$$

Hence, above inequality implies that $a = b$. Therefore, a is a unique fixed point of f. This completes the proof. □

Let $(A, \mathcal{A}, d)$ be a C^*-av symmetric space and $f : A \to A$. Then for every $a \in A$ and for all $i, j \in \mathbb{N}$, we define

$$\mathfrak{S}(d, f, a) = \sup\{d(f^i a, f^j a) : i, j \in \mathbb{N}\}. \tag{5}$$

Definition 3.4. Let $(A, \mathcal{A}, d)$ be a C^*-av symmetric space. A mapping $f : A \to A$ is said to be C^*-contraction if there exists $\rho \in \mathcal{A}$ with $\|\rho\| < 1$ such that

$$d(fa, fb) \preccurlyeq \rho^* d(a, b)\rho, \ \forall\ a, b \in A. \tag{6}$$

Now, we prove an analogue of Banach contraction principle in the setting of C^*-av symmetric space:

Theorem 3.5. *Let $(A, \mathcal{A}, d)$ be a complete C^*-av symmetric space and $f : A \to A$. Suppose the following conditions hold:*

(i) f satisfies condition (6)
(ii) there exists $a_0 \in A$ such that $\mathfrak{S}(d, f, a_0) \prec \infty$,
(iii) either
  (a) f is continuous or
  (b) $(A, \mathcal{A}, d)$ enjoys the $(AC1)$ property.

Then f has a unique fixed point $a \in A$.

Proof. Choose $a_0 \in A$ and construct an iterative sequence $\{a_n\}$ by:

$$a_1 = fa_0, \ a_2 = f^2 a_0, \ a_3 = f^3 a_0, \cdots, a_n = f^n a_0, \cdots .$$

Now, from (6), we have

$$d(f^{n+i} a_0, f^{n+j} a_0) \preccurlyeq \rho^* d(f^{n-1+i} a_0, f^{n-1+j} a_0)\rho, \ \forall\ i, j \in \mathbb{N}.$$

Since, the above inequality holds for all $i, j \in \mathbb{N}$, therefore by conditions (ii) and (5), we have

$$\mathfrak{S}(d, f, f^n a_0) \leq \|\rho\|^2 \mathfrak{S}(d, f, f^{n-1} a_0).$$

Repeating this procedure indefinitely, we get

$$\mathfrak{S}(d, f, f^n a_0) \leq \|\rho\|^{2n} \mathfrak{S}(d, f, a_0), \ \forall\ n \in \mathbb{N}. \tag{7}$$

Take $n, m \in \mathbb{N}$ such that $m = n + p$ (for some $p \in \mathbb{N}$). On using (7), we get

$$d(f^n a_0, f^{n+p} a_0) \leq \mathfrak{S}(d, f, f^n a_0) \leq \|\rho\|^{2n} \mathfrak{S}(d, f, a_0).$$

Since $\mathfrak{S}(d, f, a_0) \prec \infty$, then

$$\lim_{n,m\to\infty} d(a_n, a_m) = 0_{\mathcal{A}},$$

the sequence $\{a_n\}$ is Cauchy in A. Since, $(A, \mathcal{A}, d)$ is complete, there exists $a \in A$ such that $\{a_n\}$ converges to a. Now, we show that $a \in A$ is a fixed point of f.
Firstly, we take the continuity of f. Then

$$a = \lim_{n\to\infty} a_{n+1} = f(\lim_{n\to\infty} a_n) = fa.$$

Alternately, we assume that $(A, \mathcal{A}, d)$ enjoys the $(AC1)$ property. Now, we get

$$\begin{aligned} d(a_{n+1}, fa) &= d(fa_n, fa) \\ &\preccurlyeq \rho^* d(a_n, a)\rho, \\ \|d(a_{n+1}, fa)\| &\leq \|\rho\|^2 \|d(a_n, a)\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence, $\lim_{n\to\infty} \|d(a_{n+1}, fa)\| = 0$. Thus, from the $(A1)$ property, $fa = a$. Therefore, a is a fixed point of f. For the uniqueness part, suppose that $a, b \in A$ such that $fa = a$ and $fb = b$. Then, by the definition of C^*-contraction, we have

$$d(a, b) = d(fa, fa) \preccurlyeq \rho^* d(a, b)\rho,$$

so that

$$\begin{aligned} \|d(a, b)\| &= \|d(fa, fb)\| \\ &\leq \|\rho^* d(a, b)\rho\| \\ &\leq \|\rho^*\|\|d(a, b)\|\|\rho\| \\ &= \|\rho\|^2 \|d(a, b)\| \\ &< \|d(a, b)\| \end{aligned}$$

a contradiction. Hence, $a = b$, that is, f has a unique fixed point. This completes the proof. □

Now, we furnish the following example which illustrates Theorem 3.5.

Example 3.6. Let $A = [0, 1]$, and $\mathcal{A} = M_2(\mathbb{C})$ be the class of all bounded and linear operators on the Hilbert space $\mathbb{C}^2$. Define $d : A \times A \to \mathcal{A}$ by:

$$d(a, b) = \begin{bmatrix} |a-b|^p & 0 \\ 0 & k|a-b|^p \end{bmatrix} + \begin{bmatrix} |a-b|^q & 0 \\ 0 & k|a-b|^q \end{bmatrix}$$

where $k \geq 0$ and $p, q \geq 1$. Then, $(A, \mathcal{A}, d)$ is a complete C^*-av symmetric space.

Define a map $f : A \to A$ by:

$$fa = \frac{a}{4}, \text{ for all } a \in A.$$

Observe that, $d(fa, fb) \preccurlyeq \rho^* d(a, b)\rho$ (for all $a, b \in A$) with

$$\rho = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \in A \text{ and } \|\rho\| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} < 1.$$

Thus, all the hypothesis of Theorem 3.5 are satisfied and $a = 0$ is unique fixed point of f.

4. Application

As an application of Theorem 3.5 on complete C^*-av symmetric space, we find the existence and uniqueness results for a type of following integral equation:

$$a(\mu) = \int_E G(\mu, \nu, a(\nu))d\nu + h(\mu), \ \mu, \nu \in E, \tag{8}$$

where E is a measurable set, $G : E \times E \times \mathbb{R} \to \mathbb{R}$ and $h \in L^\infty(E)$.

Let $A = L^\infty(E)$, $H = L^2(E)$ and $L(H) = \mathcal{A}$. Define $d : A \times A \to \mathcal{A}$ by (for all $a, b \in A$, $p, q \geq 1$ and $\|\rho\| = k < 1$):

$$d(a, b) = \pi_{|a-b|^p} + \pi_{|a-b|^q},$$

where $\pi_u : H \to H$ is the multiplicative operator defined by:

$$\pi_u(\psi) = u.\psi, \text{ for all } \psi \in H.$$

Now, we state and prove our result as follows:

THEOREM 4.1. *Suppose that (for all $a, b \in A$)*

(1) there exists a continuous function $\psi : E \times E \to \mathbb{R}$ and $k \in (0, 1)$ such that

$$| G(\mu, \nu, a(\nu)) - G(\mu, \nu, b(\nu)) | \leq k \mid \psi(\mu, \nu) \mid\mid a(\nu) - b(\nu) |,$$

for all $\mu, \nu \in E$.

(2) $\sup_{\mu \in E} \int_E | \psi(\mu, \nu) | d\nu \leq 1$.

Then the integral equation (8) has a unique solution in A.

Proof. Define $f : A \to A$ by:

$$fa(\mu) = \int_E G(\mu, \nu, a(\nu))d\nu + h(\mu), \ \forall \ \mu, \nu \in E.$$

Set $\rho = kI$, then $\rho \in \mathcal{A}$. For any $u \in H$ and $p, q \geq 1$, we have

$$\begin{aligned}
\|d(fa, fb)\| &= \sup_{\|u\|=1} (\pi_{|fa-fb|^p} u, u) + \sup_{\|u\|=1} (\pi_{|fa-fb|^q} u, u) \\
&= \sup_{\|u\|=1} \int_E \left[\left| \int_E G(\mu, \nu, a(\nu)) - G(\mu, \nu, b(\nu))d\nu \right|^p \right] u(\mu)\bar{u(\mu)}d\mu \\
&\quad + \sup_{\|u\|=1} \int_E \left[\left| \int_E G(\mu, \nu, a(\nu)) - G(\mu, \nu, b(\nu))d\nu \right|^q \right] u(\mu)\bar{u(\mu)}d\mu \\
&\leq \sup_{\|u\|=1} \int_E \left[\int_E \big|G(\mu, \nu, a(\nu)) - G(\mu, \nu, b(\nu))\big| d\nu \right]^p |u(\mu)|^2 d\mu \\
&\quad + \sup_{\|u\|=1} \int_E \left[\int_E \big|G(\mu, \nu, a(\nu)) - G(\mu, \nu, b(\nu))\big| d\nu \right]^q |u(\mu)|^2 d\mu \\
&\leq \sup_{\|u\|=1} \int_E \left[\int_E \big|k\psi(\mu, \nu)(a(\nu) - b(\nu))\big| d\nu \right]^p |u(\mu)|^2 d\mu \\
&\quad + \sup_{\|u\|=1} \int_E \left[\int_E \big|k\psi(\mu, \nu)(a(\nu) - a(\nu))\big| d\nu \right]^q |u(\mu)|^2 d\nu \\
&\leq k^p \sup_{\|u\|=1} \int_E \left[\int_E |\psi(\mu, \nu)| d\nu \right]^p |u(\mu)|^2 d\mu \|a - b\|_\infty^p \\
&\quad + k^p \sup_{\|u\|=1} \int_E \left[\int_E |\psi(\mu, \nu)| d\nu \right]^q |u(\mu)|^2 d\mu \|a - b\|_\infty^q \\
&\leq k \sup_{\mu \in E} \int_E |\psi(\mu, \nu)| d\nu \sup_{\|u\|=1} \int_E |u(\mu)|^2 d\mu \|a - b\|_\infty^p \\
&\quad + k \sup_{\mu \in E} \int_E |\psi(\mu, \nu)| d\nu \sup_{\|u\|=1} \int_E |u(\mu)|^2 d\mu \|a - b\|_\infty^q \\
&\leq k\|a - b\|_\infty^p + k\|a - b\|_\infty^q \\
&= \|\rho\| \, \|d(a, b)\|.
\end{aligned}$$

Since, $\|\rho\| < 1$, so it is verified that the mapping f satisfies all the conditions of Theorem 3.5. Hence, f has a unique fixed point, means that the Fredholm integral Equation (8) has a unique solution. □

5. Conclusion

Firstly, we enlarged the class of C^*-avMS to the class of C^*-av symmetric space wherein we proved several fixed point results which include an analog of Banach contraction principle and Kannan-Ćirić fixed theorem in such spaces. We also furnished some examples exhibiting the utility of our newly established results. We further used one of our main results to examine the existence and uniqueness of a solution for the system of Fredholm integral equations.

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