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IHARA ZETA FUNCTION OF FINITE GRAPHS WITH CIRCUIT RANK TWO

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Abstract. In this paper, we give an explicit formula as a rational function for the Ihara zeta function of every finite connected graph without degree one vertices whose circuit rank is two.

1. Introduction

Ihara zeta function is a zeta function associated with graphs that resembles the Selberg's dynamical zeta function for a geodesic flow on a Riemannian manifold. In the Selberg's dynamical zeta function of a Riemannian manifold, primitive closed orbits (the primitive periodic points of the geodesic flow) of the geodesic flow play the role of primes in the Riemann's zeta function ([6]). Ihara investigated the p-adic analogue of the Riemann surfaces and found the similar idea of Selberg's zeta function can be adapted to the p-adic and positive-characteristic cases. He also showed that one can compute the zeta function effectively, via a product of polynomials and the determinant of a certain matrix ([2]).

As Serre remarked, a biregular tree is a p-adic analogue of Riemann surfaces in the sense that an F-points of a rank one semisimple algebraic group over an ultrametric local field F acts on the tree ([7]). Thus, we

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may interpret Ihara's zeta function and the determinant formula purely in terms of graphs. For instance, the zeta function of tree lattices are studied by many authors including [1] and [5]. Recently, a new weighted zeta function for a graph is also introduce in [3]. The explicit form, however, as a rational function of the zeta function of finite graphs are rarely known, although we have the determinant formula. The purpose of this paper is to provide explicit formulas of the Ihara zeta function of finite connected graphs which have circuit rank two, without degree one vertices. The *circuit rank* of an undirected graph is the minimum number of edges that must be removed from the graph to break all its cycles, making it into a tree. It follows directly from the definition that the circuit rank r of a connected graph (V, E) is equal to $|E|-|V|+1$. Up to homeomorphism, there are three types of graphs with circuit rank two which we call a *dumbbell graph* (Figure 1), a *figure eight graph* (Figure 2), and a *bicyclic graph* (Figure 3).

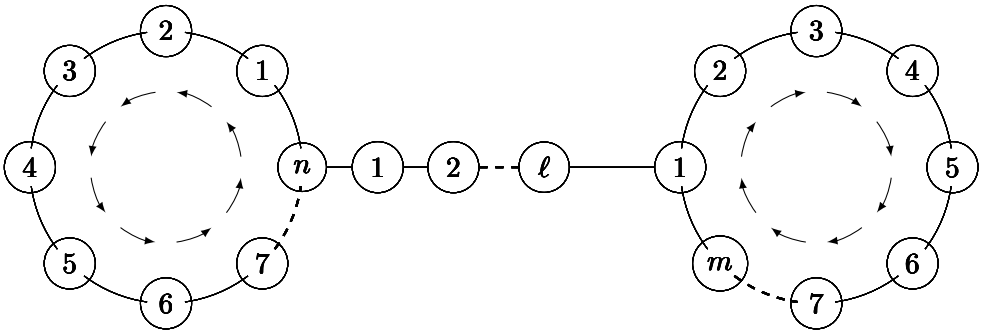


FIGURE 1. Dumbbell graph, $n, m \geqslant 3$, $\ell \geqslant 0$

Let us recall the definition of the Ihara's zeta function of a finite graph following [8]. In order to define the zeta function of graphs, we need to figure out what primes in graphs are. Let $G = (V_G, E_G)$ be a finite, connected, and undirected graph with a set V_G of vertices and a set E_G of edges. If G is any undirected finite connected graph with unorient edges set E and vertex set V, we orient its edges arbitrarily and obtain $2|E|$ oriented edges labelled by $e_1, e_2, \cdots, e_n, e_{n+1} = e_1^{-1}, e_{n+2} = e_2^{-1}, \cdots, e_{2n} = e_n^{-1}$ where $n = |E|$. Let $C = (e_1, e_2, \cdots, e_s)$ be a primitive cycle without backtracking. That is, $e_{i+1} \neq e_i^{-1}$, $e_s \neq e_1^{-1}$ for all i and $C \neq D^f$ for $f > 1$ and a closed path D in G. We say two primitive cycles

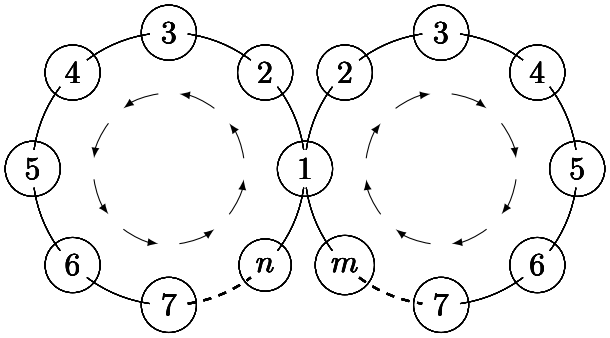


FIGURE 2. Figure eight graph, $n, m \geqslant 3$

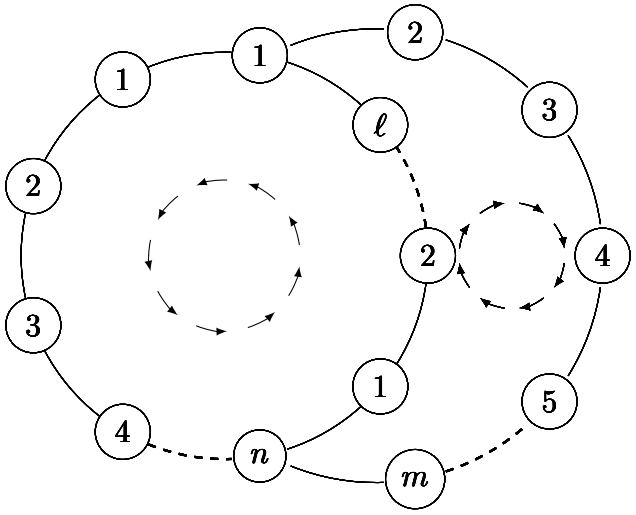


FIGURE 3. Bicyclic graph, $n, m \geqslant 2$, $\ell \geqslant 0$

are *equivalent* if we can get one from the other by changing the starting vertex. A *prime* in the graph G is an equivalence class $[C]$ of primitive cycles. The *length* of the path C is the number s of edges in C, denoted by $\nu(C)$.

DEFINITION 1.1. The Ihara zeta function of graph G is defined at a complex number u, for which $|u|$ is sufficiently small, by

$$Z_G(u) = \prod_{[P]} (1 - u^{\nu(P)})^{-1}$$

where the product is over all primes $[P]$ in G.

If $G = (V_G, E_G)$, then recall that the circuit rank r of G is equal to $|E_G| - |V_G| + 1$. Let $V_G = \{v_1, \cdots, v_n\}$. If G is a simple graph (a graph without loops and multiple edges), then the *adjacency matrix* A of G is the square matrix such that $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. Let D be the diagonal matrix with $d_{ii} = \deg_G v_i$ and $Q = D - I$.

THEOREM 1.2 ([1] (see also [2], Theorem 2)). *Let G be a connected graph (V_G, E_G), and let r be the circuit rank of G. Then, the zeta function of G is given by*

$$Z_G(u) = \frac{1}{(1-u^2)^{r-1} \det(I - Au + Qu^2)}.$$

We are ready to state our main results. We denote by $D_{n,m,\ell}$ the dumbbell graph of type (n, m, ℓ), which consists of two vertex-disjoint cycles C_m,C_n and a path P_ℓ $(n, m \geqslant 1, \ell \geqslant 0)$ joining those cycles. It has $n + m + \ell$ number of vertices and $n + m + \ell + 1$ number of edges.

THEOREM 1.3. *Let D be the dumbbell graph $D_{n,m,\ell}$ with $n, m \geqslant 3$ and $\ell \geqslant 0$. Then, the zeta function of D is given by*

$$Z_D(u) = \frac{1}{(1-u^n)(1-u^m)(1-u^n-u^m+u^{n+m}-4u^{n+m+2\ell+2})}.$$

We remark that the formula for the cases $n = 1$, $m = 1$ and $n = 1$, $m = 2$ are given in [4].

The figure eight graph of type (n, m), denoted by $E_{n,m}$, consists of two circles C_n and C_m with the same starting vertex. It has $n + m - 1$ number of vertices and $n + m$ number of edges.

THEOREM 1.4. *Let E be the figure eight graph $E_{n,m}$ with $n, m \geqslant 3$. Then, the zeta function of E is given by*

$$Z_E(u) = \frac{1}{(1-u^n)(1-u^m)(1-u^n-u^m-3u^{n+m})}.$$

The bicyclic graph $B_{n,m,\ell}$ of type (n, m, ℓ) consists of two cycles of length $n + \ell + 1$ and $m + \ell + 1$ which shares $\ell + 2$ vertices.

THEOREM 1.5. *Let B be the bicyclic graph $B_{n,m,\ell}$ with $n, m \geqslant 2$ and $\ell \geqslant 0$. Then, the zeta function $Z_B(u)$ of B is given by*

$$\frac{-1}{(2u^{n+m+\ell+1}-u^{n+m}-u^{n+\ell+1}-u^{m+\ell+1}+1)(2u^{n+m+\ell+1}+u^{n+m}+u^{n+\ell+1}+u^{m+\ell+1}-1)}.$$

2. Determinants of some tridiagonal matrices

In this section, we define some tridiagonal matrices which are frequently used in the proof of the theorems. Let

$$A_n = \begin{pmatrix} 1+u^2 & -u & & & & \\ -u & 1+u^2 & -u & & & \\ & -u & \cdots & -u & & \\ & & -u & 1+u^2 & -u & \\ & & & -u & 1+2u^2 & \end{pmatrix},$$

$$B_n = \begin{pmatrix} 1+2u^2 & -u & & & & \\ -u & 1+u^2 & -u & & & \\ & -u & \cdots & -u & & \\ & & -u & 1+u^2 & -u & \\ & & & -u & 1+u^2 & \end{pmatrix},$$

and

$$K_n = \begin{pmatrix} 1+u^2 & -u & & & & \\ -u & 1+u^2 & -u & & & \\ & -u & \cdots & -u & & \\ & & -u & 1+u^2 & -u & \\ & & & -u & 1+u^2 & \end{pmatrix}$$

be $n \times n$ matrices defined for u.

The determinant of the above tridiagonal matrices are given by

$$\det A_n = \det B_n = 1 + 2u^2 + 2u^4 + \cdots + 2u^{2n}$$

and

$$\det K_n = 1 + u^2 + u^4 + \cdots + u^{2n} = \frac{1-u^{2n+2}}{1-u^2}.$$

Let us denote by $f_n(u) = 1 + u^2 + u^4 + \cdots + u^{2n}$ so that we have $\det K_n = f_n(u)$ and $\det A_n = \det B_n = 2f_n(u) - 1$. In the sequel, we write f_n for $f_n(u)$ for simplicity.

For $n, m \geqslant 3$ and $\ell \geqslant 0$, let $H_{n,m,\ell}$, $A_{n,\ell}$, and $B_{m,\ell}$ be the matrices given by

$$H_{n,m,\ell} = \begin{bmatrix}
1+u^2 & -u & & & & & & & & & \\
-u & \cdots & -u & & & & & & & & \\
 & -u & 1+u^2 & -u & & & & & & & \\
 & & -u & 1+2u^2 & -u & & & & & & \\
 & & & -u & 1+u^2 & -u & & & & & \\
 & & & & -u & \cdots & -u & & & & \\
 & & & & & -u & 1+u^2 & -u & & & \\
 & & & & & & -u & 1+2u^2 & -u & & \\
 & & & & & & & -u & 1+u^2 & -u & \\
 & & & & & & & & -u & \cdots & -u \\
 & & & & & & & & & -u & 1+u^2
\end{bmatrix}$$

(blocks: A_n (n), K_ℓ (ℓ), B_m (m))

$$A_{n,\ell} = \begin{bmatrix}
1+u^2 & -u & & & & & \\
-u & \cdots & -u & & & & \\
 & -u & 1+u^2 & -u & & & \\
 & & -u & 1+2u^2 & -u & & \\
 & & & -u & 1+u^2 & -u & \\
 & & & & -u & \cdots & -u \\
 & & & & & -u & 1+u^2
\end{bmatrix}$$

(row/column blocks: n, ℓ)

$$B_{m,\ell} = \begin{bmatrix}
1+u^2 & -u & & & & & \\
-u & \cdots & -u & & & & \\
 & -u & 1+u^2 & -u & & & \\
 & & -u & 1+2u^2 & -u & & \\
 & & & -u & 1+u^2 & -u & \\
 & & & & -u & \cdots & -u \\
 & & & & & -u & 1+u^2
\end{bmatrix}.$$

(row/column blocks: ℓ, m)

The determinant of the above matrices are given as follows.

LEMMA 2.1. *Let $A_{n,\ell}$, $B_{m,\ell}$ and $H_{n,m,\ell}$ be the above matrices. Then, we have*

$$\begin{aligned}
\det A_{n,\ell} &= \det K_{n-1}(1-\det K_\ell) + \det A_n \det K_\ell \\
&= f_{n-1}(1-f_\ell) + (2f_n-1)f_\ell, \\
\deg B_{m,\ell} &= \det K_{m-1}(1-\det K_\ell) + \det B_m \det K_\ell \\
&= f_{m-1}\,(1-f_\ell) + (2f_m-1)\,f_\ell, \\
\det H_{n,m,\ell} &= \det B_{m,\ell} \det A_n + \det B_{m,\ell-1}(1-\det K_n) \\
&= \left[f_{m-1}\,(1-f_\ell) + (2f_m-1)\,f_\ell\right](2f_n-1) \\
&\quad + \left[f_{m-1}\,(1-f_{\ell-1}) + (2f_m-1)\,f_{\ell-1}\right](1-f_\ell)\,.
\end{aligned}$$

Proof. Using row and column operations, we have $\det B_{m,\ell+2} = (1+u^2)\det B_{m,\ell+1} - u^2 \det B_{m,\ell}$. It follows from the characteristic equation of linear recurrence relations that we have

$$\det B_{m,\ell} = c_1 + c_2 u^{2\ell}$$

for some constants c_1 and c_2. Since

$$\det B_{m,1} = (1+u^2)(1+2u^2+\cdots+2u^{2m}) - u^2(1+u^2+\cdots+u^{2m-2})$$

and

$$\det B_{m,2} = (1+u^2+u^4)(1+2u^2+\cdots+2u^{2m}) - (u^2+u^4)(1+u^2+\cdots+u^{2m-2}),$$

we have

$$c_1 = -\frac{1+u^2+\cdots+u^{2m}}{u^2-1}$$

and

$$c_2 = \frac{u^4(1+u^2+u^4+\cdots+u^{2m-4}+2u^{2m-2})}{u^2-1}.$$

This yields

$$\begin{aligned}
\det B_{m,\ell} &= \frac{-(1+u^2+\cdots+u^{2m}) + u^{2\ell+4}(1+u^2+\cdots+u^{2m-4}+2u^{2m-2})}{u^2-1} \\
&= f_{m-1}\,(1-f_\ell) + (2f_m-1)\,f_\ell.
\end{aligned}$$

The result for $A_{n,\ell}$ can be obtained analogously.

For the case of $H_{n,m,\ell}$, we have $\det H_{n+2,m,\ell} = (1+2u^2)\det H_{n+1,m,\ell} - u^2 \det H_{n,m,\ell}$ and hence $\det H_{n,m,\ell} = c_3 + c_4 u^{2n}$ for some constants c_3 and c_4. Since

$$\begin{aligned}
&\det H_{1,m,\ell} \\
&=(1+2u^2)\left(\frac{-(1+u^2+\cdots+u^{2m})+u^{2\ell+4}(1+u^2+\cdots+u^{2m-4}+2u^{2m-2})}{u^2-1}\right) \\
&\quad -u^2\left(\frac{-(1+u^2+\cdots+u^{2m})+u^{2\ell+2}(1+u^2+\cdots+u^{2m-4}+2u^{2m-2})}{u^2-1}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\det H_{2,m,\ell} \\
&=(1+2u^2+2u^4)\left(\frac{-(1+u^2+\cdots+u^{2m})+u^{2\ell+4}(1+u^2+\cdots+u^{2m-4}+2u^{2m-2})}{u^2-1}\right) \\
&\quad -(u^2+u^4)\left(\frac{-(1+u^2+\cdots+u^{2m})+u^{2\ell+2}(1+u^2+\cdots+u^{2m-4}+2u^{2m-2})}{u^2-1}\right),
\end{aligned}$$

we have

$$c_4 = \frac{-u^2(1+u^2+\cdots+u^{2m})+u^{2\ell+4}(-1+2u^2)(1+u^2+\cdots+u^{2m-4}+2u^{2m-2})}{(u^2-1)^2}$$

and

$$c_3 = b_1 - c_4u^2 = \frac{(1+u^2+\cdots+u^{2m})-u^{2\ell+6}(1+u^2+\cdots+u^{2m-4}+2u^{2m-2})}{(u^2-1)^2}.$$

It follows that the determinant of the matrix $H_{n,m,\ell}$ is equal to

$$\begin{aligned}
&\frac{(1-u^{2n+2})(1+u^2+\cdots+u^{2m})+u^{2\ell+4}(2u^{2n+2}-u^{2n}-u^2)(1+u^2+\cdots+u^{2m-4}+2u^{2m-2})}{(u^2-1)^2} \\
&=\left[f_{m-1}\left(1-f_\ell\right)+\left(2f_m-1\right)f_\ell\right]\left(2f_n-1\right)+\left[f_{m-1}\left(1-f_{\ell-1}\right)+\left(2f_m-1\right)f_{\ell-1}\right]\left(1-f_\ell\right).
\end{aligned}$$

This completes the proof of lemma. □

3. Proof of theorem 1.3

Let $D = D_{n,m,\ell}$ be a dumbbell graph with $n, m \geqslant 3$ and $\ell \geqslant 0$. We use the Ihara's determinant formula (Theorem 1.2). Let us denote by

R_D the matrix $I - Au + Qu^2$ for D given as follows.

$$R_D = \left[\begin{array}{ccccccccccc}
1+u^2 & -u & & -u & & & & & & & \\
-u & \cdots & -u & & & & & & & & \\
 & -u & 1+u^2 & -u & & & & & & & \\
-u & & -u & 1+2u^2 & -u & & & & & & \\
 & & & -u & 1+u^2 & -u & & & & & \\
 & & & & -u & \cdots & -u & & & & \\
 & & & & & -u & 1+u^2 & -u & & & \\
 & & & & & & -u & 1+2u^2 & -u & & -u \\
 & & & & & & & -u & 1+u^2 & -u & \\
 & & & & & & & & -u & \cdots & -u \\
 & & & & & & & -u & & -u & 1+u^2
\end{array}\right]$$

To show the Theorem 1.3, it is enough to calculate the determinant of R_D. Using the row and column operation we have

$$\begin{aligned}
&\det R_D \\
&= u^4 \det H_{n-2,m-2,\ell} - u^2(1+u^2) \det H_{n-2,m-1,\ell} - u^2(1+u^2) \det H_{n-1,m-2,\ell} \\
&\quad + (1+u^2) \det H_{n-1,m-1,\ell} + 4u^{m+n} \det K_\ell + 2u^{n+2} \det K_\ell \det K_{m-2} \\
&\quad + 2u^{m+2} \det K_\ell \det K_{n-2} + u^4 \det K_\ell \det K_{n-2} \det K_{m-2} + 2u^{m+2} \det A_{n-2,\ell} \\
&\quad - u^2(1+u^2) \det K_{m-2} \det A_{n-2,\ell} \\
&\quad + u^4 \det K_{m-2} \det A_{n-2,\ell} - 2u^m(1+u^2) \det A_{n-2,\ell} + 2u^{n+2} \det B_{m-2,\ell} \\
&\quad + u^4 \det K_{n-2} \det B_{m-2,\ell} \\
&\quad - 2u^n(1+u^2) \det B_{m-1,\ell} - u^2(1+u^2) \det K_{n-2} \det B_{m-1,\ell}.
\end{aligned}$$

Replacing the determinants of $H_{n,m,\ell}$, K_n, $A_{n,\ell}$, $B_{n,\ell}$ by f_n in the above expression, we have

$$\begin{aligned}
\det R_D = u^4 &\left[\left(f_{m-3}\,(1-f_\ell) + (2f_{m-2}-1)\,f_\ell\right)(2f_{n-2}-1)\right. \\
&\left. + \left(f_{m-3}\,(1-f_{\ell-1}) + (2f_{m-2}-1)\,f_{\ell-1}\right)(1-f_{n-2})\right] \\
&- u^2(1+u^2)\left[\left(f_{m-2}\,(1-f_\ell) + (2f_{m-1}-1)\,f_\ell\right)(2f_{n-2}-1)\right. \\
&\left. + \left(f_{m-2}\,(1-f_{\ell-1}) + (2f_{m-1}-1)\,f_{\ell-1}\right)(1-f_{n-2})\,\right] \\
&- u^2(1+u^2)\left[\left(f_{m-3}\,(1-f_\ell) + (2f_{m-2}-1)\,f_\ell\right)(2f_{n-1}-1)\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \left(f_{m-3}\left(1-f_{L-1}\right)+\left(2f_{m-2}-1\right)f_{\ell-1}\right)\left(1-f_{n-1}\right)\big] \\
&+ \left(1+u^2\right)\big[\left(f_{m-2}\left(1-f_{\ell}\right)+\left(2f_{m-1}-1\right)f_{\ell}\right)\left(2f_{n-1}-1\right) \\
&+ \left(f_{m-2}\left(1-f_{\ell-1}\right)+\left(2f_{m-1}-1\right)f_{\ell-1}\right)\left(1-f_{n-1}\right)\big] \\
&+ 4u^{m+n}f_{\ell}+2u^{n+2}f_{\ell}f_{m-2}+2u^{m+2}f_{\ell}f_{n-2}+u^4f_{\ell}f_{n-2}f_{m-2} \\
&+ 2u^{m+2}\left[f_{n-3}\left(1-f_{\ell}\right)+\left(2f_{n-2}-1\right)f_{\ell}\right] \\
&- u^2\left(1+u^2\right)f_{m-2}\left[f_{n-2}\left(1-f_{\ell}\right)+\left(2f_{n-1}-1\right)f_{\ell}\right] \\
&+ u^4f_{m-2}\left[f_{n-3}\left(1-f_{\ell}\right)+\left(2f_{n-2}-1\right)f_{\ell}\right] \\
&- 2u^m\left(1+u^2\right)\left[f_{n-2}\left(1-f_{\ell}\right)+\left(2f_{n-1}-1\right)f_{\ell}\right] \\
&+ 2u^{n+2}\left(f_{m-3}\left(1-f_{\ell}\right)+\left(2f_{m-2}-1\right)f_{\ell}\right) \\
&+ u^4f_{n-2}\left[f_{m-3}\left(1-f_{\ell}\right)+\left(2f_{m-2}-1\right)f_{\ell}\right] \\
&- 2u^n\left(1+u^2\right)\left[f_{m-2}\left(1-f_{\ell}\right)+\left(2f_{m-1}-1\right)f_{\ell}\right] \\
&- u^2\left(1+u^2\right)f_{n-2}\left[f_{m-2}\left(1-f_{\ell}\right)+\left(2f_{m-1}-1\right)f_{\ell}\right].
\end{aligned}
$$

Expanding the whole equation, we get

$$
\begin{aligned}
\det R_D = {} & 2u^{n+2}f_{m-3}-2u^nf_{m-2}-2u^{n+2}f_{m-2}+2u^{m+2}f_{n-3}+u^4f_{m-2}f_{n-3} \\
&+ 2u^4f_{n-2}-2u^mf_{n-2}-2u^{m+2}f_{n-2}-3u^2f_{m-2}f_{n-2}-3u^4f_{m-2}f_{n-2} \\
&- u^2f_{n-1}-u^4f_{n-1}+f_{m-2}f_{n-1}+2u^2f_{m-2}f_{n-1}+u^4f_{m-2}f_{n-1} \\
&- f_{\ell-1}\left(1-u^2+\left(1+3u^2\right)f_{m-2}-2\left(1+u^2\right)f_{m-1}\right)\left(1+u^2f_{n-2}\right. \\
&\left.-\left(1+u^2\right)f_{n-1}\right)+f_{\ell}\left[1-u^2+2u^m+2u^n-2u^{n+2}+4u^{m+n}\right. \\
&- 2u^{m+2}f_{n-3}+3u^2f_{n-2}-3u^4f_{n-2}+2u^mf_{n-2}+8u^{m+2}f_{n-2} \\
&- 2f_{n-1}+2u^4f_{n-1}-4u^mf_{n-1}-4u^{m+2}f_{n-1} \\
&- 2\left(1+u^2\right)f_{m-1}\left(1+2u^n+3u^2f_{n-2}-2\left(1+u^2\right)f_{n-1}\right) \\
&+ f_{m-2}\{-u^4f_{n-3}+u^2\left(4+13u^2\right)f_{n-2} \\
&\left.+\left(1+4u^2\right)\left(1+2u^n-2\left(1+u^2\right)f_{n-1}\right)\}\right].
\end{aligned}
$$

This can by factored as

$$
\det R_D = \left(u^m-1\right)\left(u^n-1\right)\left(4u^{n+m+2\ell+2}-u^{m+n}+u^n+u^m-1\right)/\left(u^2-1\right)
$$

which yields

$$
Z_D(u)=\frac{1}{\left(1-u^n\right)\left(1-u^m\right)\left(1-u^n-u^m+u^{n+m}-4u^{n+m+2\ell+2}\right)}.
$$

by the Ihara's determinant formula.

4. Proof of theorem 1.4

Let $E_{n,m}$ be the figure eight graph with $n, m \geqslant 3$. Let us define the $(n+m-1) \times (n+m-1)$ square matrix $H_{n,m}$ by

$$H_{n,m} = \begin{bmatrix} 1+u^2 & -u & & & & & \\ -u & \cdots & -u & & & & \\ & -u & 1+u^2 & -u & & & \\ & & -u & 1+3u^2 & -u & & \\ & & & -u & 1+u^2 & -u & \\ & & & & -u & \cdots & -u \\ & & & & & -u & 1+u^2 \end{bmatrix}$$

(the upper-left block of size n and the lower-right block of size m share the entry $1+3u^2$)

Using the row and column expansion, the determinant of $H_{n,m}$ is given by

$$\begin{aligned} \det H_{n,m} &= \left(1+3u^2\right) \det K_{n-1} \det K_{m-1} \\ &\qquad + u^2 \det K_{n-2} \det K_{m-1} + u^2 \det K_{n-1} \det K_{m-2} \\ &= \left(1+3u^2\right) f_{n-1} f_{m-1} + u^2 f_{n-2} f_{m-1} + u^2 f_{n-1} f_{m-2} \end{aligned}$$

Let R_E be the matrix $I - Au + Qu^2$ for E given as follows.

$$R_E = \begin{bmatrix} 1+u^2 & -u & & -u & & & \\ -u & \cdots & -u & & & & \\ & -u & 1+u^2 & -u & & & \\ -u & & -u & 1+3u^2 & -u & & -u \\ & & & -u & 1+u^2 & -u & \\ & & & & -u & \cdots & -u \\ & & & -u & & -u & 1+u^2 \end{bmatrix}$$

(the upper-left block of size n and the lower-right block of size m share the entry $1+3u^2$)

The determinant of R_E is given by

$$\begin{aligned}\det R_E &= (1+u^2)^2 \det H_{n-1,m-1} - (1+u^2)u^2 \det H_{n-1,m-2} \\ &\quad - (1+u^2)u^m \det K_{n-2} - (1+u^2)u^m \det K_{n-2} \\ &\quad - (1+u^2)u^2 \det K_{n-2} \det K_{m-2} - (1+u^2)u^2 \det H_{n-2,m-1} \\ &\quad + u^4 \det H_{n-2,m-2} + u^{m+2} \det K_{n-3} \\ &\quad + u^{m+2} \det K_{n-3} + u^4 \det K_{n-3} \det K_{m-2} \\ &\quad - (1+u^2)u^n \det K_{m-2} + u^{n+2} \det K_{m-3} \\ &\quad - (1+u^2) \det K_{m-2} u^{n+2} + u^{n+2} \det K_{m-3} - u^2 \det K_{n-2} \det K_{m-1}.\end{aligned}$$

As in the case of dumbbell graphs, we replace all the determinants by the functions of u and use the Ihara's determinant formula to get

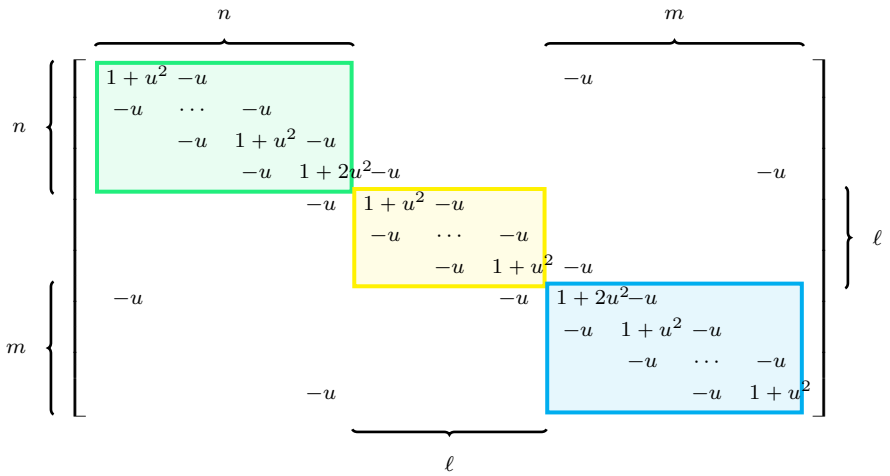
$$Z_E(u) = \frac{1}{(1-u^m)(1-u^n)(1-u^m-u^n+3u^{m+n})}.$$

This completes the proof of Theorem 1.4.

5. Proof of Theorem 1.5

Let $B_{n,m,\ell}$ be the bicyclic graph with $n, m \geqslant 2$ and $\ell \geqslant 0$. Similarly, let us denote by R_B the matrix $I - Au + Qu^2$ for B.

$R_B =$



It follows that

$$\begin{aligned}\det R_B &= (1+u^2)^2 \det H_{n-1,m-1,\ell} - (1+u^2)u^2 \det H_{n-1,m-2,\ell} \\ &\quad + (1+u^2)u^{m+\ell+1} \det K_{n-2} - (1+u^2)u^{m+\ell+1} \det K_{n-2} \\ &\quad - (1+u^2)u^2 \det K_{n-2} \det B_{m-1,\ell} \\ &\quad - (1+u^2)u^2 \det H_{n-2,m-1,\ell} + u^4 H_{n-2,m-2,\ell} + u^{m+\ell+3} \det K_{n-3} \\ &\quad + u^{m+\ell+3} \det K_{n-3} + u^4 \det K_{n-3} \det B_{m-1,\ell} - (1+u^2)u^{n+\ell+1} \det K_{m-2} \\ &\quad + u^{n+\ell+3} \det K_{m-3} - u^{n+m} \det K_\ell - (1+u^2)u^{n+\ell+1} \det K_{m-2} \\ &\quad + u^{n+\ell+3} \det K_{m-3} - u^{n+m} \det K_\ell - (1+u^2)u^2 \det A_{n-1,\ell} \det K_{m-2} \\ &\quad + u^4 \det A_{n-1,\ell} \det K_{m-3} + u^4 \det K_{n-2} \det K_\ell \det K_{m-2}.\end{aligned}$$

Expanding the whole polynomial and applying the Ihara's determinant formula, we obtain that $Z_B(u)$ is equal to

$$-\frac{1}{(2u^{n+m+\ell+1} - u^{n+m} - u^{n+\ell+1} - u^{m+\ell+1} + 1)\,(2u^{n+m+\ell+1} + u^{n+m} + u^{n+\ell+1} + u^{m+\ell+1} - 1)}.$$

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