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A NOTE ON δ-QUASI FUZZY SUBNEAR-RINGS AND IDEALS

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Abstract. In this paper, we discuss in detail some of the properties of the new kind of $(\in, \in \vee q)$-fuzzy ideals in Near-ring. The concept of $(\in, \in \vee q_0^\delta)$-fuzzy ideal of Near-ring is introduced and some of its related properties are investigated.

1. Introduction

The notion of a fuzzy set was introduced by L.A Zadeh [17], and since then this concept have been applied to various algebraic structure. Rosenfeld [16] applied this concept and introduced fuzzy subgroup. The notions of fuzzy subnear-ring and fuzzy ideals of near-rings were introduced by Abou Zaid [1]. The concept of quasi-coincidence of a fuzzy point with a fuzzy subset was introduced by P.Ming and Y.Ming [15]. Using the idea of quasi-coincidence of a fuzzy point with a fuzzy set S.Bhakat [2] defined different types of fuzzy subgroup called (α, β)-fuzzy subgroups. In particular, he introduced $(\in, \in \vee q)$-fuzzy subgroup as the only non trivial generalization of a fuzzy subgroup defined by Rosenfeld. The notions of $(\in, \in \vee q)$-fuzzy subrings and $(\in, \in \vee q)$-fuzzy ideals of a ring were introduced by S.K.Bhakat and P.Das [4]. A.Narayanan and

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T.Manikantan [14] defined $(\in, \in \vee q)$-fuzzy subnear-rings and $(\in, \in \vee q)$-fuzzy ideals of a near-ring. Y.B.Jun and M.A.Ozturk [10] introduced the concepts of $(\in, \in \vee q_0^\delta)$-fuzzy subrings and $(\in, \in \vee q_0^\delta)$-fuzzy ideals in a ring. In this paper, the notions of $(\in, \in \vee q_0^\delta)$-fuzzy subnear-rings and $(\in, \in \vee q_0^\delta)$-fuzzy ideals of a near-ring are introduced and related properties are investigated.

2. Preliminaries

We first recall the definition of near-ring. A non-empty subset N with two binary operation "+"(addition) and "·" (multiplication) is called a near-ring if it satisfies the following axioms:

i) $(N,+)$ is a group;
ii) $(N,\cdot)$ is a semigroup;
iii) $(x+y)\cdot z = x\cdot z + y\cdot z$ for all $x, y, z \in N$.

It is a right near-ring because it satisfies the right distributive law. If it satisfies left distributive law it is called left near-ring.
Unless otherwise stated, we shall consider only right near-rings throughout this paper.

Definition 2.1. Let N be a near-ring. A normal subgroup I of $(N,+)$ is called

i) a right ideal if $IN \subseteq I$
ii) a left ideal if $n(m+i) - nm \in I$ for all $n, m \in N$ and $i \in I$
iii) an ideal if it is both right and left ideal.

Definition 2.2. [15] A fuzzy set μ in a set X of the form

$$\mu(y) = \begin{cases} t \in (0,1] & \text{if } y = x; \\ 0 & \text{if } y \neq x. \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

Definition 2.3. [15] For a fuzzy point x_t and a fuzzy set μ in a set X, we say that

i) $x_t \in \mu$ (resp. $x_t q \mu$) if $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$),
ii) $x_t \in \vee q \mu$ if $x_t \in \mu$ or $x_t q \mu$.

DEFINITION 2.4. [2],[3] A fuzzy set μ of a group G is said to be an $(\in, \in \vee q)$-fuzzy subgroup of G if for all $x, y \in G$ and $t, r \in (0,1]$,

i) $x_t, y_r \in \mu \Rightarrow (xy)_{min\{t,r\}} \in \vee q\mu$ and
ii) $x_t \in \mu \Rightarrow (-x)_t \in \vee q\mu$.

DEFINITION 2.5. [14] A fuzzy set μ is said to be an $(\in, \in \vee q)$-fuzzy subnear-ring of N if $\forall x, y \in N$ and $t, r \in (0,1]$

i) $x_t, y_r \in \mu \Rightarrow (x+y)_{min\{t,r\}} \in \vee q\mu$.
ii) $x_t \in \mu \Rightarrow (-x)_t \in \vee q\mu$.
iii) $x_t, y_r \in \mu \Rightarrow (xy)_{min\{t,r\}} \in \vee q\mu$.

DEFINITION 2.6. [14] A fuzzy set μ of a near-ring N is said to be an $(\in, \in \vee q)$-fuzzy ideal of N if

i) μ is an $(\in, \in \vee q)$-fuzzy subnear-ring of N,
ii) $x_t \in \mu$ and $y \in N \Rightarrow (y+x-y)_t \in \vee q\mu$,
iii) $x_t \in \mu$ and $y \in N \Rightarrow (xy)_t \in \vee q\mu$,
iv) $a_t \in \mu$ and $x, y \in N \Rightarrow (y(x+a)-yx)_t \in \vee q\mu \ \forall x, y, a \in N$.

DEFINITION 2.7. [9] Let μ be a fuzzy set of G. Then $\forall t \in (0,1]$, the set $\mu_t = \{x \in G; \mu(x) \geq t\}$ is called level subset of μ.

DEFINITION 2.8. [5] The subset $\bar{\mu_t} = \{x \in X; \mu(x) \geq t$ or $\mu(x)+t > 1\}$ is called $(\in \vee q)$-level subset of X determined by μ and t.

Jun et al [11] generalized a quasi-coincident fuzzy point. Let $\delta \in (0,1]$. For a fuzzy point x_t and a fuzzy set μ in a set X, we say that
•x_t is a δ-quasi-coincident with μ, written as $x_t q_0^\delta \mu$, if $\mu(x)+t > \delta$.
•$x_t \in \vee q_0^\delta \mu$, if $x_t \in \mu$ or $x_t q_0^\delta \mu$.
If $\delta = 1$, then the δ-quasi-coincident with μ is the quasi-coincident with μ, i, e $x_t q_0^1 \mu = x_t q\mu$.

DEFINITION 2.9.[11] Let μ be a fuzzy set of N. Then the subset $\bar{\mu}_t^\delta = \{x \in N; \mu(x) \geq t$ or $\mu(x)+t > \delta\}$ is called $(\in \vee q_0^\delta)$-level subset of N.

DEFINITION 2.10. [11] For a subset A of N, a fuzzy set χ_A^δ in N defined by
$\chi_A^\delta : N \to [0, \delta]$ as

$$\chi_A^\delta(x) = \begin{cases} \delta & \text{if } x \in A; \\ 0 & \text{otherwise.} \end{cases}$$

is called a δ-characteristic fuzzy set of A in N.

3. Main Results

In this section, we defined the new kind of δ-quasi-coincident with fuzzy set μ in a near-ring. The properties of $(\in, \in \vee q_0^{\delta})$-fuzzy ideals in near-ring are discussed and some of these characterizations are explored. Here δ and N denote an element of (0,1] and a near-ring respectively unless otherwise specified.

Definition 3.1. A fuzzy set μ in N is called an $(\in, \in \vee q_0^{\delta})$-fuzzy subnear-ring of N if for all $x, y \in N$ and $t, r \in (0, \delta]$,

i) $x_t \in \mu, y_r \in \mu \Rightarrow (x-y)_{min\{t,r\}} \in \vee q_0^{\delta}\mu$ and

ii) $x_t \in \mu, y_r \in \mu \Rightarrow (xy)_{min\{t,r\}} \in \vee q_0^{\delta}\mu$.

Example 3.2. Let $N = \{0, a, b, c\}$ with $(N, +)$ as Klien 4-group and $(N, \cdot)$ as defined in table by

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

Then, $(N, +, \cdot)$ is a right near-ring. Define a fuzzy set μ in N by
$\mu(0) = 0.8, \mu(a) = 0.7, \mu(b) = 0.48, \mu(c) = 0.45$.
Then, μ is an $(\in, \in \vee q_0^{\delta})$-fuzzy subnear-ring of N with $\delta \in (0, 0.9]$.
If $\delta = 0.95 \in (0.9, 1]$, then $a_{0.47} \in \mu, b_{0.46} \in \mu$ but
$(a-b)_{min\{0.47,0.46\}} = c_{0.46} \overline{\in \vee q_0^{\delta}} \mu$.
Thus, μ is not an $(\in, \in \vee q_0^{\delta})$-fuzzy subnear-ring of N when $\delta \in (0.9, 1]$.
Note that every $(\in, \in \vee q_0^{\delta})$-fuzzy subnear-ring with $\delta = 1$ is an $(\in, \in \vee q)$-fuzzy subnear-ring.
If $\delta_1 > \delta_2$ in (0,1], then every $(\in, \in \vee q_0^{\delta})$-fuzzy subnear-ring of N with $\delta = \delta_1$ is also an $(\in, \in \vee q_0^{\delta})$-fuzzy subnear-ring of N with $\delta = \delta_2$. But the converse is not true as seen in example 3.2.
So, every $(\in, \in \vee q)$-fuzzy subnear-ring is an $(\in, \in \vee q_0^{\delta})$-fuzzy subnear-ring, but the converse is not true.

Analogous to result in [7],[14], the necessary and sufficient condition for determining the fuzzy set to be $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring is given here.

Theorem 3.3. *For a fuzzy set μ in N, μ is an $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring of N if and only if $\mu(x-y), \mu(xy) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\}$*

Proof. Let μ is an $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring of N.
Suppose $x, y \in N$ such that $\mu(x-y) < min\{\mu(x), \mu(y), \frac{\delta}{2}\}$
choose $t \in (0, \delta]$ such that $\mu(x-y) < t \leq min\{\mu(x), \mu(y), \frac{\delta}{2}\}$
$\Rightarrow x_t \in \mu, y_t \in \mu$ but $(x-y)_t \overline{\in \vee q_0^\delta} \mu$ which is a contradiction.
Therefore, $\mu(x-y) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\}$. for all $x, y \in N$.
Similarly, $\mu(xy) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\}$. for all $x, y \in N$.
Conversely, let us assume that $\mu(x-y), \mu(xy) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\}$. for all $x, y \in N$.
Let $x_t \in \mu$ and $y_r \in \mu$ for $x, y \in N$ and $t, r \in (0, \delta]$
Then $\mu(x) \geq t$ and $\mu(y) \geq r$.
Now, $\mu(x-y) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\} \geq min\{t, r, \frac{\delta}{2}\}$

$$\Rightarrow \mu(x-y) \geq \begin{cases} min\{t, r\} & \text{if } t \leq \frac{\delta}{2} \text{ or } r \leq \frac{\delta}{2}; \\ \frac{\delta}{2} & t > \frac{\delta}{2} \text{ and } r > \frac{\delta}{2}. \end{cases}$$

$\Rightarrow (x-y)_{min\{t,r\}} \in \vee q_0^\delta \mu$.
and $\mu(xy) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\} \geq min\{t, r, \frac{\delta}{2}\}$

$$\Rightarrow \mu(xy) \geq \begin{cases} min\{t, r\} & \text{if } t \leq \frac{\delta}{2} \text{ or } r \leq \frac{\delta}{2}; \\ \frac{\delta}{2} & \text{if } t > \frac{\delta}{2} \text{ and } r > \frac{\delta}{2}. \end{cases}$$

$\Rightarrow (xy)_{min\{t,r\}} \in \vee q_0^\delta \mu$.
Therefore, μ is an $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring of N. □

Collorary 3.4. [7],[14] *A fuzzy set μ of N is an $(\in, \in \vee q)$-fuzzy subnear-ring of N if and only if $\mu(x-y), \mu(xy) \geq min\{\mu(x), \mu(y), 0.5\} \forall x, y \in N$.*

Definition 3.5. A fuzzy set μ in N is called an $(\in, \in \vee q_0^\delta)$-fuzzy ideal in N if,

i) it is an $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring of N,
ii) $x_t \in \mu, y \in N \Rightarrow (y + x - y)_t \in \vee q_0^\delta \mu$,
iii) $x_t \in \mu, y \in N \Rightarrow (xy)_t \in \vee q_0^\delta \mu$ and
iv) $a_t \in \mu, x, y \in \mu \Rightarrow (y(x+a) - yx)_t \in \vee q_0^\delta \mu$.

A fuzzy set with condition $i), ii), iii)$ is called an $(\in, \in \vee q_0^\delta)$-fuzzy right ideal of N and if it satisfies $i), ii), iv)$, then it is called an $(\in, \in \vee q_0^\delta)$-fuzzy left ideal of N.

Example 3.2 is also an example of $(\in, \in \vee q_0^\delta)$-fuzzy ideal for $\delta \in (0, 0.9]$ but not $(\in, \in \vee q_0^\delta)$-fuzzy ideal when $\delta \in (0.9, 1]$.
Note that every $(\in, \in \vee q_0^\delta)$-fuzzy ideal with $\delta = 1$ is an $(\in, \in \vee q)$-fuzzy ideal.
If $\delta_1 > \delta_2$ in (0,1], then every $(\in, \in \vee q_0^\delta)$-fuzzy ideal of N with $\delta = \delta_1$ is also an $(\in, \in \vee q_0^\delta)$-fuzzy ideal of N with $\delta = \delta_2$. But the converse is not true as seen in example 3.2.
So, every $(\in, \in \vee q)$-fuzzy ideal is an $(\in, \in \vee q_0^\delta)$-fuzzy ideal, but the converse is not true.

Example 3.6 Let $N = \{(a, b)|a, b \in Z\}$, where Z is the integers. Then $(N, +, \cdot)$ is a near-ring under the additive operation and multiplication operation defined as follows:
$(a, b)+(c, d) = (a+c, b+d)$ and $(a, b)\cdot(c, d) = (a, b)$ for all $(a, b), (c, d) \in N$.
Define a fuzzy set μ in N as

$$\mu(x) = \begin{cases} 0.88 & \text{if } x = (1, 8), \\ 0.44 & \text{if } x \in A, \\ 0.33 & \text{if } x \in B, \\ 0.22 & \text{otherwise.} \end{cases}$$

where $A = \{(a, 4b)|a, b \in Z\} \setminus \{(1, 8)\}$ and
$B = \{(a, 2b)|a, b \in Z\}\setminus\{(a, 4b)|a, b \in Z\}$. Then, μ is an $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring for all $\delta \in (0, 1]$. It is not an $(\in, \in \vee q_0^\delta)$-fuzzy ideal. since $(1, 8)_{0.45} \in \mu, (1, 2), (1, 3) \in N$ but
$((1, 2) \cdot ((1, 3) + (1, 8)) - (1, 2) \cdot (1, 3))_{0.45} = ((1, 2) \cdot (2, 11) - (1, 2))_{0.45}$
$= ((1, 2) - (1, 2))_{0.45} = (0, 0)_{0.45} \overline{\in \vee q \mu_0^\delta}$ when $\delta = 0.9$.

Theorem 3.7. *Let μ be a fuzzy set of a near-ring N. Then μ is an $(\in, \in \vee q_0^\delta)$-fuzzy ideal of N if and only if*

i) $\mu(x - y) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\}$
ii) $\mu(xy) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\}$
iii) $\mu(y + x - y) \geq min\{\mu(x), \frac{\delta}{2}\}$

iv) $\mu(xy) \geq min\{\mu(x), \frac{\delta}{2}\}$
v) $\mu(y(x+a) - yx) \geq min\{\mu(a), \frac{\delta}{2}\}$ for all $x, y, a \in N$

Proof. The proof is similar to the proof of theorem 3.3. □

Note: If μ is an $(\in, \in \vee q_0^\delta)$-fuzzy ideal of N then,
$\mu(x) = \mu(-y+y+x-y+y) \geq min\{\mu(y+x-y), \frac{\delta}{2}\}$ [by condition iii)]
$\Rightarrow \mu(x) \geq min\{\mu(y+x-y), \frac{\delta}{2}\}$ for all $x, y \in N$.

As discussed in [7], the properties of characteristic function of subset A of N is now replaced by the δ-characteristic function of A.

THEOREM 3.8. *A non-empty subset A of N is a subnear-ring(ideal) of N if and only if χ_A^δ is an $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring(ideal) of N.*

Proof. Let A be an ideal of N, and let $x, y \in N$, if $x, y \in A$ then $x-y, xy \in A$. Therefore, $\chi_A^\delta(x-y) = \delta > min\{\chi_A^\delta(x), \chi_A^\delta(y), \frac{\delta}{2}\}$ and $\chi_A^\delta(xy) = \delta > min\{\chi_A^\delta(x), \chi_A^\delta(y), \frac{\delta}{2}\}$. If at least one of $x, y \notin A$, then $\chi_A^\delta(x-y) \geq 0 = min\{\chi_A^\delta(x), \chi_A^\delta(y), \frac{\delta}{2}\}$
and $\chi_A^\delta(xy) \geq 0 = min\{\chi_A^\delta(x), \chi_A^\delta(y), \frac{\delta}{2}\}$
Let $x \in A$, then $y+x-y \in A$ and so $\chi_A(y+x-y) = \delta > min\{\chi_A^\delta(x), \frac{\delta}{2}\}$
and if $x \notin A$, then $\chi_A^\delta(y+x-y) \geq 0 = min\{\chi_A^\delta(x), \frac{\delta}{2}\}$
Let $x, u, v \in N$, if $x \in A$ then $xu, u(v+x) - uv \in A$. Therefore,
$\chi_A^\delta(xu) = \delta > min\{\chi_A^\delta(x), \frac{\delta}{2}\}$ and $\chi_A^\delta(u(v+x) - uv) = \delta > min\{\chi_A^\delta(x), \frac{\delta}{2}\}$
If $x \notin A$, then $\chi_A^\delta(xu) \geq 0 = min\{\chi_A^\delta(x), \frac{\delta}{2}\}$ and
$\chi_A^\delta(u(v+x) - uv) \geq 0 = min\{\chi_A^\delta(x), \frac{\delta}{2}\}$
Hence, χ_A^δ is an $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring(ideal) of N.

Conversely, Let χ_A^δ is an $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring(ideal) of N. Let $x, y \in A$, Now $\chi_A^\delta(x-y) \geq min\{\chi_A^\delta(x), \chi_A^\delta(y), \frac{\delta}{2}\} = min\{\delta, \frac{\delta}{2}\} = \frac{\delta}{2} \neq 0$
so, $x-y \in A$. Similarly, we can show that $u+x-u, xu, u(v+x)-uv \in A$ for all $x, y \in A$ and $u, v \in N$. Therefore, A is an ideal of N. □

The level sets have important aspects in respect to the connection of the fuzzy sets and crisp sets. As discussed in [5], the $(\in \vee q)$-level set $\bar{\mu}_t$ is a generalized level set of μ_t. It was found that μ_t is subnear-ring(ideal) if $t \in (0, 0.5)$ and $\bar{\mu}_t$ is subnear-ring(ideal) if $t \in (0, 1)$. Here we attempt to develope this kind of connections in regard to the level set $\bar{\mu}_t^\delta$ as well.

THEOREM 3.9. *A fuzzy set μ in N is an $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring(ideal) of N if and only if the $(\in \vee q_0^\delta)$-level subset $\bar{\mu}_t^\delta$ is a subnear-ring(ideal) of N for all $t \in (0, \delta]$ and $\delta \in (0, 1]$.*

Proof. Let μ be an $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring(ideal) of N and let $x, y \in \bar{\mu}_t^\delta$ for $t \in (0, \delta]$. Then, $x_t \in \vee q_0^\delta \mu$ or $y_t \in \vee q_0^\delta \mu$
that is, $\mu(x) \geq t$ or $\mu(x) + t > \delta$ and $\mu(y) \geq t$ or $\mu(y) + t > \delta$.
Since μ is an $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring(ideal) of N, we have
$\mu(x - y) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\}$.

Case 1. $\mu(x) \geq t$ and $\mu(y) \geq t$.
a) if $t > \frac{\delta}{2}$, then $\mu(x - y) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\} \geq min\{t, \frac{\delta}{2}\} = \frac{\delta}{2}$
thus, $\mu(x - y) + t > \delta \Rightarrow (x - y)_t q_0^\delta \mu$.
b) if $t \leq \frac{\delta}{2}$, then $\mu(x - y) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\} \geq min\{t, \frac{\delta}{2}\} = t$
$\Rightarrow (x - y)_t \in \mu$.

Case 2. Let $\mu(x) \geq t$ and $\mu(y) + t > \delta$ or $\mu(x) + t > \delta$ and $\mu(y) \geq t$.
a) if $t > \frac{\delta}{2}$, then $\mu(x-y) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\} > min\{t, \delta-t, \frac{\delta}{2}\} = \delta-t$.
$\Rightarrow \mu(x - y) + t > \delta \Rightarrow (x - y)_t q_0^\delta \mu$.
b) if $t \leq \frac{\delta}{2}$, then $\mu(x - y) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\} > min\{t, \delta - t, \frac{\delta}{2}\} = t$
$\Rightarrow (x - y)_t \in \mu$.

Case 3. $\mu(x) + t > \delta$ and $\mu(y) + t > \delta$.
a) if $t > \frac{\delta}{2}$, then $\mu(x - y) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\} > min\{\delta - t, \frac{\delta}{2}\} = \delta - t$
$\Rightarrow \mu(x - y) + t > \delta \Rightarrow (x - y)_t q_0^\delta \mu$.
b) if $t \leq \frac{\delta}{2}$, then $\mu(x - y) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\} > min\{\delta - t, \frac{\delta}{2}\} = \frac{\delta}{2} \geq t$
$\Rightarrow (x-y)_t \in \mu$. Thus, in all cases, we have $(x-y)_t \in \vee q_0^\delta \mu$. $\Rightarrow x-y \in \bar{\mu}_t^\delta$.

Similarly, we can show that $a + x - a, xa, a(b + x) - ab \in \bar{\mu}_t^\delta$ for all $a, b, x \in N$.
Thus, $\bar{\mu}_t^\delta$ is a subnear-ring(ideal) of N for all $t \in (0, \delta]$ and $\delta \in (0, 1]$.

Conversely, let $\bar{\mu}_t^\delta$ is an ideal of N.
Suppose, $\mu(x-y) < t \leq min\{\mu(x), \mu(y), \frac{\delta}{2}\}$, then $\mu(x) \geq t$ and $\mu(y) \geq t$
$\Rightarrow x_t \in \mu, y_t \in \mu \Rightarrow x, y \in \bar{\mu}_t^\delta \Rightarrow x - y \in \bar{\mu}_t^\delta$ [since $\bar{\mu}_t^\delta$ is an ideal],
which is a contradiction to $\mu(x - y) < t \leq \frac{\delta}{2}$
Hence, $\mu(x - y) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\}$.

Similarly, we can show that
$\mu(xy) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\}$.
$\mu(a + x - a) \geq min\{\mu(x), \frac{\delta}{2}\}$
$\mu(xy) \geq min\{\mu(x), \frac{\delta}{2}\}$

$\mu(a(b+x)-ab) \geq min\{\mu(x), \frac{\delta}{2}\}$ for all $a, b, x, y \in N$.
Hence, μ is an $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring(ideal) of N. □

COLLORARY 3.10. [11] *A fuzzy set μ in a group N is an $(\in, \in \vee q_0^\delta)$-fuzzy subgroup of N if and only if the $(\in \vee q_0^\delta)$-level subset $\bar{\mu}_t^\delta$ is a subgroup of N for all $t \in (0, \delta]$.*

COLLORARY 3.11. [5] *A fuzzy set μ in a group N is an $(\in, \in \vee q)$-fuzzy subgroup of N if and only if the $(\in \vee q)$-level subset $\bar{\mu}_t$ is a subgroup of N for all $t \in (0, 1]$.*

COLLORARY 3.12. [8],[12]. *A fuzzy set μ of N is an $(\in, \in \vee q)$-fuzzy subnear-ring(ideal) of N if and only if the $(\in \vee q)$-level subset $\bar{\mu}_t$ is a subnear-ring(ideal) of N for all $t \in (0, 1]$.*

THEOREM 3.13. *A fuzzy set μ in N is an $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring(ideal) of N if and only if the $(\in \vee q)$-level subset $\bar{\mu}_t$ is a subnear-ring(ideal) of N for all $t \in (0, \frac{\delta}{2}]$ and $\delta \in (0, 1]$.*

Proof. Assume that μ is an $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring(ideal) of N. Let $x, y \in \bar{\mu}_t$. Then, $x_t \in \vee q\mu$ or $y_t \in \vee q\mu$
that is, $\mu(x) \geq t$ or $\mu(x)+t > 1$ and $\mu(y) \geq t$ or $\mu(y)+t > 1$.
$\Rightarrow \mu(x) \geq t$ and $\mu(y) \geq t$ [since if $\mu(x) < t \leq \frac{\delta}{2} \leq 0.5 \Rightarrow \mu(x)+t < 1$ and $\mu(y) < t \leq \frac{\delta}{2} \leq 0.5 \Rightarrow \mu(y)+t < 1 \Rightarrow x, y \notin \bar{\mu}_t$, which is a contradiction].
Since μ is an $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring(ideal) of N, we have
$\mu(x-y) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\} \geq min\{t, \frac{\delta}{2}\} = t. \Rightarrow x-y \in \bar{\mu}_t$, and
$\mu(xy) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\} \geq min\{t, \frac{\delta}{2}\} = t, \Rightarrow xy \in \bar{\mu}_t$.
Therefore, $\bar{\mu}_t$ is a subnear-ring of N for all $t \in (0, \frac{\delta}{2}]$. Let $a, b \in N$. Then,
$\mu(a+x-a) \geq min\{\mu(x), \frac{\delta}{2}\} \geq min\{t, \frac{\delta}{2}\} = t$,
$\mu(xa) \geq min\{\mu(x), \frac{\delta}{2}\} \geq min\{t, \frac{\delta}{2}\} = t$ and
$\mu(a(b+x)-ab) \geq min\{\mu(x), \frac{\delta}{2}\} \geq min\{t, \frac{\delta}{2}\} = t$.
Therefore, $a+x-a, xa, a(b+x)-ab \in \bar{\mu}_t$ for all $a, b \in N$ and for all $x \in \bar{\mu}_t$.
Hence, $\bar{\mu}_t$ is an ideal of N for all $t \in (0, \frac{\delta}{2}]$.
Proof of the converse part is similar to theorem 3.9. □

THEOREM 3.14. *A fuzzy set μ in N is an $(\in, \in \vee q_0^{\delta})$-fuzzy subnear-ring(ideal) of N if and only if the set $\mu_t = \{x \in N | \mu(x) \geq t\}$ is a subnear-ring(ideal) of N for all $t \in (0, \frac{\delta}{2}]$ and $\delta \in (0, 1]$.*

Proof. It is similar to the proof of theorem 3.13. □

REMARK 3.15. The above theorem 3.14. may not be true if $t \in (\frac{\delta}{2}, 1]$. In the example 3.2., μ is an $(\in, \in \vee q_0^{\delta})$-fuzzy subnear-ring of N for $\delta \in (0, 0.9]$.
Take $\delta = 0.9$ and let $t = 0.46 \in (\frac{\delta}{2}, 1]$. Then $\mu_t = \{0, a, b\}$.
Now $a, b \in \mu_t$ but $a - b = c \notin \mu_t$. Therefore μ_t is not a subnear-ring of N.

COLLORARY 3.16. [11] *A fuzzy set μ of a group N is an $(\in, \in \vee q_0^{\delta})$-fuzzy subgroup of N if and only if the set $\mu_t = \{x \in N | \mu(x) \geq t\}$ is a subgroup of N for all $t \in (0, \frac{\delta}{2}]$.*

REMARK 3.17. [3],[14] A fuzzy set μ of a group N is an $(\in, \in \vee q)$-fuzzy subgroup of N if and only if the level subset $\mu_t = \{x \in N | \mu(x) \geq t\}$ is a subgroup of $N \forall t \in (0, 0.5]$. But the level set $\mu_t, t \in (0.5, 1]$ may not be a subgroup of N.

THEOREM 3.18. *Let A be a non-empty subset of N and μ_A be a fuzzy set in N defined by*

$$\mu_A(x) = \begin{cases} \frac{\delta}{2}, & \text{if } x \in A; \\ t, & \text{otherwise.} \end{cases}$$

for all $x \in N$ and $t < \frac{\delta}{2}$. Then μ_A is a $(\in, \in \vee q_0^{\delta})$-fuzzy ideal of N if and only if A is an ideal of N.

Proof. Let μ_A be an $(\in, \in \vee q_0^{\delta})$-fuzzy ideal of N and let $x, y \in A$, Then
$\mu_A(x - y) \geq min\{\mu_A(x), \mu_A(y), \frac{\delta}{2}\} = \frac{\delta}{2} \Rightarrow x - y \in A$
$\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y), \frac{\delta}{2}\} = \frac{\delta}{2} \Rightarrow xy \in A$.
Let $x \in A$, Now $\mu_A(y + x - y) \geq min\{\mu_A(x), \frac{\delta}{2}\} = \frac{\delta}{2}$ and
$\mu_A(xy) \geq min\{\mu_A(x), \frac{\delta}{2}\} = \frac{\delta}{2}$ for any $y \in N$. $\Rightarrow y + x - y, xy \in A$.
Let $x \in A$ and $u, v \in N$. Now, $\mu_A(u(v + x) - uv) \geq min\{\mu_A(x), \frac{\delta}{2}\} = \frac{\delta}{2}$
$\Rightarrow u(v + x) - uv \in A$. Therefore, A is an ideal of N.

Conversely, Let A is an ideal of N. If $x, y \in A$ then $x-y, xy \in A$ so, $\mu_A(x-y) = \frac{\delta}{2} = min\{\mu_A(x), \mu_A(y), \frac{\delta}{2}\}$
$\mu_A(xy) = \frac{\delta}{2} = min\{\mu_A(x), \mu_A(y), \frac{\delta}{2}\}$
If at least one of x and y does not belong to A, Then
$\mu_A(x-y) \geq t = min\{\mu_A(x), \mu_A(y), \frac{\delta}{2}\}$ and
$\mu_A(xy) \geq t = min\{\mu_A(x), \mu_A(y), \frac{\delta}{2}\}$.
Let $x \in A$ and $u, v \in N$ then $u+x-u, xu, u(v+x)-uv \in A$. so,
$\mu_A(u+x-u) = \frac{\delta}{2} = min\{\mu_A(x), \frac{\delta}{2}\}$
$\mu_A(xu) = \frac{\delta}{2} = min\{\mu_A(x), \frac{\delta}{2}\}$
and $\mu_A(u(v+x)-uv) = \frac{\delta}{2} = min\{\mu_A(x), \mu_A(y), \frac{\delta}{2}\}$.
If $x \notin A$, then $\mu_A(u+x-u) \geq t = min\{\mu_A(x), \frac{\delta}{2}\}$,
$\mu_A(xu) \geq t = min\{\mu_A(x), \frac{\delta}{2}\}$
and $\mu_A(u(v+x)-uv) \geq t = min\{\mu_A(x), \mu_A(y), \frac{\delta}{2}\}$
Hence, μ_A is an $(\in, \in \vee q_0^\delta)$-fuzzy ideal of N. □

Collorary 3.19.*Let A be a non-empty subset of N and μ_A be a fuzzy set in N defined by*

$$\mu_A(x) = \begin{cases} t, & \text{if } x \in A; \\ 0, & \text{otherwise.} \end{cases}$$

for all $x \in N$ with $t \in (0, \frac{\delta}{2}]$, Then μ_A is a $(\in, \in \vee q_0^\delta)$-fuzzy ideal of N if and only if A is an ideal of N.

Let $x \in N$ be such that $\mu(x) \geq \frac{\delta}{2}$, then
$\mu(0) = \mu(x-x) \geq min\{\mu(x), \mu(x), \frac{\delta}{2}\} = \frac{\delta}{2}$.
$\Rightarrow \mu(0) \geq \frac{\delta}{2}$. Again if $\mu(0) < \frac{\delta}{2}$, then $\mu(x) < \frac{\delta}{2} \forall x \in N$ then μ is fuzzy subgroup in the sense of Rosenfeld. In order to see a nontrivial generalization of fuzzy subgroup, we assume that $\mu_{\frac{\delta}{2}} \neq \{0\}$.
Henceforth, unless otherwise mentioned by $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring(ideal) of N, we shall mean an $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring(ideal) of N with $\mu_{\frac{\delta}{2}} \neq \{0\}$.

Lemma 3.20. *Let μ be an $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring of N. Let $x, y \in N$ be such that $\mu(x) < \mu(y)$, then*

- i) $\mu(x+y) = \mu(y+x) = \mu(x)$ if $\mu(x) < \frac{\delta}{2}$.
- ii) $\mu(xy), \mu(yx) \geq \frac{\delta}{2}$ if $\mu(x) \geq \frac{\delta}{2}$.

Proof. i) Let $x, y \in N$ be such that $\mu(x) < \mu(y)$ and $\mu(x) < \frac{\delta}{2}$.
Then, $\mu(x+y) = \mu(x-(-y)) \geq min\{\mu(x), \mu(-y), \frac{\delta}{2}\}$
$\geq min\{\mu(x), \mu(y), \frac{\delta}{2}\} = \mu(x) \Rightarrow \mu(x+y) \geq \mu(x)$.
and $\mu(x) = \mu(x+y-y) \geq min\{\mu(x+y), \mu(y), \frac{\delta}{2}\} = \mu(x+y)$[since it is given $\mu(x) < \mu(y)$ and $\mu(x) < \frac{\delta}{2}$].
$\Rightarrow \mu(x) \geq \mu(x+y)$. Therefore, $\mu(x+y) = \mu(x)$.

Similarly, we can show that $\mu(y+x) = \mu(x)$.
Hence, $\mu(x+y) = \mu(y+x) = \mu(x)$.
ii) Let $x, y \in N$ be such that $\mu(x) < \mu(y)$ and $\mu(x) \geq \frac{\delta}{2}$.
Then, $\mu(xy) \geq min\{\mu(x), \mu(y), \frac{\delta}{2}\} = \frac{\delta}{2}$
and $\mu(yx) \geq min\{\mu(y), \mu(x), \frac{\delta}{2}\} = \frac{\delta}{2}$.
Hence, $\mu(xy), \mu(yx) \geq \frac{\delta}{2}$ if $\mu(x) \geq \frac{\delta}{2}$.

□

Lemma 3.21. *Let μ be an $(\in, \in \vee q_0^\delta)$-fuzzy ideal of N. Then $\mu_a = \mu_b$ if and only if $\mu(a-b), \mu(b-a) \geq \frac{\delta}{2} \forall a, b \in N$.*

Proof. Suppose that $\mu(a-b), \mu(b-a) \geq \frac{\delta}{2}$.
Let $x \in N$, then $\mu_a(x) = min\{\mu(x-a), \frac{\delta}{2}\} = min\{\mu((x-b)-(a-b)), \frac{\delta}{2}\}$
$\geq min\{\mu(x-b), \mu(a-b), \frac{\delta}{2}\} \geq min\{\mu(x-b), \frac{\delta}{2}\} = \mu_b(x)$ for all $x \in N$.
$\Rightarrow \mu_a \geq \mu_b$. Similarly, we can show that $\mu_b \geq \mu_a$, thus $\mu_a = \mu_b$.

Conversely, suppose that $\mu_a = \mu_b$. Then $\mu_a(a) = \mu_b(a)$
$\Rightarrow min\{\mu(0), \frac{\delta}{2}\} = min\{\mu(a-b), \frac{\delta}{2}\}$
$\Rightarrow \frac{\delta}{2} = min\{\mu(a-b), \frac{\delta}{2}\} \Rightarrow \mu(a-b) \geq \frac{\delta}{2}$.
And $\mu_a(b) = \mu_b(b) \Rightarrow min\{\mu(b-a), \frac{\delta}{2}\} = min\{\mu(0), \frac{\delta}{2}\}$
$\Rightarrow min\{\mu(b-a), \frac{\delta}{2}\} = \frac{\delta}{2} \Rightarrow \mu(b-a) \geq \frac{\delta}{2}$. □

4. Quasi δ-fuzzy cosets

In this section, we introduce and discuss about quasi δ-fuzzy cosets of a $(\in, \in \vee q_0^\delta)$-fuzzy ideal in a near-ring N and prove fundamental theorem under isomorphism between two near-rings with respect to the structure induced by quasi δ-fuzzy cosets.

Definition 4.1. Let μ be an $(\in, \in \vee q_0^\delta)$-fuzzy ideal in N. Given $a \in N$, a fuzzy set μ_a in N defined by $\mu_a(x) = min\{\mu(x-a), \frac{\delta}{2}\}$ is called the $(\in, \in \vee q_0^\delta)$-fuzzy coset of μ in N determined by a and μ.

DEFINITION 4.2. Let μ be an $(\in, \in \vee q_0^\delta)$-fuzzy ideal of N and $N_\delta^\mu = \{\mu_a | a \in N\}$ is the set of all $(\in, \in \vee q_0^\delta)$-fuzzy cosets of μ in N. We provide two operations $\bigoplus$ and $\bigodot$ into N_δ^μ as follows
$\mu_x \bigoplus \mu_y = \mu_{x+y}$ and $\mu_x \bigodot \mu_y = \mu_{xy}$ for all $\mu_x, \mu_y \in N_\delta^\mu$
We first show that the compositions are well defined.
Let $a, b, x, y \in N$ be such that $\mu_a = \mu_x$ and $\mu_b = \mu_y$,
now, $\mu(a+b-y-x) = \mu(-x+a+b-y) = \mu((-x+a)-(y-b)$
$\geq min\{\mu(-x+a), \mu(y-b), \frac{\delta}{2}\} \geq min\{\mu(a-x), \mu(y-b), \frac{\delta}{2}\}$
$\geq \frac{\delta}{2}$. [By lemma 3.21.]
$\Rightarrow \mu((a+b)-(x+y)) \geq \frac{\delta}{2}$.
Therefore, by lemma 3.21., $\mu_{a+b} = \mu_{x+y}$. $\Rightarrow \mu_a \bigoplus \mu_b = \mu_x \bigoplus \mu_y$.
Again, $\mu(ab-xy) = \mu(ab-xb+xb-xy) = \mu((a-x)b-(xy-xb)$
$\geq min\{\mu((a-x)b), \mu(xy-xb), \frac{\delta}{2}\} \geq min\{\mu(a-x), \mu(x(b-b+y)-xb), \frac{\delta}{2}\}$
$\geq min\{\mu(a-x), \mu(-b+y), \frac{\delta}{2}\} \geq \frac{\delta}{2}$. [By lemma 3.21.]
Therefore, by lemma 3.21., $\mu_{ab} = \mu_{xy}$. $\Rightarrow \mu_a \bigodot \mu_b = \mu_x \bigodot \mu_y$.
Hence, the composition are well defined.

THEOREM 4.3. *For any $(\in, \in \vee q_0^\delta)$-fuzzy ideal μ of N, the set of all $(\in, \in \vee q_0^\delta)$-fuzzy cosets of μ in N i,e $N_\delta^\mu = \{\mu_a | a \in N\}$ is a near-ring under operation $\bigoplus$ and $\bigodot$.*

The Proof of Theorem 4.3 is straight foward.

For a fuzzy set μ in N, we define a fuzzy set $\bar{\mu}$ in N_δ^μ by
$\bar{\mu}(\mu_x) = \mu(x)$ for all $x \in N$.

THEOREM 4.4. *If μ is an $(\in, \in \vee q_0^\delta)$-fuzzy ideal of N, then $\bar{\mu}$ is an $(\in, \in \vee q_0^\delta)$-fuzzy ideal in N_δ^μ.*

Proof. Suppose μ is an $(\in, \in \vee q_0^\delta)$-fuzzy ideal of N. Let $a, b \in N$. Now,
$\bar{\mu}(\mu_a \ominus \mu_b) = \bar{\mu}(\mu_{a-b}) = \mu(a-b) \geq min\{\mu(a), \mu(b), \frac{\delta}{2}\} = min\{\bar{\mu}(\mu_a), \bar{\mu}(\mu_b), \frac{\delta}{2}\}$.
$\bar{\mu}(\mu_a \odot \mu_b) = \bar{\mu}(\mu_{ab}) = \mu(ab) \geq min\{\mu(a), \mu(b), \frac{\delta}{2}\} = min\{\bar{\mu}(\mu_a), \bar{\mu}(\mu_b), \frac{\delta}{2}\}$.
$\bar{\mu}(\mu_a \oplus \mu_b \ominus \mu_a) = \bar{\mu}(\mu_{a+b-a}) = \mu(a+b-a) \geq min\{\mu(b), \frac{\delta}{2}\} = min\{\bar{\mu}(\mu_b), \frac{\delta}{2}\}$.
$\bar{\mu}(\mu_a \odot \mu_b) = \bar{\mu}(\mu_{ab}) = \mu(ab) \geq min\{\mu(a), \frac{\delta}{2}\} = min\{\bar{\mu}(\mu_a), \frac{\delta}{2}\}$.
$\bar{\mu}(\mu_a \odot (\mu_b \oplus \mu_c) \ominus (\mu_a \odot \mu_b)) = \bar{\mu}(\mu_a \odot \mu_{b+c} \ominus \mu_{ab}) = \bar{\mu}(\mu_{a(b+c)} \ominus \mu_{ab}) =$

$\bar{\mu}(\mu_{a(b+c)-ab}) = \mu(a(b+c)-ab) \geq min\{\mu(c), \frac{\delta}{2}\} = min\{\bar{\mu}(\mu_c), \frac{\delta}{2}\}$.
Therefore, $\bar{\mu}$ is an $(\in, \in \vee q_0^{\delta})$-fuzzy ideal of N_{δ}^{μ}. □

Definition 4.5. Let N and N' be near-rings. A map $\theta : N \to N'$ is called a near-ring homomorphism if $\theta(x+y) = \theta(x) + \theta(y)$ and $\theta(xy) = \theta(x)\theta(y)$ for all $x, y \in N$.

Theorem 4.6. *If μ is an $(\in, \in \vee q_0^{\delta})$-fuzzy ideal of N, then the mapping $f : N \to N_{\delta}^{\mu}$ as $f(x) = \mu_x$ is a homomorphism with $kerf = \mu_{\frac{\delta}{2}}$.*

Proof. Let $x, y \in N$, now
$f(x+y) = \mu_{x+y} = \mu_x \oplus \mu_y = f(x) \oplus f(y)$ and
$f(xy) = \mu_{xy} = \mu_x \odot \mu_y = f(x) \odot f(y)$.
Therefore f is a homomorphism. And
$kerf = \{x \in N | f(x) = f(0)\} = \{x \in N | \mu_x = \mu_0\} = \{x \in N | \mu_x(x) = \mu_0(x)\}$
$= \{x \in N | min\{\mu(0), \frac{\delta}{2}\} = min\{\mu(x), \frac{\delta}{2}\}\} = \{x \in N | \frac{\delta}{2} = min\{\mu(x), \frac{\delta}{2}\}\}$
$= \{x \in N | \mu(x) \geq \frac{\delta}{2}\} = \mu_{\frac{\delta}{2}}$.
□

Thoerem 4.7. *For a near-ring homomorphism $f : N \to N'$, Let μ and ν be $(\in, \in \vee q_0^{\delta})$-fuzzy ideals of N and N' respectively. Then the mapping $\phi : N_{\delta}^{\mu} \to N_{\delta}'^{\nu}$ as $\phi(\mu_x) = \nu_{f(x)}$ for $x \in N$ is a homomorphism.*

Proof. Let $x, y \in N$, now
$\phi(\mu_x \oplus \mu_y) = \phi(\mu_{x+y}) = \nu_{f(x+y)} = \nu_{f(x)+f(y)} = \nu_{f(x)} \oplus \nu_{f(y)} = \phi(\mu_x) \oplus \phi(\mu_y)$ and $\phi(\mu_x \odot \mu_y) = \phi(\mu_{xy}) = \nu_{f(xy)} = \nu_{f(x)f(y)} = \nu_{f(x)} \odot \nu_{f(y)} = \phi(\mu_x) \odot \phi(\mu_y)$.
Therefore, ϕ is a homomorphism.
□

Theorem 4.8. *If μ is an $(\in, \in \vee q_0^{\delta})$-fuzzy subnear-ring(ideal) of N, then the fuzzy set $\nu : N \to [0, \delta]$ as $\nu(x) = \bar{\mu}(\mu_x)$ is an $(\in, \in \vee q_0^{\delta})$-fuzzy subnear-ring(ideal) of N.*

Proof. Let μ is an $(\in, \in \vee q_0^{\delta})$-fuzzy subnear-ring(ideal) of N,
Then by theorem 4.4., $\bar{\mu}$ is an $(\in, \in \vee q_0^{\delta})$-fuzzy subnear-ring(ideal) of N_{δ}^{μ}.
Let $x, y \in N$. now
$\nu(x-y) = \bar{\mu}(\mu_{x-y}) = \bar{\mu}(\mu_x \ominus \mu_y) \geq min\{\bar{\mu}(\mu_x), \bar{\mu}(\mu_y), \frac{\delta}{2}\} = min\{\nu(x), \nu(y), \frac{\delta}{2}\}$.
$\nu(xy) = \bar{\mu}(\mu_{xy}) = \bar{\mu}(\mu_x \odot \mu_y) \geq min\{\bar{\mu}(\mu_x), \bar{\mu}(\mu_y), \frac{\delta}{2}\} = min\{\nu(x), \nu(y), \frac{\delta}{2}\}$.

$\nu(y+x-y) = \bar{\mu}(\mu_{y+x-y}) = \bar{\mu}(\mu_y \oplus \mu_x \ominus \mu_y) \geq min\{\bar{\mu}(\mu_x), \frac{\delta}{2}\} = min\{\nu(x), \frac{\delta}{2}\}$.
$\nu(xy) = \bar{\mu}(\mu_{xy}) = \bar{\mu}(\mu_x \odot \mu_y) \geq min\{\bar{\mu}(\mu_x), \frac{\delta}{2}\} = min\{\nu(x), \frac{\delta}{2}\}$.
$\nu(y(x+a)-yx) = \bar{\mu}(\mu_{y(x+a)-yx}) = \bar{\mu}\{\mu_y \odot \mu_{(x+a)} \ominus \mu_{yx}\}$
$= \bar{\mu}\{\mu_y \odot (\mu_x \oplus \mu_a) - \mu_y \odot \mu_x\} \geq min\{\bar{\mu}(\mu_a), \frac{\delta}{2}\} = min\{\nu(a), \frac{\delta}{2}\}$.
Therefore, ν is an $(\in, \in \vee q_0^\delta)$-fuzzy subnear-ring(ideal) of N. □

Definition 4.9. [15] If μ is a fuzzy set in N and f is a function defined on N, then the fuzzy set ν in $f(N)$ defined by

$$\nu(y) = \sup_{x \in f^{-1}(y)} \mu(x)$$

for all $y \in f(N)$ is called the image of μ under f. Similarly, if ν is a fuzzy set in $f(N)$, then the fuzzy set $\mu = f \circ \nu$ in N (that is, the fuzzy set defined by $\mu(x) = \nu(f(x))$ for all $x \in N$ is called the preimgae of ν under f.
We say that a fuzzy set μ in N has the sup property if for any subset T of N, there exists $t_0 \in T$ such that

$$\mu(t_0) = \sup_{t \in T} \mu(t).$$

Theorem 4.10. *A near-ring homomorphic preimage of an $(\in, \in \vee q_0^\delta)$-fuzzy ideal is an $(\in, \in \vee q_0^\delta)$-fuzzy ideal.*

Proof. Let $\theta : N \to N'$ be a near-ring homomorphism.
Let ν be an $(\in, \in \vee q_0^\delta)$-fuzzy ideal of N' and μ be the preimage of ν under θ. Let $x, y, a \in N$. Now
$\mu(x-y) = \nu(\theta(x-y)) = \nu(\theta(x) - \theta(y)) \geq min\{\nu(\theta(x)), \nu(\theta(y)), \frac{\delta}{2}\}$
$= min\{\mu(x), \mu(y), \frac{\delta}{2}\}$
$\mu(xy) = \nu(\theta(xy)) = \nu(\theta(x)\theta(y)) \geq min\{\nu(\theta(x)), \nu(\theta(y)), \frac{\delta}{2}\}$
$= min\{\mu(x), \mu(y), \frac{\delta}{2}\}$
$\mu(y+x-y) = \nu(\theta(y+x-y)) = \nu(\theta(y)+\theta(x)-\theta(y)) \geq min\{\nu(\theta(x)), \frac{\delta}{2}\}$
$= min\{\mu(x), \frac{\delta}{2}\}$
$\mu(xy) = \nu(\theta(xy)) = \nu(\theta(x)\theta(y)) \geq min\{\nu(\theta(x)), \frac{\delta}{2}\} = min\{\mu(x), \frac{\delta}{2}\}$
$\mu(y(x+a)-yx) = \nu(\theta(y(x+a)-yx)) = \nu(\theta(y(x+a) - \theta(yx))$
$= \nu(\theta(y)(\theta(x)+\theta(a) - \theta(y)\theta(x)) \geq min\{\nu(\theta(a)), \frac{\delta}{2}\} = min\{\mu(a), \frac{\delta}{2}\}$
Therefore, μ is an $(\in, \in \vee q_0^\delta)$-fuzzy ideal. □

Theorem 4.11. *A near-ring homomorphic image of an $(\in, \in \vee q_0^\delta)$-fuzzy ideal having the sup property is an $(\in, \in \vee q_0^\delta)$-fuzzy ideal.*

Proof. Let $\theta : N \to N'$ be a near-ring homomorphism and μ be an $(\in, \in \vee q_0^\delta)$-fuzzy ideal of N having the sup property and ν be the image of μ under θ.
Let $\theta(x), \theta(y) \in \theta(N)$ and $x_0 \in \theta^{-1}(\theta(x)), y_0 \in \theta^{-1}(\theta(y))$ be such that

$$\mu(x_0) = \sup_{t\in\theta^{-1}(\theta(x))} \mu(t), \mu(y_0) = \sup_{t\in\theta^{-1}(\theta(y))} \mu(t)$$

respectively. Then,

$$\nu(\theta(x) - \theta(y)) = \sup_{t\in\theta^{-1}(\theta(x)-\theta(y))} \mu(t) \geq \mu(x_0 - y_0)[\text{by sup property}]$$

$$\geq min\{\mu(x_0), \mu(y_0), \frac{\delta}{2}\} = min\{\sup_{t\in\theta^{-1}(\theta(x))} \mu(t), \sup_{t\in\theta^{-1}(\theta(y))} \mu(t), \frac{\delta}{2}\}$$

$$= min\{\nu(\theta(x)), \nu(\theta(y)), \frac{\delta}{2}\}.$$

$$\nu(\theta(x)\theta(y)) = \sup_{t\in\theta^{-1}(\theta(x)\theta(y))} \mu(t) \geq \mu(x_0 y_0)$$

$$\geq min\{\mu(x_0), \mu(y_0), \frac{\delta}{2}\} = min\{\sup_{t\in\theta^{-1}(\theta(x))} \mu(t), \sup_{t\in\theta^{-1}(\theta(y))} \mu(t), \frac{\delta}{2}\}$$

$$= min\{\nu(\theta(x)), \nu(\theta(y)), \frac{\delta}{2}\}.$$

$$\nu(\theta(y) + \theta(x) - \theta(y)) = \sup_{t\in\theta^{-1}(\theta(y)+\theta(x)-\theta(y))} \mu(t) \geq \mu(y_0 + x_0 - y_0).$$

$$\geq min\{\mu(x_0), \frac{\delta}{2}\} = min\{\sup_{t\in\theta^{-1}(\theta(x))} \mu(t), \frac{\delta}{2}\} = min\{\nu(\theta(x)), \frac{\delta}{2}\}.$$

$$\nu(\theta(x)\theta(y)) = \sup_{t\in\theta^{-1}(\theta(x)\theta(y))} \mu(t) \geq \mu(x_0 y_0)$$

$$\geq min\{\mu(x_0), \frac{\delta}{2}\} = min\{\sup_{t\in\theta^{-1}(\theta(x))} \mu(t), \frac{\delta}{2}\} = min\{\nu(\theta(x)), \frac{\delta}{2}\}.$$

$$\text{and } \nu((\theta(x) + \theta(a))\theta(y) - \theta(x)\theta(y)) = \sup_{t\in\theta^{-1}((\theta(x)+\theta(a))\theta(y)-\theta(x)\theta(y)}$$

$$\geq \mu((x_0 + a_0)y_0 - x_0 y_0) \geq min\{\mu(a_0), \frac{\delta}{2}\} = min\{\sup_{t\in\theta^{-1}(\theta(a)} \mu(t), \frac{\delta}{2}\}$$

$$= min\{\nu(\theta(a)), \frac{\delta}{2}\}.$$

Therefore, ν is an $(\in, \in \vee q_0^\delta)$-fuzzy ideal.

□

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