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DOMAIN OF EULER-TOTIENT MATRIX OPERATOR IN THE SPACE $\mathcal{L}_p$

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Abstract. The most apparent aspect of the present study is to introduce a new sequence space $\Phi^\star(\mathcal{L}_p)$ derived by double Euler-Totient matrix operator. We examine its topological and algebraic properties and give an inclusion relation. In addition to those, the $\alpha-$, $\beta(bp)-$ and $\gamma-$duals of the space $\Phi^\star(\mathcal{L}_p)$ are determined and finally, some 4-dimensional matrix mapping classes related to this space are characterized.

1. Introduction, Definitions and Notations

Before presenting our main results, let us give all the necessary definitions and notions which are going to be used in the rest of this study. As it is well known, the set of all double sequences is symbolizes by Ω which is a vector space with coordinatewise addition and scalar multiplication. Any linear subspace of Ω is called as *double sequence space*. The set of all bounded complex valued double sequences is represented by $\mathcal{M}_u$, that is,

$$\mathcal{M}_u = \left\{ x = (x_{tu}) \in \Omega : \|x\|_\infty = \sup_{t,u\in\mathbb{N}} |x_{tu}| < \infty \right\},$$

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where $\mathbb{N} = \{1, 2, ...\}$. It should be noted that $\mathcal{M}_u$ is a Banach space with the norm $\|x\|_\infty$. We say that the double sequence $x = (x_{tu})$ is *convergent in the Pringsheim's sense* (or briefly *p-convergent*) provided that for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $|x_{tu} - L| < \varepsilon$ whenever $t, u > n_\varepsilon$. $L \in \mathbb{C}$ is called the *Pringsheim limit* of x and stated by $p - \lim_{t,u\to\infty} x_{tu} = L$; where $\mathbb{C}$ denotes the complex field. $\mathcal{C}_p$ represents the space of all convergent double sequences in the Pringsheim's sense. Of particular interest is unlike single sequences, *p-convergent* double sequences need not be bounded. For example, consider the sequence $x = (x_{tu})$ identified by

$$x_{tu} := \begin{cases} t & , \quad t \in \mathbb{N}, u = 1, \\ u & , \quad u \in \mathbb{N}, t = 1, \\ 0 & , \quad t, u \in \mathbb{N}. \end{cases}$$

Then, it can be easily seen that $p - \lim x_{tu} = 0$ but $\sup_{t,u\in\mathbb{N}} |x_{tu}| = \infty$. As a conclusion $x \in \mathcal{C}_p \backslash \mathcal{M}_u$. The bounded sequences which are also p-convergent are indicated by $\mathcal{C}_{bp}$, that is, $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$. A double sequence $x = (x_{tu}) \in \mathcal{C}_p$ is called as *regularly convergent* if the limits $x_t := \lim_u x_{tu}$, $(t \in \mathbb{N})$ and $x_u := \lim_t x_{tu}$, $(u \in \mathbb{N})$ exist, and the limits $\lim_t \lim_u x_{tu}$ and $\lim_u \lim_t x_{tu}$ exist and are equivalent to the $p - \lim$ of x. The space of all such double sequences is denoted by $\mathcal{C}_r$. Obviously, the regular convergence of a double sequence x implies the convergence in Pringsheim's sense as well as the boundedness of the terms of x, but the converse implication fails. A sequence $x = (x_{tu})$ is called *double null sequence* if it converges to zero. Additionally, all double null sequences in the spaces $\mathcal{C}_{bp}$ and $\mathcal{C}_r$ are represent by $\mathcal{C}_{bp0}$ and $\mathcal{C}_{r0}$, respectively. Móricz [17] showed that the spaces $\mathcal{C}_{bp}$, $\mathcal{C}_{bp0}$, $\mathcal{C}_r$ and $\mathcal{C}_{r0}$ are Banach spaces endowed with the norm $\|.\|_\infty$.

Let us take any $x \in \Omega$ and consider the sequence $K = (k_{rs})$ defined by

$$k_{rs} := \sum_{t=1}^{r} \sum_{u=1}^{s} x_{tu}, \quad (r, s \in \mathbb{N}).$$

Thus, the pair $((x_{rs}), (k_{rs}))$ and the sequence $K = (k_{rs})$ are entitled as *double series* and the *sequence of partial sums* of the double series, respectively.

Let Ψ be a space of double sequences, converging with respect to some linear convergence rule $v - \lim : \Psi \to \mathbb{C}$. The sum of a double series $\sum_{t,u} x_{tu}$ relating to this rule is defined by $v - \sum_{t,u} x_{tu} = v - \lim_{r,s\to\infty} k_{rs}$.

Here and thereafter, when needed we will use the summation $\sum_{t,u}$ instead of $\sum_{t=1}^{\infty}\sum_{u=1}^{\infty}$, assume that $v \in \{p, bp, r\}$ and p' denotes the conjugate of p, that is, $p' = p/(p-1)$ for $1 < p < \infty$.

The $\alpha-dual$ Ψ^{α}, $\beta(v)-dual$ $\Psi^{\beta(v)}$ with respect to the $v-$convergence and the $\gamma-dual$ Ψ^{γ} of a double sequence space Ψ are respectively described as

$$\begin{aligned}
\Psi^{\alpha} &:= \left\{c = (c_{tu}) \in \Omega : \sum_{t,u} |c_{tu}x_{tu}| < \infty \quad \text{for all} \quad (x_{tu}) \in \Psi\right\},\\
\Psi^{\beta(v)} &:= \left\{c = (c_{tu}) \in \Omega : v - \sum_{t,u} c_{tu}x_{tu} \quad \text{exists for all} \quad (x_{tu}) \in \Psi\right\},\\
\Psi^{\gamma} &:= \left\{c = (c_{tu}) \in \Omega : \sup_{r,s\in\mathbb{N}} \left|\sum_{t,u=0}^{r,s} c_{tu}x_{tu}\right| < \infty \quad \text{for all} \quad (x_{tu}) \in \Psi\right\}.
\end{aligned}$$

It can be easily seen that if $\Psi \subset \Lambda$, then $\Lambda^{\alpha} \subset \Psi^{\alpha}$ and $\Psi^{\alpha} \subset \Psi^{\gamma}$ for the double sequence spaces Ψ and Λ.

Now, we shall deal with matrix mapping. Let us consider double sequence spaces Ψ and Λ and the 4-dimensional complex infinite matrix $B = (b_{rstu})$. If for every $x = (x_{tu}) \in \Psi$, $Bx = \{(Bx)_{rs}\}_{t,u\in\mathbb{N}}$, the B-transform of x, is in Λ, where

$$(Bx)_{rs} = v - \sum_{t,u} b_{rstu}x_{tu}, \tag{1}$$

provided that the double series converges in the sense of v for each $r, s \in \mathbb{N}$, then it is said that B is a matrix mapping from Ψ into Λ and is written as $B : \Psi \to \Lambda$.

(Ψ, Λ) stands for the class of all 4-dimensional complex infinite matrices from a double sequence space Ψ into a double sequence space Λ. Then, $B \in (\Psi, \Lambda)$ if and only if $B_{rs} \in \Psi^{\beta(v)}$, where $B_{rs} = (b_{rstu})_{t,u\in\mathbb{N}}$ for all $r, s \in \mathbb{N}$.

The v-summability domain $\Psi_B^{(v)}$ of a 4-dimensional complex infinite matrix B in a double sequence space Ψ consists of whose B-transforms are in Ψ; that is,

$$\Psi_B^{(v)} := \left\{x = (x_{tu}) \in \Omega : Bx := \left(v - \sum_{tu} b_{rstu}x_{tu}\right)_{r,s\in\mathbb{N}} \text{exists and is in } \Psi\right\}.$$

In the rest of the article, we will only interest in bp-summability domain.

If $B = (b_{rstu}) = 0$ for $t > r$ or $u > s$ or both, then any 4-dimensional matrix is called as *triangular*, [1]. Moreover, a triangular matrix $B = (b_{rstu})$ is called as *triangle* if $b_{rsrs} \neq 0$ for every $r, s \in \mathbb{N}$. It should be noted by [6] that, every triangle matrix has a unique inverse which is also a triangle.

In the past, double sequence spaces have been studied by many authors. Now, let us give some information about these studies. In her doctoral dissertation, Zeltser [31] has fundamentally examined both the topological structure and the theory of summability of double sequences. Recently, Altay and Başar [2] defined the double sequence spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double series whose sequences of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those spaces. Later, in [4], Başar and Sever have defined the Banach space $\mathcal{L}_p$ by

$$\mathcal{L}_p := \left\{ (x_{tu}) \in \Omega : \sum_{t,u} |x_{tu}|^p < \infty \right\}, \quad (1 \leq p < \infty)$$

with the norm $\|.\|_{\mathcal{L}_p}$, which is defined in the following way:

$$\|.\|_{\mathcal{L}_p} = \left(\sum_{t,u} |x_{tu}|^p \right)^{\frac{1}{p}}.$$

It is also significant to say that the double sequence space $\mathcal{L}_u$ which was defined by Zeltser [32] is the special case of the space $\mathcal{L}_p$ for $p = 1$. For more details about the double sequences and related topics, the reader may refer to Altay and Başar [2], Başar [3], Başar and Sever [4], Çapan and Başar [7], Demiriz and Duyar [8], Mursaleen [18], Talebi [27], Tuğ [28] and Yeşilkayagil and Başar [30].

Throughout the article, φ and μ denote the Euler function and the Möbius function, respectively. For every $r \in \mathbb{N}$ with $r > 1$, $\varphi(r)$ is the number of positive integers less than r which are coprime with r and $\varphi(1) = 1$. If ${a_1}^{b_1}{a_2}^{b_2}{a_3}^{b_3}...{a_m}^{b_m}$ is the prime factorization of a naturel number $r > 1$, then

$$\varphi(r) = r(1 - \frac{1}{a_1})(1 - \frac{1}{a_2})(1 - \frac{1}{a_3})...(1 - \frac{1}{a_m}).$$

Also, the equality

$$r = \sum_{t|r} \varphi(t)$$

holds for every $r \in \mathbb{N}$ and $\varphi(r_1 r_2) = \varphi(r_1)\varphi(r_2)$, where $r_1, r_2 \in \mathbb{N}$ are coprime. Given any $r \in \mathbb{N}$ with $r > 1$, μ is defined as

$$\mu(r) := \begin{cases} (-1)^m & , \quad \text{if } r = a_1 a_2 ... a_m, \text{ where } a_1 a_2 ... a_m \text{ are} \\ & \quad \text{non-equivalent prime numbers} \\ 0 & , \quad \text{if } a^2 \mid r \text{ for some prime number} \quad a, \end{cases}$$

and $\mu(1) = 1$. If ${a_1}^{b_1} {a_2}^{b_2} {a_3}^{b_3} ... {a_m}^{b_m}$ is the prime factorization of a naturel number $r > 1$, in this fact,

$$\sum_{t|r} t\mu(t) = (1 - a_1)(1 - a_2)(1 - a_3)...(1 - a_m).$$

Also, the equality

$$\sum_{t|r} \mu(t) = 0$$

holds except for $r = 1$ and $\mu(r_1 r_2) = \mu(r_1)\mu(r_2)$, where $r_1, r_2 \in \mathbb{N}$ are coprime.

The 2-dimensional Euler-Totient matrix $\Phi = (\phi_{rt})$ which is regular is defined as following way:

$$\phi_{rt} := \begin{cases} \dfrac{\varphi(t)}{r} & , \quad \text{if} \quad t \mid r, \\ \\ 0 & , \quad \text{if} \quad t \nmid r. \end{cases}$$

By using the 2-dimensional Euler-Totient matrix Φ, the Euler-Totient sequence spaces $\ell_p(\Phi)$ and $\ell_\infty(\Phi)$ which consist of all sequences whose Φ-transforms are in the spaces ℓ_p of absolutely p-summable and ℓ_∞ of bounded single sequences are introduced by İlkhan and Kara. For more details on the spaces $\ell_p(\Phi)$ and $\ell_\infty(\Phi)$, the reader may refer to [10].

The general frame of the rest of the study can be given as follows: In the second section, at the beginning the double Euler-Totient matrix $\Phi^\star$ is defined and the double Euler-Totient sequence space $\Phi^\star(\mathcal{L}_p)$ is introduced by using this matrix. Also, some of the algebraic and topological properties of this space are examined and a inclusion relation is given. Nextly, in the third section, we determine the $\alpha-$, $\beta(bp)-$ and $\gamma-$duals of the space $\Phi^\star(\mathcal{L}_p)$. Finally, in the last section, some matrix classes on this new space are characterized.

2. The double Euler-Totient sequence space

In this section, we introduce the double sequence space $\Phi^\star(\mathcal{L}_p)$ by using the 4-dimensional Euler-Totient matrix $\Phi^\star$ and give some properties and results on this new space.

We define the 4-dimensional Euler-Totient matrix $\Phi^\star = (\phi^\star_{rstu})$ which is a triangle as follows:

$$\phi^\star_{rstu} := \begin{cases} \dfrac{\varphi(t)\varphi(u)}{rs} & , \quad t \mid r \ , \ u \mid s, \\ \\ 0 & , \quad \text{otherwise}, \end{cases} \tag{2}$$

for every $r, s, t, u \in \mathbb{N}$. Thus, by keeping in mind the 4-dimensional Euler-Totient matrix, the $\Phi^\star$-transform of a double sequence $x = (x_{rs})$ is given by

$$y_{rs} := \{\Phi^\star x\}_{rs} = \frac{1}{rs} \sum_{t|r,u|s} \varphi(t)\varphi(u)x_{tu}, \tag{3}$$

for every $r, s \in \mathbb{N}$. Throughout the article, we will suppose that the terms of the double sequences $x = (x_{rs})$ and $y = (y_{rs})$ are connected with the relation (3).

The inverse $\{\Phi^\star\}^{-1} = (\phi^{\star -1}_{rstu})$ of the triangle matrix $\Phi^\star$ can be found by a simple computation and is described as

$$\phi^{\star -1}_{rstu} := \begin{cases} \dfrac{\mu(\frac{r}{t})\mu(\frac{s}{u})}{\varphi(r)\varphi(s)} tu & , \quad t \mid r \ , \ u \mid s, \\ \\ 0 & , \quad \text{otherwise}, \end{cases}$$

for every $r, s, t, u \in \mathbb{N}$. We introduce the sequence space $\Phi^\star(\mathcal{L}_p)$ as the set of all double sequences such that $\Phi^\star$-transforms of them are in the space $\mathcal{L}_p$, that is,

$$\Phi^\star(\mathcal{L}_p) = \left\{ x = (x_{rs}) \in \Omega : \sum_{r,s} \left| \frac{1}{rs} \sum_{t|r,u|s} \varphi(t)\varphi(u)x_{tu} \right|^p < \infty \right\}$$

for $0 < p < \infty$. $\Phi^\star(\mathcal{L}_p)$ can be rewritten as $\Phi^\star(\mathcal{L}_p) = \{\mathcal{L}_p\}_{\Phi^\star}$ with the notation of (1). If Ψ is any normed double sequence space, then we call the matrix domain $\Psi_{\Phi^\star}$ as the double Euler-Totient sequence space.

DEFINITION 2.1. [9], [23] A 4-dimensional matrix B is said to be RH-regular if it maps every bounded p-convergent sequence into a p-convergent sequence with the same p-limit.

LEMMA 2.2. *[9], [23] A 4-dimensional triangle matrix $B = (b_{rstu})$ is RH-regular if and only if*

$$RH_1 \quad : \quad p - \lim_{r,s\to\infty} b_{rstu} = 0 \quad \textit{for each} \quad t, u \in \mathbb{N},$$

$$RH_2 \quad : \quad p - \lim_{r,s\to\infty} \sum_{t,u} b_{rstu} = 1,$$

$$RH_3 \quad : \quad p - \lim_{r,s\to\infty} \sum_{t} |b_{rstu}| = 0 \quad \textit{for each} \quad u \in \mathbb{N},$$

$$RH_4 \quad : \quad p - \lim_{r,s\to\infty} \sum_{u} |b_{rstu}| = 0 \quad \textit{for each} \quad t \in \mathbb{N},$$

RH_5 : *There exists finite positive integers* M_1 *and* M_2 *such that*

$$\sum_{t,u>M_1} |b_{rstu}| < M_2.$$

THEOREM 2.3. *The 4-dimensional Euler-Totient matrix* $\Phi^\star$ *described by (2) is RH-regular.*

Proof. Since $\frac{\varphi(t)}{r} \to 0$, as $r \to \infty$ and $\frac{\varphi(u)}{s} \to 0$, as $s \to \infty$ from the regularity of the 2-dimensional Euler-Totient matrix, then it can be easily seen that $\phi^\star_{rstu} \to 0$, as $r, s \to \infty$ for each $t, u \in \mathbb{N}$, that is, RH_1 satisfies. By taking into account the equality

$$\sum_{t,u} \phi^\star_{rstu} = \sum_{t|r,u|s} \phi^\star_{rstu} = 1, \tag{4}$$

we obtain that $p-\lim_{r,s\to\infty} \sum_{t,u} \phi^\star_{rstu} = 1$. So, the condition RH_2 holds. We deduce from the equation $\sum_t |\phi^\star_{rstu}| = \frac{\varphi(u)}{s}$ that RH_3 satisfies. With the similar way, the condition RH_4 holds. Using the relation (4) and the positivity of the 4-dimensional matrix $\Phi^\star$, i.e., $\phi^\star_{rstu} \geq 0$ for every $r, s, t, u \in \mathbb{N}$, it is clear that the condition RH_5 satisfies. This step concludes the proof. □

Now, we may continue with the following theorem which is the essential in the text.

Theorem 2.4. *The set $\Phi^{\star}(\mathcal{L}_p)$ becomes a linear space with coordinatewise addition and scalar multiplication for double sequences and the following statements hold:*

(i): *If $0 < p < 1$, then $\Phi^{\star}(\mathcal{L}_p)$ is a complete $p-$normed space with*

$$\|x\|^{\wr}_{\Phi^{\star}(\mathcal{L}_p)} = \|\Phi^{\star}x\|^{\wr}_{\mathcal{L}_p} = \sum_{r,s}\left|\frac{1}{rs}\sum_{t|r,u|s}\varphi(t)\varphi(u)x_{tu}\right|^{p}, \tag{5}$$

which is $p-$norm isomorphic to the space $\mathcal{L}_p$.

(ii): *If $1 \leq p < \infty$, then $\Phi^{\star}(\mathcal{L}_p)$ is a Banach space with*

$$\|x\|_{\Phi^{\star}(\mathcal{L}_p)} = \|\Phi^{\star}x\|_{\mathcal{L}_p} = \left(\sum_{r,s}\left|\frac{1}{rs}\sum_{t|r,u|s}\varphi(t)\varphi(u)x_{tu}\right|^{p}\right)^{\frac{1}{p}}, \tag{6}$$

which is norm isomorphic to the space $\mathcal{L}_p$.

Proof. Part (i) can be proved in the similar way, therefore we give the proof only for Part (ii)

Since the initial assertion is routine verification and is easy to prove, we ignore its proof in here. To confirm the fact that $\Phi^{\star}(\mathcal{L}_p)$ is norm isomorphic to the space $\mathcal{L}_p$, we need to be sure the existence of a linear and norm preserving bijection between the spaces $\Phi^{\star}(\mathcal{L}_p)$ and $\mathcal{L}_p$ for $1 \leq p < \infty$. For this purpose, let us take the transformation B defined from $\Phi^{\star}(\mathcal{L}_p)$ into $\mathcal{L}_p$ by $x \mapsto y = Bx$, where $y = (y_{rs})$ is the $\Phi^{\star}$-transform of the sequence $x = (x_{tu})$. The linearity of B is clear. Consider the equality

$$Bx = \begin{bmatrix} x_{11} & \frac{x_{11}+x_{12}}{2} & \cdots & \frac{\sum_{u|s}\varphi(u)x_{1u}}{s} & \cdots \\ \frac{x_{11}+x_{21}}{2} & \frac{x_{11}+x_{12}+x_{21}+x_{22}}{4} & \cdots & \frac{\sum_{u|s}\varphi(u)(x_{1u}+x_{2u})}{2s} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\sum_{t|r}\varphi(t)x_{t1}}{r} & \frac{\sum_{t|r}\varphi(t)(x_{t1}+x_{t2})}{2r} & \cdots & \frac{\sum_{t|r,u|s}\varphi(t)\varphi(u)x_{tu}}{rs} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} = \theta$$

which yields us that $x_{tu} = 0$ for every $t, u \in \mathbb{N}$. So, $x = \theta$. Therefore, B is injective. Let us consider $y \in \mathcal{L}_p$ for $1 \leq p < \infty$ and describe the double sequence $x = (x_{rs})$ by

$$x_{rs} = \sum_{t|r,u|s} \frac{\mu(\frac{r}{t})\mu(\frac{s}{u})}{\varphi(r)\varphi(s)} tuy_{tu} \tag{7}$$

for every $r, s \in \mathbb{N}$. In that case, for $1 \leq p < \infty$, it is seen that

$$\begin{aligned}
\|x\|_{\Phi^\star(\mathcal{L}_p)} &= \|\Phi^\star x\|_{\mathcal{L}_p} \\
&= \left(\sum_{r,s} \left| \frac{1}{rs} \sum_{t|r,s|u} \varphi(t)\varphi(u)x_{tu} \right|^p \right)^{1/p} \\
&= \left(\sum_{r,s} \left| \frac{1}{rs} \sum_{t|r,s|u} \varphi(t)\varphi(u) \sum_{m|t,n|u} \frac{\mu(\frac{t}{m})\mu(\frac{u}{n})}{\varphi(t)\varphi(u)} mny_{mn} \right|^p \right)^{1/p} \\
&= \left(\sum_{r,s} |y_{rs}|^p \right)^{1/p} = \|y\|_{\mathcal{L}_p} < \infty.
\end{aligned}$$

Thus, we have that $x \in \Phi^\star(\mathcal{L}_p)$ for $1 \leq p < \infty$ and consequently B is surjective and norm preserving. Hence, B is a linear and norm preserving bijection which means that the spaces $\Phi^\star(\mathcal{L}_p)$ and $\mathcal{L}_p$ are norm isomorphic, as desired.

Now, let us prove that $\Phi^\star(\mathcal{L}_p)$ is a Banach space with the norm $\|.\|_{\Phi^\star(\mathcal{L}_p)}$ described by (6). To do this, it can be used Section (b) of Corollary 6.3.41 in [5] which says that "Let $(\Psi, \digamma_1)$ and $(\Lambda, \digamma_2)$ be seminormed spaces and $B : (\Psi, \digamma_1) \to (\Lambda, \digamma_2)$ be an isometric isomorphism. Then, $(\Psi, \digamma_1)$ is complete if and only if $(\Lambda, \digamma_2)$ is complete. In particular, $(\Psi, \digamma_1)$ is a Banach space if and only if $(\Lambda, \digamma_2)$ is a Banach space." Since the map B described in the proof of this theorem from $\Phi^\star(\mathcal{L}_p)$ to $\mathcal{L}_p$ is an isometric isomorphism and the double sequence space $\mathcal{L}_p$ is a Banach space from Theorem 2.1 in [4], it can be obviously seen that the space $\Phi^\star(\mathcal{L}_p)$ is Banach space. In fact, this is exactly what we want to prove. □

Let us remember the definition of barelled space in [5] which says that "Let Ψ be a locally convex space. Then, a subset of Ψ is called barrel if it is absolutely convex, absorbing and closed in Ψ. Moreover, Ψ is called a barrelled space if each barrel is a neighborhood of zero."

Lemma 2.5. *[24] If the sequence space Ψ is a Banach space or a Fréchet space, then it is a barelled space.*

Theorem 2.6. *The double sequence space $\Phi^\star(\mathcal{L}_p)$ is a barelled space for $1 \leq p < \infty$.*

Proof. The proof is obvious while keeping in mind Theorem 2.4 and Lemma 2.5. $\square$

With the notation Zeltzer [31], we define the double sequence $e^{tu} = (e^{tu}_{rs})$ by

$$e^{tu}_{rs} = \begin{cases} 1 \quad , & (r,s) = (t,u), \\ \\ 0 \quad , & \text{otherwise}, \end{cases}$$

for every $r, s, t, u \in \mathbb{N}$. "A non-empty subset Λ of a locally convex space Ψ is called fundamental if the closure of the linear span of Λ equals Ψ, [5]." From previous description, Yeşilkayagil and Başar [29] have showed that E is the fundamental set of $\mathcal{L}_p$, where $E := \{e^{tu} : t, u \in \mathbb{N}\}$. In the light of this fact, let us describe the double sequences $f^{tu} = (f^{tu}_{rs})$ by

$$f^{tu}_{rs} := \begin{cases} \dfrac{\mu(\frac{r}{t})\mu(\frac{s}{u})}{\varphi(r)\varphi(s)} tu \quad , & t \mid r \ , \ u \mid s, \\ \\ \qquad 0 \qquad\quad , & \text{otherwise}, \end{cases}$$

for every $r, s, t, u \in \mathbb{N}$, Thus, $\{f^{tu} : t, u \in \mathbb{N}\}$ is the fundamental set of the space $\Phi^\star(\mathcal{L}_p)$; because $\Phi^\star f^{tu} = e^{tu}$.

Now, we may continue the following theorem about inclusion relation:

Theorem 2.7. *Let $1 \leq p < p_1 < \infty$. Then the inclusion $\Phi^\star(\mathcal{L}_p) \subset \Phi^\star(\mathcal{L}_{p_1})$ holds.*

Proof. Suppose that $x = (x_{tu}) \in \Phi^\star(\mathcal{L}_p)$ is an arbitrary double sequence. Then, $\Phi^\star x \in \mathcal{L}_p$. Since the inclusion $\mathcal{L}_p \subset \mathcal{L}_{p_1}$ for $1 \leq p < p_1 < \infty$ from Başar and Sever [4], it is concluded that $\Phi^\star x \in \mathcal{L}_{p_1}$. Hence $x \in \Phi^\star(\mathcal{L}_{p_1})$, as desired. $\square$

3. The $\alpha-$, $\beta(bp)-$ and $\gamma-$duals of the space $\Phi^\star(\mathcal{L}_p)$

In the present section, we will determine the $\alpha-$, $\beta(bp)-$ and $\gamma-$duals of the space $\Phi^\star(\mathcal{L}_p)$. For this purpose, firstly we need to give a lemma.

LEMMA 3.1. *[29] Suppose that $B = (b_{rstu})$ be a 4-dimensional infinite matrix. At that time, the following statements hold:*

(i) *Assume that $0 < p \leq 1$. In that case, $B \in (\mathcal{L}_p : \mathcal{M}_u)$ iff*

$$M_3 = \sup_{r,s,t,u\in\mathbb{N}} |b_{rstu}| < \infty. \tag{8}$$

(ii) *Assume that $1 < p < \infty$. In that case, $B \in (\mathcal{L}_p : \mathcal{M}_u)$ iff*

$$M_4 = \sup_{r,s\in\mathbb{N}} \sum_{t,u} |b_{rstu}|^{p'} < \infty. \tag{9}$$

(iii) *Assume that $0 < p \leq 1$ and $1 \leq p_1 < \infty$. In that case, $B \in (\mathcal{L}_p : \mathcal{L}_{p_1})$ iff*

$$\sup_{t,u\in\mathbb{N}} \sum_{r,s} |b_{rstu}|^{p_1} < \infty. \tag{10}$$

(iv) *Assume that $0 < p \leq 1$. In that case, $B \in (\mathcal{L}_p : \mathcal{C}_{bp})$ iff the condition (8) holds and there exists a $(\alpha_{tu}) \in \Omega$ such that*

$$bp - \lim_{r,s\to\infty} b_{rstu} = \alpha_{tu}. \tag{11}$$

(v) *Assume that $1 < p < \infty$. In that case, $B \in (\mathcal{L}_p : \mathcal{C}_{bp})$ iff (9) and (11) hold.*

THEOREM 3.2. *Consider the set w_1 defined by*

$$w_1 = \left\{ c = (c_{rs}) \in \Omega : \sup_{t,u\in\mathbb{N}} \sum_{r\in\mathbb{N},t|r} \sum_{s\in\mathbb{N},u|s} \left| \frac{\mu(\frac{r}{t})\mu(\frac{s}{u})}{\varphi(r)\varphi(s)} tuc_{rs} \right| < \infty \right\}.$$

Then, $(\Phi^\star(\mathcal{L}_p))^\alpha = w_1$ for $0 < p \leq 1$.

Proof. Consider the 4-dimensional matrix $G = (g_{rstu})$ defined by

$$g_{rstu} := \begin{cases} \dfrac{\mu(\frac{r}{t})\mu(\frac{s}{u})}{\varphi(r)\varphi(s)} tuc_{rs} & , \quad t \mid r \quad , \quad u \mid s, \\ \\ 0 & , \quad \text{otherwise,} \end{cases}$$

for every $r, s, t, u \in \mathbb{N}$. In that case, by using the relation (7) we obtain that

$$\begin{aligned} c_{rs}x_{rs} &= c_{rs}\sum_{t|r,u|s}\frac{\mu(\frac{r}{t})\mu(\frac{s}{u})}{\varphi(r)\varphi(s)}tuy_{tu} \\ &= \sum_{t|r,u|s}\left\{\frac{\mu(\frac{r}{t})\mu(\frac{s}{u})}{\varphi(r)\varphi(s)}tuc_{rs}\right\}y_{tu} \\ &= (Gy)_{rs} \end{aligned} \tag{12}$$

for every $r, s \in \mathbb{N}$. In this fact, we conclude from relation (12) that $cx = (c_{rs}x_{rs}) \in \mathcal{L}_u$ whenever $x \in \Phi^\star(\mathcal{L}_p)$ iff $Gy \in \mathcal{L}_u$ whenever $y \in \mathcal{L}_p$. This means that $c = (c_{rs}) \in (\Phi^\star(\mathcal{L}_p))^\alpha$ iff $G \in (\mathcal{L}_p : \mathcal{L}_u)$. Then, we derive by using part (iii) of Lemma 3.1 with $p_1 = 1$ that

$$\sup_{t,u\in\mathbb{N}}\sum_{r\in\mathbb{N},t|r}\sum_{s\in\mathbb{N},u|s}\left|\frac{\mu(\frac{r}{t})\mu(\frac{s}{u})}{\varphi(r)\varphi(s)}tuc_{rs}\right| < \infty.$$

This yields the desired consequence that $(\Phi^\star(\mathcal{L}_p))^\alpha = w_1$ for $0 < p \leq 1$. □

Theorem 3.3. *Consider the sets w_2, w_3 and w_4 defined by*

$$\begin{aligned} w_2 &= \left\{c = (c_{rs}) \in \Omega : \sup_{r,s,t,u\in\mathbb{N}}\left|\sum_{m=t,t|m}^{r}\sum_{n=u,u|n}^{s}\frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(m)\varphi(n)}tuc_{mn}\right| < \infty\right\}, \\ w_3 &= \left\{c = (c_{rs}) \in \Omega : bp-\lim_{r,s\to\infty}\sum_{m=t,t|m}^{r}\sum_{n=u,u|n}^{s}\frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(m)\varphi(n)}tuc_{mn} \quad \textit{exists}\right\}, \\ w_4 &= \left\{c = (c_{rs}) \in \Omega : \sup_{r,s\in\mathbb{N}}\sum_{t,u}\left|\sum_{m=t,t|m}^{r}\sum_{n=u,u|n}^{s}\frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(m)\varphi(n)}tuc_{mn}\right|^{p'} < \infty\right\}. \end{aligned}$$

In that case, following statements are satisfied:

(i) *Assume that $0 < p \leq 1$. In that case, $(\Phi^\star(\mathcal{L}_p))^{\beta(bp)} = w_2 \cap w_3$,*

(ii) *Assume that $1 < p < \infty$. In that case, $(\Phi^\star(\mathcal{L}_p))^{\beta(bp)} = w_3 \cap w_4$.*

Proof. Let $c = (c_{rs}) \in \Omega$ and $x \in \Phi^\star(\mathcal{L}_p)$ be given. Then, we can conclude from Theorem 2.4 that there exists a double sequence $y =$

$(y_{rs}) \in \mathcal{L}_p$. Define the 4-dimensional matrix $A = (A_{rstu})$ by

$$a_{rstu} := \begin{cases} \displaystyle\sum_{m=t,t|m}^{r} \sum_{n=u,u|n}^{s} \frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(m)\varphi(n)} tuc_{mn} & , \quad t \mid m, u \mid n, \\ 0 & , \quad \text{otherwise}, \end{cases}$$

for every $r, s, t, u \in \mathbb{N}$. Therefore, we obtain by the relation (7) that,

$$\begin{aligned} z_{rs} &= \sum_{t|r,u|s} c_{tu} x_{tu} \\ &= \sum_{t|r,u|s} c_{tu} \left\{ \sum_{m|t} \sum_{n|u} \frac{\mu(\frac{t}{m})\mu(\frac{u}{n})}{\varphi(t)\varphi(u)} mny_{mn} \right\} \\ &= \sum_{t|r,u|s} \left\{ \sum_{m=t,t|m}^{r} \sum_{n=u,u|n}^{s} \frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(m)\varphi(n)} tuc_{mn} \right\} y_{tu} \\ &= (Ay)_{rs} \end{aligned}$$

for every $r, s \in \mathbb{N}$. Then by considering the equality above, we deduce that $cx = (c_{rs}x_{rs}) \in \mathcal{CS}_{bp}$ whenever $x = (x_{rs}) \in \Phi^\star(\mathcal{L}_p)$ iff $z = (z_{rs}) \in \mathcal{C}_{bp}$ whenever $y = (y_{rs}) \in \mathcal{L}_p$. This leads us to the fact that $c = (c_{rs}) \in (\Phi^\star(\mathcal{L}_p))^{\beta(bp)}$ iff $A \in (\mathcal{L}_p : \mathcal{C}_{bp})$. Hence;

(i) If $0 < p \leq 1$, then from Part (iv) of Lemma 3.1, we achieve that $(\Phi^\star(\mathcal{L}_p))^{\beta(bp)} = w_2 \cap w_3$,

(ii) If $1 < p < \infty$, then from Part (v) of Lemma 3.1, we have $(\Phi^\star(\mathcal{L}_p))^{\beta(bp)} = w_3 \cap w_4$.

In fact, they are exactly what we want to prove. □

Theorem 3.4.

(i) *If* $0 < p \leq 1$, *then* $(\Phi^\star(\mathcal{L}_p))^\gamma = w_2$,

(ii) *If* $1 < p < \infty$, *then* $(\Phi^\star(\mathcal{L}_p))^\gamma = w_4$.

Proof. This can be obtained by analogy with the proof of Theorem 3.3 with Parts (i) and (ii) of Lemma 3.1 instead of Parts (iv) and (v), respectively. Therefore, we leave the details. □

4. Some matrix transformations related to the space $\Phi^{\star}(\mathcal{L}_p)$

In this section, we give the characterization of the classes $(\Phi^{\star}(\mathcal{L}_p) : \Lambda)$, where $\Lambda \in \{\mathcal{M}_u, \mathcal{C}_{bp}, \mathcal{L}_q\}$ and $0 < p < \infty$.

THEOREM 4.1. *Assume that $B = (b_{rstu})$ be an arbitrary 4-dimensional infinite matrix. In that case:*

(i) *If $0 < p \leq 1$, then $B \in (\Phi^{\star}(\mathcal{L}_p) : \mathcal{M}_u)$ iff*

$$B_{rs} \in (\Phi^{\star}(\mathcal{L}_p))^{\beta(bp)}, \tag{13}$$

and

$$\sup_{r,s,t,u \in \mathbb{N}} \left| \sum_{m=t,t|m}^{\infty} \sum_{n=u,u|n}^{\infty} \frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(t)\varphi(u)} tub_{rsmn} \right| < \infty. \tag{14}$$

(ii) *If $1 < p < \infty$, then $B \in (\Phi^{\star}(\mathcal{L}_p) : \mathcal{M}_u)$ iff (13) holds and*

$$\sup_{r,s \in \mathbb{N}} \sum_{t,u} \left| \sum_{m=t,t|m}^{\infty} \sum_{n=u,u|n}^{\infty} \frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(t)\varphi(u)} tub_{rsmn} \right|^{p'} < \infty. \tag{15}$$

Proof. In case of $0 < p \leq 1$, the theorem can be proved by using similar method of the proof of the second part, to avoid the repetition of similar statements, we give the proof only for $1 < p < \infty$.

(ii) Let $1 < p < \infty$ and take the sequence $x \in \Phi^{\star}(\mathcal{L}_p)$. So, we can say that there exists a double sequence $y = (y_{rs}) \in \mathcal{L}_p$ from Theorem 2.4. By taking into account the equality (3), the (i,j)th rectangular partial sum of the series $\sum_{t,u} b_{rstu}x_{tu}$ obtained as

$$\begin{aligned} (Bx)_{rs}^{[i,j]} &= \sum_{t,u}^{i,j} b_{rstu}x_{tu} \\ &= \sum_{t,u=1}^{i,j} \left[\sum_{m=t,t|m}^{i} \sum_{n=u,u|n}^{j} \frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(t)\varphi(u)} tub_{rsmn} \right] y_{tu} \end{aligned} \tag{16}$$

for every $r, s, i, j \in \mathbb{N}$. By defining the 4-dimensional matrix $H_{rs} = \left(h^{rs}_{ijtu}\right)$ as

$$h^{rs}_{ijtu} := \begin{cases} \displaystyle\sum_{m=t, t|m}^{i} \sum_{n=u, u|n}^{j} \frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(t)\varphi(u)} tub_{rsmn} & , \quad t \mid m \quad , \quad u \mid n, \\ 0 & , \quad \text{otherwise,} \end{cases}$$

for every $i, j, t, u \in \mathbb{N}$, it is obvious that (16) can be rewritten as $(Bx)^{[i,j]}_{rs} = (H_{rs}y)_{[i,j]}$ which clearly indicates that the *bp*-convergence of $(Bx)^{[i,j]}_{rs}$ and the statement $H_{rs} \in (\mathcal{L}_p : \mathcal{C}_{bp})$ are equivalent for all $x \in \Phi^\star(\mathcal{L}_p)$ and $r, s \in \mathbb{N}$. Therefore, the condition (13) must be provided for each fixed $r, s \in \mathbb{N}$.
Let us take *bp*-limit in the terms of the matrix $H_{rs} = \left(h^{rs}_{ijtu}\right)$ while $i, j \to \infty$, that is,

$$bp - \lim_{i,j\to\infty} h^{rs}_{ijtu} = \sum_{m=t, t|m}^{\infty} \sum_{n=u, u|n}^{\infty} \frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(t)\varphi(u)} tub_{rsmn}. \tag{17}$$

By using the relation (17), we can define the 4-dimensional matrix $H = (h_{rstu})$ as

$$h_{rstu} = \sum_{m=t, t|m}^{\infty} \sum_{n=u, u|n}^{\infty} \frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(t)\varphi(u)} tub_{rsmn},$$

for all $r, s, t, u \in \mathbb{N}$. Therefore, we obtain by the equations (16) and (17) that

$$bp - \lim_{i,j\to\infty} (Bx)^{[i,j]}_{rs} = bp - \lim (Hy)_{rs}.$$

So, if we take into account the fact that "$B = (b_{rstu}) \in (\Phi^\star(\mathcal{L}_p) : \mathcal{M}_u)$ if and only if $H \in (\mathcal{L}_p : \mathcal{M}_u)$" with Part (*ii*) of Lemma 3.1, then it is obvious that

$$\sup_{r,s\in\mathbb{N}} \sum_{t,u} \left| \sum_{m=t, t|m}^{\infty} \sum_{n=u, u|n}^{\infty} \frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(t)\varphi(u)} tub_{rsmn} \right|^{p'} < \infty.$$

Thus, we can conclude that $B = (b_{rstu}) \in (\Phi^\star(\mathcal{L}_p) : \mathcal{M}_u)$ iff conditions (13) and (15) are satisfy. $\square$

To avoid the repetition of the similar statements, we give the following two theorems without proof since they may be proved in the similar way used in proving Theorem 4.1.

THEOREM 4.2. *Let $B = (b_{rstu})$ be any 4-dimensional matrix. In that case, following statements are hold:*

(i) *Let $0 < p \leq 1$. Then, $B \in (\Phi^{\star}(\mathcal{L}_p) : \mathcal{C}_{bp})$ iff the conditions (13) and (14) satisfy and there exists $(\alpha_{tu}) \in \Omega$ such that*

$$bp - \lim_{r,s\to\infty} \sum_{m=t,t|m}^{\infty} \sum_{n=u,u|n}^{\infty} \frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(t)\varphi(u)} tub_{rsmn} = \alpha_{tu}, \tag{18}$$

(ii) *Let $1 < p < \infty$. Then, $B \in (\Phi^{\star}(\mathcal{L}_p) : \mathcal{C}_{bp})$ iff the conditions (13), (15) and (18) satisfy.*

THEOREM 4.3. *Let $0 < p \leq 1$, $1 \leq q < \infty$ and $B = (b_{rstu})$ be any 4-dimensional matrix. Then, $B \in (\Phi^{\star}(\mathcal{L}_p) : \mathcal{L}_q)$ iff the condition (13) satisfies and*

$$\sup_{r,s\in\mathbb{N}} \sum_{t,u} \left| \sum_{m=t,t|m}^{\infty} \sum_{n=u,u|n}^{\infty} \frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(t)\varphi(u)} tub_{rsmn} \right|^q < \infty.$$

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