Korean J. Math. **27** (2019), No. 4, pp. 1119–1131
https://doi.org/10.11568/kjm.2019.27.4.1119

ON GENERALIZED f-DERIVATIONS OF LATTICE IMPLICATION ALGEBRAS

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Abstract. In this paper, we introduce the notion of generalized f-derivation of lattice implication algebra and investigate some related properties. Also, we prove that if D is a generalized f-derivation associated with an f-derivation d of L, then $D(x \to y) = f(x) \to D(y)$ for all $x, y \in L$.

1. Introduction

The concept of lattice implication algebra was proposed by Y. Xu [11], in order to establish an alternative logic knowledge representation. Also, in [12], Y. Xu and K. Y. Qin discussed the properties lattice H implication algebras, and gave some equivalent conditions about lattice H implication algebras. Y. Xu and K. Y. Qin [13] introduced the notion of filters in a lattice implication, and investigated their properties. The present author [5, 14] introduced the notion of derivation and f-derivation in lattice implications algebras and obtained some related results. In this paper, we introduce the notion of generalized f-derivation of lattice implication algebra and investigate some related properties. Also, we prove that if D is a generalized f-derivation associated with an f-derivation d of L, then $D(x \to y) = f(x) \to D(y)$ for all $x, y \in L$.

Received October 8, 2019. Revised September 18, 2019. Accepted December 3, 2019.

2010 Mathematics Subject Classification: 16Y30, 06B35, 06B99.

Key words and phrases: Lattice implication algebra, generalized f-derivation, isotone, $Fix_D(L)$, $KerD$..

2. Preliminaries

Definition 2.1. A *lattice implicationalgebra* is an algebra $(L; \wedge, \vee, ', \rightarrow, 0, 1)$ of type $(2, 2, 1, 2, 0, 0)$, where $(L; \wedge, \vee, 0, 1)$ is a bounded lattice, " $'$ " is an order-reversing involution and " $\rightarrow$ " is a binary operation, satisfying the following axioms, for all $x, y, z \in L$,

(L1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(L2) $x \rightarrow x = 1$,
(L3) $x \rightarrow y = y' \rightarrow x'$,
(L4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,
(L5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
(L6) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
(L7) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$.

If L satisfies conditions (I1) – (I5), we say that L is a *quasi lattice implicationalgebra*. A lattice implication algebra L is called a *lattice H implication algebra* if it satisfies $x \vee y \vee ((x \wedge y) \rightarrow z) = 1$ for all $x, y, z \in L$.

In the sequel the binary operation " $\rightarrow$ " will be denoted by juxtaposition. We can define a partial ordering " $\leq$ " on a lattice implicationalgebra L by $x \leq y$ if and only if $x \rightarrow y = 1$ for all $x, y \in L$.

Proposition 2.2. *In a lattice implicationalgebra L, the following hold, for all $x, y, z \in L$,* (see [11])

(u1) $0 \rightarrow x = 1$, $1 \rightarrow x = x$ *and* $x \rightarrow 1 = 1$,
(u2) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,
(u3) $x \leq y$ *implies* $y \rightarrow z \leq x \rightarrow z$ *and* $z \rightarrow x \leq z \rightarrow y$,
(u4) $x' = x \rightarrow 0$.
(u5) $x \vee y = (x \rightarrow y) \rightarrow y$,
(u6) $((y \rightarrow x) \rightarrow y')' = x \wedge y = ((x \rightarrow y) \rightarrow x')'$,
(u7) $x \leq (x \rightarrow y) \rightarrow y$.

Definition 2.3. In a lattice H implication algebra L, the following hold, for all $x, y, z \in L$,

(u8) $x \rightarrow (x \rightarrow y) = x \rightarrow y$,
(u9) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$(see [11]).

Definition 2.4. A subset F of a lattice implication algebra L is called a *filter* of L it satisfies,

(F1) $1 \in F$,
(F2) $x \in F$ and $x \to y \in F$ imply $y \in F$, for all $x, y \in L$(see [11]).

Definition 2.5. Let L_1 and L_2 be lattice implication algebras.
(1) A mapping $f : L_1 \to L_2$ is an *implication homomorphism* if $f(x \to y) = f(x) \to f(y)$ for all $x, y \in L_1$.
(2) A mapping $f : L_1 \to L_2$ is an *lattice implication homomorphism* if $f(x \vee y) = f(x) \vee f(y), f(x \wedge y) = f(x) \wedge f(y), f(x') = f(x)'$ for all $x, y \in L_1$(see [11]).

Definition 2.6. Let L be a lattice implication algebra and let $f : L \to L$ be an implication homomorphism on L. A mapping $d : L \to L$ is called an *f-derivation* of L if there exists an implication homomorphism f such that

$$d(x \to y) = (f(x) \to d(y)) \vee (d(x) \to f(y))$$

for all $x, y \in L$(see [11]).

Proposition 2.7. *Let d be a f-derivation on L. Then the following conditions hold.*
(1) $d(1) = 1$.
(2) $d(x) = d(x) \vee f(x)$ *for every* $x \in L$.
(3) $f(x) \leq d(x)$ *for every* $x \in L$.
(4) $f(x) \vee f(y) \leq d(x) \vee d(y)$ *for every* $x, y \in L$.
(5) $d(x \to y) = f(x) \to d(y)$ *for every* $x, y \in L$.

3. Generalized f-derivations of lattice implication algebras

In what follows, let L denote a lattice implication algebra and let f be an implication homomorphism on L unless otherwise specified.

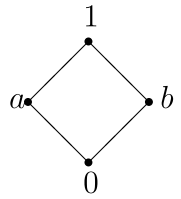
Definition 3.1. Let L be a lattice implication algebra and let $f : L \to L$ be an implication homomorphism on L. A map $D : L \times L \to L$ is called a *generalized f-derivation* of L if there exists an f-derivation $d : L \to L$ satisfying the the following condition

$$D(x \to y) = (f(x) \to D(y)) \vee (d(x) \to f(y))$$

for all $x, y \in L$.

Let L be a lattice implication algebra and let f be an implication homomorphism on L. If $D = d$, then D is an f-derivation on L.

Example 3.2. Let $X = \{x, y\}$. Then
$$L = \mathcal{P}(X) = \{\emptyset, \{x\}, \{y\}, X\}.$$
Let $0 = \emptyset$, $a = \{x\}$, $b = \{y\}$, $1 = X$. Then $L = \{0, a, b, 1\}$ is a bounded lattice with above Hasse diagram.



We can make an implication $\to$ on L such as
$$a \to b = \{x\}^C \cup \{y\} = \{y\} \cup \{y\} = \{y\} = b.$$
Hence we have the operation table of the implication :

x	x'
0	1
a	b
b	a
1	0

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

If we define a map $f : L \to L$ by
$$f(x) = \begin{cases} 0 & \text{if } x = 0, a \\ 1 & \text{if } x = b, 1 \end{cases}$$
then this map f is an implication homomorphism. Define a map $d : L \to L$ and $D : L \to L$ by
$$d(x) = \begin{cases} b & \text{if } x = 0, a \\ 1 & \text{if } x = b, 1 \end{cases} \qquad D(x) = \begin{cases} 0 & \text{if } x = 0, a \\ 1 & \text{if } x = b, 1 \end{cases}$$
Then it is easy to check that d is an f-derivation on L and D is a generalized f-derivation associated with d.

Example 3.3. In Example 3.2, if we define a map $f : L \to L$ by
$$f(x) = \begin{cases} 0 & \text{if } x = 0, b \\ 1 & \text{if } x = a, 1 \end{cases}$$
then this map f is an implication homomorphism on L. Define a map $d : L \to L$ and $D : L \to L$ by

$$d(x) = \begin{cases} 1 & \text{if } x = a, 1 \\ a & \text{if } x = 0, b \end{cases} \qquad D(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x = 1, a \\ a & \text{if } x = b \end{cases}$$

Then it is easy to check that d is an f-derivation on L and D is a generalized f-derivation associated with d.

Proposition 3.4. *Let d be an f-derivation on L and let D be a generalized f-derivation associated with d. Then the following conditions hold.*

(1) $D(1) = 1$.
(2) $D(x) = D(x) \vee f(x)$ *for every* $x \in L$.
(3) $f(x) \leq D(x)$ *for every* $x \in L$.
(4) $f(x) \to y \leq D(x) \to y$ *for every* $x, y \in L$.

Proof. (1) Let D be a generalized f-derivation associated with d. Then

$$\begin{aligned} D(1) &= D(1 \to 1) = (f(1) \to D(1)) \vee (d(1) \to f(1)) \\ &= (1 \to D(1)) \vee (1 \to 1) = D(1) \to 1 = 1. \end{aligned}$$

(2) For every $x \in L$, we have

$$\begin{aligned} D(x) &= D(1 \to x) = (f(1) \to D(x)) \vee (d(1) \to f(x)) \\ &= (1 \to D(x)) \vee (1 \to f(x)) = D(x) \vee f(x). \end{aligned}$$

(3) For all $x \in L$, by part (2), we obtain

$$\begin{aligned} f(x) \to D(x) &= f(x) \to (D(x) \vee f(x)) = f(x) \to (D(x) \to f(x)) \to f(x)) \\ &= (D(x) \to f(x)) \to (f(x) \to f(x)) = (D(x) \to f(x)) \to 1 \\ &= 1. \end{aligned}$$

This implies $D(x) \leq f(x)$ for every $x \in L$.

(4) For every $x, y \in L$, we have $D(x) \leq f(x)$ for every $x \in L$ by part (3). Hence we get $f(x) \to y \leq D(x) \to y$ for every $x, y \in L$ by (u3). □

Proposition 3.5. *Let d be an f-derivation on L and let D be a generalized f-derivation associated with d and $f(D(x)) = D(x)$ for every $x \in L$. Then $D(D(x) \to x) = 1$ for every $x \in L$.*

Proof. Let D be a generalized f-derivation associated with d. Then

$$\begin{aligned} D(D(x) \to x) &= (f(D(x)) \to D(x)) \vee (d(D(x)) \to f(x)) \\ &= (D(x) \to D(x)) \vee (d(D(x)) \to f(x)) = 1 \vee (d(D(x)) \to f(x)) \\ &= 1. \end{aligned}$$

□

PROPOSITION 3.6. *Let d be an f-derivation on L and let D be a generalized f-derivation associated with d on L. Then the following conditions hold:*

(1) $D(x) \to D(y) \leq D(x \to y)$ *for all* $x, y \in L$.
(2) $D(x) \to f(y) \leq f(x) \to D(y)$ *for all* $x, y \in L$.
(3) $f(x) \to f(y) \leq D(x \to y)$ *for all* $x, y \in L$.

Proof. (1) For all $x, y \in L$, we have $f(x) \to D(y) \leq (f(x) \to D(y)) \vee (d(x) \to f(y)) = D(x \to y)$ from (u7). Now from $f(x) \leq D(x)$, we get $D(x) \to D(y) \leq f(x) \to D(y)$ by using (u3). Hence $D(x) - D(y) \leq D(x \to y)$.

(2) For any $x, y \in L$, from $f(x) \leq D(x)$ and $f(y) \leq D(y)$, we get $D(x) \to f(y) \leq f(x) \to f(y)$ and $f(x) \to f(y) \leq f(x) \to D(y)$ by using (u3). Hence we obtain $D(x) \to f(y) \leq f(x) \to D(y)$ for all $x, y \in L$.

(3) From Definition 3.1 and (u7), for all $x, y \in L$, we have $f(x) \to D(y) \leq (f(x) \to D(y)) \vee (d(x) \to f(y)) = D(x \to y)$ for all $x, y \in L$. Since $f(y) \leq D(y)$, we get $f(x) \to f(y) \leq f(x) \to D(y)$, which implies $f(x) \to f(y) \leq D(x \to y)$.

□

THEOREM 3.7. *Let d be an f-derivation on L. If D is a generalized f-derivation associated with d on L, we get $D(x \to y) = f(x) \to D(y)$ for all $x, y \in L$.*

Proof. Suppose that D is a generalized f-derivation associated with a derivation d on L. Then for any $x, y \in L$, we have $d(x) \to f(y) \leq f(x) \to f(y)$ since $f(x) \leq d(x)$ and $f(x) \to f(y) \leq f(x) \to D(y)$ since $f(y) \leq D(y)$. Hence we have $d(x) \to f(y) \leq f(x) \to D(y)$ and

$$\begin{aligned} D(x \to y) &= (f(x) \to D(y)) \vee (d(x) \to f(y)) \\ &= ((f(x) \to D(y)) \to (d(x) \to f(y))) \to (d(x) \to f(y)) \\ &= ((d(x) \to f(y)) \to (f(x) \to D(y))) \to (f(x) \to D(y)) \\ &= 1 \to (f(x) - D(y)) = f(x) - D(y) \end{aligned}$$

from (L5) and (u3). This completes the proof.

$\square$

THEOREM 3.8. *Let d be an f-derivation on L and let D be a generalized f-derivation associated with d. If it satisfies $D(x \to y) = D(x) \to f(y)$ for every $x, y \in L$, we have $D(x) = f(x)$.*

Proof. Let d be an f-derivation on L and let D be a generalized f-derivation associated with d. If it satisfies $D(x \to y) = D(x) \to f(y)$ for all $x, y \in L$, we have

$$\begin{aligned} D(x) &= D(1 \to x) = D(1) \to f(x) \\ &= 1 \to f(x) = f(x). \end{aligned}$$

This completes the proof.

$\square$

THEOREM 3.9. *Let D be a generalized f-derivation associated with an f-derivation d on L and let D be lattice implication homomorphism on L. Then we have $D(x \vee y) = D(f(x)) \vee D(f(y))$ for every $x, y \in L$.*

Proof. For every $x, y \in L$, we obtain, by (L7)

$$\begin{aligned} D(x \vee y) &= D(x'' \vee y'') = D((x' \wedge y') \to 0) \\ &= f(x' \wedge y') \to D(0) = (f'(x) \to D(0)) \vee (f'(y) \to D(0)) \\ &= D(f'(x) \to 0) \vee D(f'(y) \to 0) = D(f(x)) \vee D(f(y)). \end{aligned}$$

$\square$

THEOREM 3.10. *Let D be a generalized f-derivation associated with an f-derivation d on L. Then the following conditions are equivalent:*

(1) *D is an isotone generalized f-derivation associate with d.*
(2) *$D(x) \vee D(y) \leq D(x \vee y)$ for all $x, y \in L$.*

Proof. (1) $\Rightarrow$ (2): Suppose that D is an isotone generalized f-derivation associated with an f-derivation d of L. We know that $x \leq x \vee y$ and $y \leq x \vee y$ for all $x, y \in L$. Since D is isotone, $D(x) \leq D(x \vee y)$ and $D(y) \leq D(x \vee y)$. Hence we obtain $D(x) \vee D(y) \leq D(x \vee y)$.

(2)$\Rightarrow$ (1): Suppose that $D(x) \vee D(y) \leq D(x \vee y)$ and $x \leq y$. Then we have $D(x) \leq D(x) \vee D(y) \leq D(x \vee y) = D(y)$.

$\square$

DEFINITION 3.11. Let d be an f-derivation on L and let D be a generalized f-derivation associated with d.

(1) D is called a *monomorphic generalized f-derivation* associate with d if D is one-to- one.
(2) D is called an *epic generalized generalized f-derivation* associate with d if D is onto.

THEOREM 3.12. *Let D be a generalized f-derivation associated with an f-derivation d on L and let D is idempotent, that is, $D^2 = D$. Then the following conditions are equivalent:*

(1) *$D(x) = x$ for all $x \in L$.*
(2) *D is a monomorphic generalized f-derivation associate with an f-derivation d of L.*
(3) *D is an epic generalized f-derivation associate with an f-derivation d of L.*

Proof. (1) $\Rightarrow$(2) is clear.
(2) $\Rightarrow$(1) Let D be a monomorphic generalized f-derivation associate with d and $x \in L$. By hypothesis, we have $D(D(x)) = D(x)$ for every $x \in L$. Since D is monomorphic, we get $D(x) = x$ for all $x \in L$.
(1) $\Rightarrow$(3) is trivial.
(3) $\Rightarrow$(1) Let D be an epic generalized f-derivation associate with d and $x \in L$. Then there exists $y \in L$ such that $D(y) = x$. Hence we have $D(x) = D(D(y)) = D^2(y) = D(y) = x$.

□

Let d be an f-derivation of L and let D be a generalized f-derivation associated with d. Define a set $Fix_D(L)$ by

$$Fix_D(L) := \{x \in L \mid D(x) = f(x)\}$$

for all $x \in L$. Clearly, $1 \in Fix_D(L)$.

PROPOSITION 3.13. *Let d be an f-derivation on L and let D be a generalized f-derivation associated with d. Then the following properties hold.*

(1) *If $x \in L$ and $y \in Fix_D(L)$, we have $x \to y \in Fix_D(L)$.*
(2) *If $x \in L$ and $y \in Fix_D(L)$, we have $x \vee y \in Fix_D(L)$.*

Proof. (1) Let $x \in L$ and $y \in Fix_D(L)$. Then we have $D(y) = f(y)$. Hence we get

$$\begin{aligned} D(x \to y) &= f(x) \to D(y) = f(x) \to f(y) \\ &= f(x \to y) \end{aligned}$$

from Theorem 3.7. This completes the proof.

(2) Let $x, y \in Fix_D(L)$. Then we get

$$\begin{aligned} D(x \vee y) &= D((x \to y) \to y) = f(x \to y) \to D(y) \\ &= f(x \to y) \to f(y) = f((x \to y) \to y) \\ &= f(x \vee y) \end{aligned}$$

from Theorem 3.7. This completes the proof. □

PROPOSITION 3.14. *Let d be an f-derivation of L and let D be a generalized f-derivation associated with d. If $x \leq y$ and $x \in Fix_D(L)$, we have $y \in Fix_D(L)$.*

Proof. Let $x \leq y$ and $x \in Fix_D(L)$. Then we have $x \to y = 1$, and so $f(x) \to f(y) = f(x \to y) = f(1) = 1$. This means $f(x) \leq f(y)$. By hypothesis, $D(x) = f(x)$ for every $x \in L$. Thus we get

$$\begin{aligned} D(y) &= D((1 \to y) = D((x \to y) \to y) \\ &= D((y \to x) \to x) = f(y \to x) \to D(x) \\ &= f(y \to x) \to f(x) = (f(y) \to f(x)) \to f(x) \\ &= (f(x) \to f(y)) \to f(y) = f(x) \vee f(y) = f(y), \end{aligned}$$

from Theorem 3.7. Hence $y \in Fix_D(L)$. □

DEFINITION 3.15. Let L be a lattice implication algebra. A non-empty set F of L is called a *normal filter* if it satisfies the following conditions:

(1) $1 \in F$.
(2) $x \in L$ and $y \in F$ imply $x \to y \in F$.

EXAMPLE 3.16. In Example 3.3, let $F = \{1, a\}$. Then F is a normal filter of a lattice implication algebra L.

PROPOSITION 3.17. *Let d be an f-derivation on L and let D be a generalized f-derivation associated with d. Then $Fix_D(L)$ is a normal filter of L.*

Proof. Clearly, $1 \in Fix_D(L)$. By Proposition 3.13 (1), we know tat $x \in L$ and $y \in F$ imply $x \to y \in F$. This completes the proof. □

Let d be an f-derivation on L and let D be a generalized f-derivation associated with d of L. Define a set $KerD$ by

$$KerD = \{x \in L \mid D(x) = 1\}.$$

PROPOSITION 3.18. *Let d be an f-derivation on L and let D be a generalized f-derivation associated with d. Then*

(1) *If $y \in KerD$, then we have $x \vee y \in KerD$ for all $x \in L$.*
(2) *If $x \leq y$ and $x \in KerD$, then $y \in KerD$.*
(3) *If $y \in KerD$, we have $x \to y \in KerD$ for all $x \in L$.*

Proof. (1) Let D be a generalized f-derivation on L and $y \in KerD$. Then we get $D(y) = 1$, and so

$$D(x \vee y) = D((x \to y) \to y) = f(x \to y) \to D(y) = f(x \to y) \to 1 = 1$$

from Theorem 3.7. Hence we have $x \vee y \in KerD$.

(2) Let $x \leq y$ and $x \in KerD$. Then we get $x \to y = 1$ and $D(x) = 1$, and so

$$\begin{aligned} D(y) &= D(1 \to y) = D((x \to y) \to y) \\ &= D((y \to x) \to x) = f(y \to x) \to D(x) \\ &= f(y \to x) \to 1 = 1 \end{aligned}$$

from Theorem 3.7. Hence we have $y \in KerD$.

(3) Let $y \in KerD$. Then $D(y) = 1$. Thus we have

$$D(x \to y) = f(x) \to D(y) = f(x) \to 1 = 1$$

from Theorem 3.7. Hence we get $x \to y \in KerD$.

□

THEOREM 3.19. *Let d be an f-derivation on L and let D be a generalized f-derivation associated with a derivation d. Then $KerD$ is a normal filter of L.*

Proof. Clearly, $1 \in KerD$. Let $x \in L$ and $y \in KerD$. Then we have $d(y) = 1$, and so

$$\begin{aligned} D(x \to y) &= f(x) \to D(y) \\ &= f(x) \to 1 = 1, \end{aligned}$$

which implies $x \to y \in KerD$ from Theorem 3.7. Hence $KerD$ is a normal filter of L.

□

Definition 3.20. Let d be an f-derivation on L and let D be a generalized f-derivation associated with d. A normal filter F of L is called a *D-normal filter* if $D(F) = F$.

Since $D(1) = 1$, it can be easily observed that the normal filter $\{1\}$ is a D-normal filter of L. If L is onto, then $D(L) = L$, which implies L is an D-normal filter of L.

Example 3.21. In Example 3.3, let $F = \{1, a, b\}$. Then F is a normal filter of D. It can be verified that $D(F) = F$. Therefore, F is an D-normal filter of L.

Lemma 3.22. *Let d be an f-derivation on L and let D be a generalized f-derivation associated with d and let I, J be any two D-normal filters of L. Then we have $I \subseteq J$ implies $D(I) \subseteq D(J)$.*

Proof. Let $I \subseteq J$ and $x \in D(I)$. Then we have $x = D(y)$ for some $y \in I \subseteq J$. Hence we get $x = D(y) \in D(J)$. Therefore, $D(I) \subseteq D(J)$. $\square$

Proposition 3.23. *Let d be an f-derivation on L and let D be a generalized f-derivation associated with an f-derivation d of L. Then an intersection of any two D-normal filters is also an D-normal filter of L.*

Proof. Let $x \in D(I \cap J)$. Then $x = D(a)$ for some $a \in I$ and $a \in J$. Hence $x = D(a) \in D(I) = I$ and $x = D(a) \in D(J) = J$, which implies $x \in I \cap J$. Now let $x \in I \cap J$. Then $x \in I = D(I)$ and $x \in J = D(J)$. Hence we have $x \in D(I) \cap D(J)$. Hence $I \cap J$ is a D-normal filter of L. $\square$

Definition 3.24. Let D be a generalized f-derivation associated with a f-derivation d of L. A normal filter F of L is called an *injective normal filter* with respect to D if for $x, y \in L$, $D(x) = D(y)$ and $x \in F$ implies $y \in F$.

Evidently, $KerD$ is an injective normal filter of L. Though the normal filter $\{1\}$ is a D-normal filter, there is no guarantee that it is injective normal filter.

Theorem 3.25. *Let D be a generalized f-derivation associated with an f-derivation d of L. Then the following conditions are equivalent.*

(1) $\{1\}$ *is injective with respect to* D.
(2) $KerD = \{1\}$.
(3) $D(x) = 1$ *implies that* $x = 1$ *for all* $x \in L$.

Proof. (1) $\Rightarrow$ (2). Suppose that $\{1\}$ is injective with respect to D. Let $x \in KerD$. Then $D(x) = D(1)$. Since $\{1\}$ is injective, we can get $x \in \{1\}$. Therefore, $KerD = \{1\}$.
(2) $\Rightarrow$ (3). The proof is trivial.
(3) $\Rightarrow$ (1). Let $D(x) = D(y)$ and $x \in \{1\}$. Hence $D(y) = D(x) = D(1) = 1$, which implies $y = 1 \in \{1\}$.

□

THEOREM 3.26. *Let* D *be a generalized* f*-derivation associated with an* f*-derivation* d *of* L *and let* D *be idempotent. Then an* D*-normal filter* F *of* L *is injective with respect to* D *if and only if for any* $x \in L, D(x) \in F$ *implies* $x \in F$.

Proof. Let F be a D-normal filter of L and let F be injective with respect to D. Suppose that $D(x) \in F = D(F)$ and $x \in L$. Then $D(x) = D(a)$ for some $a \in F$. Since F is injective and $a \in F$, we get that $x \in F$. Conversely, let $x, y \in L, D(x) = D(y)$ and $x \in F$. Since $x \in D(F)$, we get $x = D(a)$ for some $a \in F$. Hence $D(y) = D(x) = D(D(a)) = D(a) \in D(F)$, which implies that $y \in F$. Therefore, F is an injective normal filter of L with respect to D.

□

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