

POWERS OF INTEGERS WITH ARITHMETIC TABLES

EUNMI CHOI AND MYUNGJIN CHOI

ABSTRACT. Any powers of 11 are easily obtained from the Pascal triangle. In this work we study powering and negative powering of any k digit integers by means of certain arithmetic tables.

1. Introduction

Pascal triangle is an arithmetic table(AT) of $(x + 1)^m$ which gives connections among various areas in mathematics. One of the interesting features is a relationship with powers of 11 ([5], [6], [7] and [8]), indeed $11^9 = 2357947691$ is obtained immediately from the 9th row (1, 9, 36, 84, 126, 126, 84, 36, 9, 1) of Pascal triangle. However it seems no one has asked about powering of any integer by certain AT.

A purpose of the work is to investigate powers and even negative exponent powers of any integers by utilizing AT of certain polynomials. We denote by $T^{(a_k, \dots, a_0)}$ the AT of $(a_k x^k + \dots + a_1 x + a_0)^m$ expanded in descending order. We may refer [1], [3] and [4] for the AT $T^{(1, \dots, 1)}$. As an analog of powering of 11 through Pascal triangle $T^{(1, 1)}$, we ask about powers of $k + 1$ digit integer $a_k \dots a_0$ via $T^{(a_k, \dots, a_0)}$. To be precise, for a three digit integer $cba = c10^2 + b10 + a$ ($0 \leq a, b, c \leq 9$, $0 \neq c$), consider $(cx^2 + bx + a)^m$ and its AT $T^{(c, b, a)}$. We first express $T^{(c, b, a)}$ by means of $T^{(1, 1)}$ and discuss powers of cba . This consideration is extended to powering integers having negative exponent $(cba)^{-t}$ ($t > 0$). Finally for powers of any $(k + 1)$ digit integers $a_k \dots a_1 a_0$, the AT of

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$(a_k x^k + \dots + a_1 x + a_0)^m$ and its relationships with the AT $T^{(1,1)}$ are investigated. A feature of the work is to provide an easy way to have, for example 357^4 , 11112^5 and even $111^{-3} \approx (0.7311913813)10^{-6}$.

Throughout the work, $r_n^{(a_1, a_0)}$ and $e_{n,j}^{(a_1, a_0)}$ denote the n th row and the (n, j) th entry of AT $T^{(a_1, a_0)}$ of $(a_1 x + a_0)^m$, respectively ($n, j \geq 0$). We simply write $T^{(1,1)} = T$ the Pascal triangle, $r_n^{(1,1)} = r_n$ and $e_{n,j}^{(1,1)} = e_{n,j}$. Moreover $r_n^{(a_k, \dots, a_0)}$ and $e_{n,j}^{(a_k, \dots, a_0)}$ denote the corresponding ones in $T^{(a_k, \dots, a_0)}$ of $(a_k x^k + \dots + a_0)^m$. Let $[a^i]_{i \geq 0}$ be a diagonal matrix $\text{diag}(1, a, a^2, \dots)$ having entries a^i , and a multiplication of ordered tuples is defined by $(u_1, \dots, u_k)(v_1, \dots, v_k) = (u_1 v_1, \dots, u_k v_k)$.

2. Powering of any integers

For $2 \leq n \leq 4$, $11^n = 121, 1331$ and 14641 follow from n th row r_n of T . And $11^5 = 161051$ is due to $r_5 = (1, 5, 10, 10, 5, 1)$ by Table 1 or Table 2.

<p style="margin: 0;">Table 1</p> <table style="margin: 0 auto; border-collapse: collapse;"> <tr><td style="border: 1px solid black; padding: 2px;">1</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">5</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">① ①</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">1 0</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">5 1</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">S₁₀^{r₅} 1 6 1 0 5 1</td></tr> </table>	1	5	① ①	1 0	5 1	S ₁₀ ^{r₅} 1 6 1 0 5 1	<p style="margin: 0;">Table 2</p> <table style="margin: 0 auto; border-collapse: collapse;"> <tr><td style="border: 1px solid black; padding: 2px;">1 5 ① 0 5 1</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">① 1</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">S₁₀^{r₅} 1 6 1 0 5 1</td></tr> </table>	1 5 ① 0 5 1	① 1	S ₁₀ ^{r₅} 1 6 1 0 5 1	<p style="margin: 0;">Table 3</p> <table style="margin: 0 auto; border-collapse: collapse;"> <tr><td style="border: 1px solid black; padding: 2px;">100</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">500</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">1000</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">1000</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">500</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">1</td></tr> <tr><td style="border: 1px solid black; padding: 2px;">S_{10²}^{r₅} 10510100501</td></tr> </table>	100	500	1000	1000	500	1	S _{10²} ^{r₅} 10510100501
1																		
5																		
① ①																		
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5 1																		
S ₁₀ ^{r₅} 1 6 1 0 5 1																		
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S _{10²} ^{r₅} 10510100501																		

The last row $S_{10}^{r_5} = 161051$ is the sum of entries of each columns, that equals 11^5 . The both Table 1 and 2 will be called column sum tables of r_5 with base 10. As a generalization, powering of integers beginning and ending with 1 where all intermediate digits are 0s, such as 101, 1001, 10001, etc, are obtained by the column sum tables with base 10^k ($k > 0$). In fact, $101^5 = 10510100501$ follows from the column sum table of r_5 with base 10^2 (Table 3).

In order to obtain powers of two digit integers $ba = b \cdot 10 + a$ ($0 \leq a, b \leq 9$), we begin to consider AT $T^{(b,a)}$ of $(bx + a)^m$.

THEOREM 1. $T^{(b,a)} = T^b [a^i]_{i \geq 0} = T^{b-1} T^{(1,a)} = T^{b-s} T^{(s,a)}$ for any $0 \leq s \leq b$, and $r_n^{(b,a)} = r_n(b^{n-1}, \dots, b, 1)(1, a, \dots, a^{n-1})$.

Proof. When $b = 1$, the AT of $(x + a)^m$ forms

$$T^{(1,a)} = \begin{bmatrix} 1 \\ 1a \\ 12aa^2 \\ 13a3a^2a^3 \dots \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \\ 121 \\ 1331 \dots \end{bmatrix} \begin{bmatrix} 1 \\ a \\ a^2 \\ a^3 \dots \end{bmatrix} = T[a^i]_{i \geq 0}$$

so $r_n^{(1,a)} = r_n(1, \dots, a^n)$ and $e_{n,j}^{(1,a)} = e_{n,j}a^j$. Let $q = \frac{a}{b}$ ($b \neq 0$). Then

$$(bx + a)^n = b^n(x + q)^n = b^n r_n^{(1,q)} \begin{bmatrix} x^n \\ \dots \\ 1 \end{bmatrix}$$

implies

$$\begin{aligned} r_n^{(b,a)} &= b^n r_n^{(1,q)} = b^n r_n(1, q, \dots, q^n) \\ &= r_n(b^n, b^{n-1}a, \dots, a^n) = r_n(b^n, \dots, 1)(1, \dots, a^n), \end{aligned}$$

hence we have

$$T^{(b,a)} = \begin{bmatrix} r_0^{(b,a)} \\ r_1^{(b,a)} \\ r_2^{(b,a)} \\ r_3^{(b,a)} \end{bmatrix} = \begin{bmatrix} r_0 \\ r_1(b, 1)(1, a) \\ r_2(b^2, b, 1)(1, a, a^2) \\ r_3(b^3, \dots, 1)(1, \dots, a^3) \end{bmatrix} = \begin{bmatrix} 1 \\ b & 1 \\ b^2 & 2b & 1 \\ b^3 & 3b^2 & 3b & 1 \end{bmatrix} [a^i]_{i \geq 0}.$$

But since

$$\begin{bmatrix} 1 \\ b & 1 \\ b^2 & 2b & 1 \\ b^3 & 3b^2 & 3b & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ b-1 & & & \\ (b-1)^2 & & 2(b-1) & \\ (b-1)^3 & & 3(b-1)^2 & 3(b-1) & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 121 \\ 1331 \end{bmatrix},$$

it is inductively equal to

$$\begin{bmatrix} 1 & & & \\ b-2 & & & \\ (b-2)^2 & & 2(b-2) & \\ (b-2)^3 & & 3(b-2)^2 & 3(b-2) & 1 \end{bmatrix} T^2 = \dots = \begin{bmatrix} 1 \\ 11 \\ 121 \\ 1331 \end{bmatrix} T^{b-1} = T^b.$$

Thus we conclude that, for any $s \leq b$,

$$T^{(b,a)} = T^b [a^i]_{i \geq 0} = T^{b-1} T[a^i]_{i \geq 0} = T^{b-1} T^{(1,a)} = T^{b-s} T^{(s,a)}. \quad \square$$

Clearly $T^{(b,a)} = T^{b-a} T^{(a,a)} = T^{b-a} [a^i]_{i \geq 0} T$ if $b \leq a$. So

$$T^{(2,2)} = \begin{bmatrix} 1 \\ 22 \\ 484 \end{bmatrix} = [2^i]_{i \geq 0} T \text{ and } \begin{bmatrix} 1 \\ (1, 1)(3, 1)(1, 7) \\ (1, 2, 1)(3^2, 3, 1)(1, 7, 7^2) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 & 7 \\ 9 & 42 & 49 \end{bmatrix} =$$

$T^{(3,7)}$.

Thus

$r_5^{(3,7)} = r_5(3^5, \dots, 1)(1, \dots, 7^5) = (243, 2835, 13230, 30870, 36015, 16807)$, which is the set of coefficients of $(3x + 7)^5$. Moreover its column sum table with base 10 gives rise to the column sum $S_{10}^{r_5^{(3,7)}} = 69343957 = 37^5$ (Table 4).

Table 4	Table 5
$\begin{array}{r} 350057 \\ 433710 \\ 282808 \\ 23066 \\ 1331 \\ \hline S_{10}^{r_5^{(3,7)}} \end{array}$	$\begin{array}{r} 24300 \\ 283500 \\ 1323000 \\ 3087000 \\ 3601500 \\ 16807 \\ \hline S_{10^3}^{r_5^{(3,7)}} \end{array}$

The next theorem is about powers of integers $10^k b + a$ ($k > 0, 1 \leq a, b \leq 9$) filled with zeros except the first and last digits.

THEOREM 2. *Let $U = 10^k b + a$ ($k > 0$). Then any power U^n ($n > 0$) can be obtained by the column sum table of $r_n^{(b,a)}$ with base 10^k .*

Proof. Considering $U^n = (X + a)^n$ with $X = 10^k b$, we have

$$U^n = r_n^{(1,a)} \begin{bmatrix} b^n 10^{kn} \\ \dots \\ b 10^k \\ 1 \end{bmatrix} = r_n^{(1,a)}(b^n, \dots, b, 1) \begin{bmatrix} 10^{kn} \\ \dots \\ 10^k \\ 1 \end{bmatrix} = r_n^{(b,a)} \begin{bmatrix} 10^{kn} \\ \dots \\ 10^k \\ 1 \end{bmatrix}$$

by Theorem 1, so the column sum table of $r_n^{(b,a)}$ with base 10^k yields U^n . □

For example, $3007^5 = 245848260906031807$ is obtained easily from the column sum table of $r_5^{(3,7)}$ with base 10^3 (Table 5).

For three digit integers, consider the AT $T^{(a_2, a_1, a_0)}$ of $(a_2 x^2 + a_1 x + a_0)^m$. Then its i th row $r_i^{(a_2, a_1, a_0)}$ is the set of coefficients of $(a_2 x^2 + a_1 x + a_0)^i$ and can be regarded as a $1 \times (2i + 1)$ row matrix. If we write the k tuple $(0, \dots, 0)$ by $0^{[k]}$ then $(0^{[k]}; r_i^{(a_2, a_1, a_0)}; 0^{[j]})$ is a $1 \times (2i + k + j + 1)$ row matrix consisting of k zeros followed by $r_i^{(a_2, a_1, a_0)}$ and then j zeros.

THEOREM 3. $r_n^{(a_2, a_1, a_0)} = r_n^{(1, a_0)} \begin{bmatrix} r_n^{(a_2, a_1)}; 0^{[n]} \\ \dots \\ 0^{[2(n-i)}; r_i^{(a_2, a_1)}; 0^{[i]} \\ \dots \\ 0^{[2n]}; r_0^{(a_2, a_1)} \end{bmatrix}$ for $n \geq 0$.

Proof. Consider $(a_2 x^2 + a_1 x + a_0)^n = r_n^{(a_2, a_1, a_0)} \begin{bmatrix} x^{2n} \\ \dots \\ 1 \end{bmatrix}$. (1)

Assume $a_2 = 1$ and let $X = x(x + a_1)$. Then

$$X^i = x^i r_i^{(1,a_1)} \begin{bmatrix} x^i \\ \dots \\ 1 \end{bmatrix} = r_i^{(1,a_1)} \begin{bmatrix} x^{2i} \\ \dots \\ x^i \end{bmatrix} = (0^{[2(n-i)}]; r_i^{(1,a_1)}; 0^{[i]}) \begin{bmatrix} x^{2n} \\ \dots \\ x^{2i} \\ \dots \\ x^i \\ \dots \\ 1 \end{bmatrix}$$

for all $0 \leq i \leq n$, so we have

$$(x^2 + a_1x + a_0)^n = (X + a_0)^n = r_n^{(1,a_0)} \begin{bmatrix} X^n \\ \dots \\ 1 \end{bmatrix} = r_n^{(1,a_0)} \begin{bmatrix} r_n^{(1,a_1)}; 0^{[n]} \\ \dots \\ 0^{[2(n-i)}]; r_i^{(1,a_1)}; 0^{[i]} \\ \dots \\ 0^{[2n]}; r_0^{(1,a_1)} \end{bmatrix} \begin{bmatrix} x^{2n} \\ \dots \\ x^{2i} \\ \dots \\ x^i \\ \dots \\ 1 \end{bmatrix}. \quad (2)$$

Hence we have $r_n^{(1,a_1,a_0)} = r_n^{(1,a_0)} \begin{bmatrix} r_n^{(1,a_1)}; 0^{[n]} \\ 0^{[2]}; r_{n-1}^{(1,a_1)}; 0^{[n-1]} \\ \dots \\ 0^{[2n]}; r_0^{(1,a_1)} \end{bmatrix}$ by (1) and (2).

When $a_2 \neq 0$, let $X = a_2x^2 + a_1x = a_2x(x + q)$ with $q = \frac{a_1}{a_2}$. Then

$$X^i = a_2^i x^i r_i^{(1,q)} \begin{bmatrix} x^i \\ \dots \\ 1 \end{bmatrix} = a_2^i r_i^{(1,q)} \begin{bmatrix} x^{2i} \\ \dots \\ x^i \end{bmatrix} = (0^{[2(n-i)}]; a_2^i r_i^{(1,q)}; 0^{[i]}) \begin{bmatrix} x^{2n} \\ \dots \\ x^{2i} \\ \dots \\ x^i \\ \dots \\ 1 \end{bmatrix},$$

since $a_2^i r_i^{(1,q)} = r_i^{(a_2,a_1)}$ by Theorem 1. Thus

$$(a_2x^2 + a_1x + a_0)^n = (X + a_0)^n = r_n^{(1,a_0)} \begin{bmatrix} r_n^{(a_2,a_1)}; 0^{[n]} \\ 0^{[2]}; r_{n-1}^{(a_2,a_1)}; 0^{[n-1]} \\ \dots \\ 0^{[2n]}; r_0^{(a_2,a_1)} \end{bmatrix} \begin{bmatrix} x^{2n} \\ \dots \\ x^{2i} \\ \dots \\ x^i \\ \dots \\ 1 \end{bmatrix}. \quad (3)$$

So we have $r_n^{(a_2,a_1,a_0)} = r_n^{(1,a_0)} \begin{bmatrix} r_n^{(a_2,a_1)}; 0^{[n]} \\ 0^{[2(n-i)}]; r_i^{(a_2,a_1)}; 0^{[i]} \\ \dots \\ 0^{[2n]}; r_0^{(a_2,a_1)} \end{bmatrix}$ by (1) and (3). \square

THEOREM 4. For $1 \leq a, b, c \leq 9$, the column sum table of $r_n^{(c,b,a)}$ with base 10^k ($k > 0$) yields the n th power of $10^{2k}c + 10^kb + a$. If $k = 1$, the column sum table of $r_n^{(c,b,a)}$ with base 10 gives $S_{10}^{r_n^{(c,b,a)}}$, the n th power of three digit integer cba .

The proof is similar to Theorem 2. For instance with $r_5 = (1, 5, 10, 10, 5, 1)$,

$$r_5^{(1,1,1)} = r_5 \begin{bmatrix} r_5; 0^{[5]} \\ 0^{[2]}; r_4; 0^{[4]} \\ \dots \\ 0^{[10]}; r_0 \end{bmatrix} = r_5 \begin{bmatrix} 15 & 10 & 10 & 5 & 1 \\ & 1 & 4 & 6 & 4 & 1 \\ & & & 13 & 3 & 1 \\ & & & & 12 & 1 \\ & & & & & 1 & 1 \\ & & & & & & & 1 \end{bmatrix}$$

$$= (1, 5, 15, 30, 45, 51, 45, 30, 15, 5, 1),$$

which is the set of coefficients of $(x^2 + x + 1)^5$. Moreover by Theorem 1 and 3, the $r_5^{(3,5,7)}$ of $T^{(3,5,7)}$ equals

$$r_5^{(1,7)} \begin{bmatrix} r_5^{(3,5)}; 0^{[5]} \\ 0^{[2]}; r_4^{(3,5)}; 0^{[3]} \\ \dots \\ 0^{[8]}; r_1^{(3,5)}; 0^{[1]} \\ 0^{[10]}; r_0^{(3,5)} \end{bmatrix} = r_5(1, 7, \dots, 7^5) \begin{bmatrix} r_5(3^5, \dots, 1)(1, \dots, 5^5); 0^{[5]} \\ 0^{[2]}; r_4(3^4, \dots, 1)(1, \dots, 5^4); 0^{[4]} \\ \dots \\ 0^{[8]}; r_1(3, 1)(1, 5); 0^{[1]} \\ 0^{[10]}; r_0 \end{bmatrix}$$

$$= (1, 35, 490, 3430, 12005, 16807) \begin{bmatrix} 243 & 2025 & 6750 & 11250 & 9375 & 3125 \\ & 81 & 540 & 1350 & 1500 & 625 \\ & & & 27 & 135 & 225 & 125 \\ & & & & 9 & 30 & 25 \\ & & & & & 3 & 5 \\ & & & & & & & 1 \end{bmatrix}$$

(4)
 $= (243, 2025, 9585, 30150, 69855, 121775, 162995, 164150, 121765, 60025, 16807)$,
 which is the set of coefficients of $(3x^2 + 5x + 7)^5$. So the column sum table of $r_5^{(3,5,7)}$ with base 10 shows $357^5 = 5798839393557$ (Table 6). Moreover $30507^5 = 26423936089214873669307$ is due to Table 7.

Table 6	Table 7
35505550557 42855795620 20518791708 2909124106 36266261 1111	0 0 0 0 0 0 0 0 0 07 3 5 5 0 5 5 5 0 5 50 4 2 8 5 5 7 9 5 6 28 2 0 5 1 8 7 9 1 7 06 2 9 0 9 1 2 4 1 01 3 6 2 6 6 2 6 1 1 1 1
$S_{10}^{r_5^{(3,5,7)}}$ 5798839393557	$S_{10^2}^{r_5^{(3,5,7)}}$ 26423936089214873669307

The table in (4) yields not only $r_5^{(3,5,7)}$ but also $r_i^{(3,5,7)}$ ($1 \leq i \leq 5$). In fact,

$$r_4^{(3,5,7)} = r_4(1, \dots, 7^4) \begin{bmatrix} 81 & 540 & 1350 & 1500 & 625 \\ & 27 & 135 & 225 & 125 \\ & & & 9 & 30 & 25 \\ & & & & 3 & 5 \\ & & & & & 1 \end{bmatrix}$$

$$= (81, 540, 2106, 5280, 9571, 12320, 11466, 6860, 2401),$$

which is the coefficients set of $(3x^2 + 5x + 7)^4$. So $357^4 = S_{10}^{r_4^{(3,5,7)}} = 16243247601$ is obtained through the column sum table of $r_4^{(3,5,7)}$ with

base 10. Now we are ready to have powers of any $k + 1$ digit integer by AT $T^{(a_k, \dots, a_0)}$ of $(a_k x^k + \dots + a_0)^m$ and its i th row $r_i^{(a_k, \dots, a_0)}$ for $k, i \geq 0$.

THEOREM 5. $r_n^{(a_k, \dots, a_0)} = r_n^{(1, a_0)} \begin{bmatrix} r_n^{(a_k, \dots, a_1)}; 0^{[n]} \\ \dots \\ 0^{[(n-i)k]}; r_i^{(a_k, \dots, a_1)}; 0^{[i]} \\ \dots \\ 0^{[nk]}; r_0^{(a_k, \dots, a_1)} \end{bmatrix}$. So the col-

umn sum table of $r_n^{(a_k, \dots, a_0)}$ with base 10 yields the n th power of $k + 1$ digit integer $a_k \dots a_1 a_0$ ($1 \leq a_i \leq 9, 0 \leq i \leq k$).

Proof. Let $f(x) = (a_k x^k + \dots + a_1 x + a_0)^n = (X + a_0)^n$ with $X = x(a_k x^{k-1} + \dots + a_1)$. Then $r_n^{(a_k, \dots, a_0)} \begin{bmatrix} x^{nk} \\ \dots \\ 1 \end{bmatrix} = r_n^{(1, a_0)} \begin{bmatrix} X^n \\ \dots \\ 1 \end{bmatrix}$ and

$$X^i = r_i^{(a_k, \dots, a_1)} \begin{bmatrix} x^{ik} \\ \dots \\ x^i \\ 1 \end{bmatrix} = (0^{[(n-i)k]}; r_i^{(a_k, \dots, a_1)}; 0^{[i]}) \begin{bmatrix} x^{nk} \\ \dots \\ x^{ik} \\ \dots \\ 1 \end{bmatrix}.$$

Thus we have $r_n^{(a_k, \dots, a_0)} = r_n^{(1, a_0)} \begin{bmatrix} r_n^{(a_k, \dots, a_1)}; 0^{[n]} \\ \dots \\ 0^{[(n-i)k]}; r_i^{(a_k, \dots, a_1)}; 0^{[i]} \\ \dots \\ 0^{[nk]}; r_0^{(a_k, \dots, a_1)} \end{bmatrix}$.

Since $r_n^{(a_k, \dots, a_0)}$ is the set of coefficients of $f(x)$, we have $f(10) = (a_k 10^k + \dots + a_1 10 + a_0)^n = \underbrace{(a_k \dots a_0)}_{k+1 \text{ digit}}^n$. So n th power of the $k + 1$ digit

integer $a_k \dots a_1 a_0$ is obtained through the column sum table of $r_n^{(a_k, \dots, a_0)}$ with base 10. □

From $r_5^{(1,1,1,1)} = (1, 5, 15, 35, 65, 101, 135, 155, 155, 135, 101, 65, 35, 15, 5, 1)$ by Theorem 5, the power $1111^5 = 1692662195786551$ is obtained by the column sum table of $r_5^{(1,1,1,1)}$ with base 10 (Table 8).

Moreover the coefficients of $(x^4 + x^3 + x^2 + x + 2)^5$ equals

$$r_5(1, 1, 1, 1, 2) = r_5^{(1,2)} \begin{bmatrix} r_5^{(1,1,1,1)}, 0^{(5)} \\ 0^{[4]}; r_4^{(1,1,1,1)}, 0^{[4]} \\ \dots \\ 0^{[20]}; r_0^{(1,1,1,1)} \end{bmatrix}$$

$$= (1, 5, 10, 10, 5, 1)(1, 2, \dots, 2^5) \begin{bmatrix} 1515356510113515515513510165351551 \\ 1410203140444031201041 \\ 136035530631 \\ 111111 \\ 11111 \\ 1 \end{bmatrix}$$

$= (1, 5, 15, 35, 75, 141, 235, 355, \dots, 855, 745, 601, 450, 280, 160, 80, 32)$.
So the power $11112^5 = 169418629274859896832$ is obtained from the column sum table of $r_5^{(1,1,1,1,2)}$ with base 10 (Table 9).

Table 8		Table 9	
1555515555155551	136035530631	155551555515555100002	1374350586054058683
111111	1692662195786551	12356789876421	169418629274859896832
$S_{10}^{r_5^{(1,1,1,1,1)}}$		$S_{10}^{r_5^{(1,1,1,1,2)}}$	

3. Negative powering of integers

We turn our attention to powers of integers with negative exponent. Let $T^{(b,a)}$ be the AT of $(bx + a)^{-m}$ for $m > 0$, and $r_n^{(b,a)}$ and $e_{n,j}^{(b,a)}$ ($n \geq 1, j \geq 0$) be the n th row and the (n, j) th component of $T^{(b,a)}$, respectively. We write $T^{(1,1)} = T'$, $r_n^{(1,1)} = r'_n$ and $e_{n,j}^{(1,1)} = e'_{n,j}$ for short. In $T' = [e'_{n,j}]$, $(x + 1)^{-m}$ is expanded in ascending powers of x , and the series $\sum_{j=0}^{\infty} e'_{n,j}x^j$ of $r'_n = (e'_{n,0}, e'_{n,1}, e'_{n,2}, \dots)$ converges if $|x| < 1$.

m	AT T' of $(x + 1)^{-m}$									
1	1	-1	1	-1	1	-1	1	-1	1	...
2	1	-2	3	-4	5	-6	7	-8	9	...
3	1	-3	6	-10	15	-21	28	-36	45	...
4	1	-4	10	-20	35	-56	84	-120	165	...
5	1	-5	15	-35	70	-126	210	-330	495	...

Let $r'_{n,A} = (e'_{n,0}, \dots, e'_{n,2l}, \dots)$ and $r'_{n,B} = (e'_{n,1}, \dots, e'_{n,2l+1}, \dots)$ be subsets consisting of even and odd entries of r'_n respectively, and consider the column sum tables of each. Indeed, from $r'_6 = (1, -6, 21, -56, 126, -252, 462, -792, 1287, \dots)$, the column sum subtables of subset $r'_{6,A} = (1, 21, 126, 462, 1287, \dots)$ and $r'_{6,B} = (-6, -56, -252, -792, \dots)$ with base 10^{-2} are

$S_{10^{-2}}^{r'_{6,A}}$	1.2 1 2 6 ⑥ ② 8 7	and	$-S_{10^{-2}}^{r'_{6,B}}$	0.6 5 6 ⑤ ② 9 2
	1 ④ 1 2			② 7
	1.2 2 3 0 7 4 8 7			0.6 5 8 5 9 9 2

So the columns sums show

$S_{10^{-2}}^{r'_{6,A}} + S_{10^{-2}}^{r'_{6,B}} \approx (1.22307487) - (0.6585992) \approx 0.56447567 \approx \frac{10^6}{11^6}$, (\approx denotes approximation), and $11^{-6} \approx (0.56447)10^{-6}$ is obtained. With a help of the example, we have an effective way for powering of 11 with negative exponent, which may not be obtained easily by algebraic manipulations.

THEOREM 6. *Let $S_{10^{-2}}^{r'_{n,A}}$ and $S_{10^{-2}}^{r'_{n,B}}$ be the column sums of $r'_{n,A}$ and $r'_{n,B}$ respectively. Then $11^{-n} \approx (S_{10^{-2}}^{r'_{n,A}} + S_{10^{-2}}^{r'_{n,B}})10^{-n}$.*

Proof. Let $f(x) = (x + 1)^{-n}$. Then with $x = 0.1 < 1$, $f(0.1) = \frac{10^n}{11^n}$.

By Taylor series expansion, the coefficient of x^j in $f(x)$ equals $e'_{n,j} = \binom{-n}{j} = (-1)^j \frac{n(n+1)\cdots(n+j-1)}{j!}$. So $e'_{n,j} > 0$ if j is even, otherwise $e'_{n,j} < 0$. Hence the subsets $r'_{n,A} = (e'_{n,0}, \dots, e'_{n,2l}, \dots)$ and $r'_{n,B} = (e'_{n,1}, \dots, e'_{n,2l+1}, \dots)$ are classified according to the sign of $e'_{n,j}$. And the column sum subtable of each $r'_{n,A}$ and $r'_{n,B}$ with base 10^{-2} produce $S_{10^{-2}}^{r'_{n,A}}$ and $S_{10^{-2}}^{r'_{n,B}}$ respectively. It thus follows that

$$\frac{10^n}{11^n} = f(0.1) \approx S_{10^{-2}}^{r'_{n,A}} + S_{10^{-2}}^{r'_{n,B}}, \text{ i.e., } 11^{-n} \approx (S_{10^{-2}}^{r'_{n,A}} + S_{10^{-2}}^{r'_{n,B}})10^{-n}. \quad \square$$

For instance, with $r'_1 = (1, -1, 1 - 1, 1, \dots)$, the column sum table of $r'_{1,A} = (1, 1, 1, \dots)$ and $r'_{1,B} = (-1, -1, -1, \dots)$ with base 10^{-2} yield $S_{10^{-2}}^{r'_{1,A}} \approx 1.0101010$ and $S_{10^{-2}}^{r'_{1,B}} \approx -0.1010101$. Thus we have

$$S_{10^{-2}}^{r'_{1,A}} + S_{10^{-2}}^{r'_{1,B}} \approx 0.9090909 \approx \frac{10}{11}, \text{ and } 11^{-1} \approx (0.9090909)10^{-1}.$$

Moreover from $r'_{10} = (1, -10, 55, -220, 715, -2002, 5005, -11440, 24310, \dots)$, $r'_{10,A} = (1, 55, 715, 5005, 24310, \dots)$ and $r'_{10,B} = (-10, -220, -2002, -11440, \dots)$, we have $S_{10^{-2}}^{r'_{10,A}} \approx 1.62674810$ and $S_{10^{-2}}^{r'_{10,B}} \approx -1.2411640$ by the tables

1.55150510		1.0200240
75043		22014
2	and	1
$S_{10^{-2}}^{r'_{10,A}}$		$-S_{10^{-2}}^{r'_{10,B}}$
1.62674810		1.2411640

Thus we immediately have

$$11^{-10} = (S_{10^{-2}}^{r'_{10,A}} + S_{10^{-2}}^{r'_{10,B}})10^{-10} \approx (0.3855)10^{-10}.$$

A generalization to two digit integer $ba = 10b + a$ ($0 \leq a, b \leq 9$) with negative exponent is as follows. We need a lemma in [2].

LEMMA 7. *Let $T^{(b,a)} = [e'_{i,j}^{(b,a)}]$ be an AT of $(bx + a)^{-m}$. Then $r_n^{(b,a)} = r'_n(\frac{1}{a^n}, \frac{b}{a^{n+1}}, \frac{b^2}{a^{n+2}}, \dots)$ and $e'_{n,j}^{(b,a)} = e'_{n,j} \frac{b^j}{a^{n+j}}$.*

Let $r_{n,A}^{(b,a)} = (e_{n,0}^{(b,a)}, \dots, e_{n,2l}^{(b,a)}, \dots)$ and $r_{n,B}^{(b,a)} = (e_{n,1}^{(b,a)}, \dots, e_{n,2l+1}^{(b,a)}, \dots)$ be subsets of the n th row $r_n^{(b,a)}$ of $T^{(b,a)}$ defined as above.

THEOREM 8. *Let $S_{10^{-2}}^{r_{n,A}^{(b,a)}}$ and $S_{10^{-2}}^{r_{n,B}^{(b,a)}}$ be the sums of column sum tables of $r_{n,A}^{(b,a)}$ and $r_{n,B}^{(b,a)}$ with base 10^{-2} , respectively. Then the negative n th power of two digit integer ba equals $(ba)^{-n} \approx (S_{10^{-2}}^{r_{n,A}^{(b,a)}} + S_{10^{-2}}^{r_{n,B}^{(b,a)}})10^{-n}$.*

The proof follows immediately from Lemma 7 and Theorem 6. For example, in order to get 13^{-5} , consider $f(x) = (3x+1)^{-5}$. For $x = 0.1 < 1$, $f(x) = \frac{10^5}{13^5}$. And the coefficients of $f(x) = (3x+1)^{-5}$ equals

$$\begin{aligned} r_5^{(3,1)} &= r_5'(1, 3, 3^2, \dots) \\ &= (1, -5, 15, -35, 70, -126, 210, -330, 495, -715, 1001, -1365, 1820, -2380, 3060, \\ &\quad -3876, 4845, -5985, 7315, -8855, \dots)(1, 3, 3^2, \dots, 3^{19}, \dots) \\ &= (1, -15, 135, -945, 5670, -30618, 153090, -721710, 3247695, -14073345, 59108049, \\ &\quad -241805655, 967222620, -3794488740, 14635885140, \dots). \end{aligned}$$

With two subsets $r_{5,A}^{(3,1)}$ and $r_{5,B}^{(3,1)}$ of $r_5^{(3,1)}$, the column sum tables

$$\begin{array}{c|c} \begin{array}{l} 1.3570909549204045 \\ 1.56307680265132 \\ 152410228836 \\ 359673561 \\ 94685 \\ 120 \end{array} & \text{and} \\ \hline S_{10^{-2}}^{r_{5,A}^{(3,1)}} & \begin{array}{l} 1.54518104555403255 \\ 906173356873555 \\ 3720780483687 \\ 1441941603 \\ 2375629 \\ 577 \end{array} \\ \hline S_{10^{-2}}^{r_{5,A}^{(3,1)}} & -S_{10^{-2}}^{r_{5,B}^{(3,1)}} \end{array} \begin{array}{l} 3.1096121925077245 \\ 2.84028519582628755 \end{array}$$

with base 10^{-2} yield

$$\frac{10^5}{13^5} \approx S_{10^{-2}}^{r_{5,A}^{(3,1)}} + S_{10^{-2}}^{r_{5,B}^{(3,1)}} \approx 0.269326997, \text{ so } 13^{-5} \approx (0.269326997)10^{-5}.$$

THEOREM 9. *Consider $T^{(1,1,a)}$ of $(x^2 + x + a)^{-m}$ and its n th row $r_n^{(1,1,a)}$. Then $r_n^{(1,1,a)} = r_n^{(1,a)} \begin{bmatrix} r_0; 0^{[\infty]} \\ \dots \\ 0^{[i]}; r_i; 0^{[\infty]} \\ \dots \end{bmatrix}$.*

Proof. With $X = x(x+1)$, we clearly have

$$r_n^{(1,1,a)} \begin{bmatrix} 1 \\ x \\ \dots \end{bmatrix} = (x^2 + x + a)^{-n} = (X + a)^{-n} = r_n^{(1,a)} \begin{bmatrix} 1 \\ X \\ \dots \end{bmatrix},$$

and $X^i = x^i r_i \begin{bmatrix} 1 \\ \dots \\ x^i \end{bmatrix} = r_i \begin{bmatrix} x^i \\ \dots \\ x^{2i} \end{bmatrix} = (0^{[i]}; r_i; 0^{[\infty]}) \begin{bmatrix} 1 \\ \dots \\ x^i \\ \dots \end{bmatrix}$. Thus we have

$$r_n'^{(1,1,a)} \begin{bmatrix} 1 \\ x \\ \dots \\ x^i \\ \dots \end{bmatrix} = r_n'^{(1,a)} \begin{bmatrix} r_0, 0^{[\infty]} \\ 0^{[1]}; r_1; 0^{[\infty]} \\ \dots \\ 0^{[i]}; r_i; 0^{[\infty]} \\ \dots \end{bmatrix} \begin{bmatrix} 1 \\ x \\ \dots \\ x^i \\ \dots \end{bmatrix}. \quad \square$$

For example, with $r'_1 = (1, -1, 1, -1, \dots)$, Theorem 9 shows

$$r_1'^{(1,1,1)} = r_1' \begin{bmatrix} r_0 \\ \dots \\ 0^{[i]}; r_i \\ \dots \end{bmatrix} = r_1' \begin{bmatrix} 1 \\ 11 \\ 121 \\ 1331 \\ \dots \end{bmatrix} = (1, -1, 0, 1, -1, 0, 1, -1, 0, \dots).$$

Similarly, $r_2'^{(1,1,1)}$ also follows from $r'_2 = (1, -2, 3, -4, 5, -6, \dots)$ that

$$r_2'^{(1,1,1)} = r_2' \begin{bmatrix} 1 \\ 11 \\ 121 \\ 1331 \\ 14641 \\ \dots \end{bmatrix} = (1, -2, 1, 2, -4, 2, 3, -6, 3, 4, -8, \dots).$$

Continuing with r'_n , the AT $T'^{(1,1,1)}$ of $(x^2 + x + 1)^{-m}$ forms as follows.

m	$T'^{(1,1,1)}$ of $(x^2 + x + 1)^{-m}$																
1	1	-1	0	1	-1	0	1	-1	0	1	-1	0	1	-1	0	1	...
2	1	-2	1	2	-4	2	3	-6	3	4	-8	4	5	-10	5	6	...
3	1	-3	3	2	-9	9	3	-18	18	4	-30	30	5	-45	45	6	...
4	1	-4	6	0	-15	24	-6	-36	60	-20	-70	120	-45	-120	210	-84	...
5	1	-5	10	-5	-20	49	-35	-50	145	-115	-100	335	-280	-175	665	-574	...

Now for 111^{-2} , consider $f(x) = (x^2 + x + 1)^{-2}$. Then $f(0.1) = (1.11)^{-2} = 111^{-2} \cdot 10^4$. From $r_2'^{(1,1,1)} = (1, -2, 1, 2, -4, 2, 3, -6, 3, 4, \dots)$, the expansion

$$(x^2 + x + 1)^{-2} = 1 - 2x + x^2 + 2x^4 - 4x^4 + 2x^5 + 3x^6 + \dots$$

follows easily. Moreover we consider three subsets of $r_2'^{(1,1,1)}$ that

$$\begin{aligned} r_{2,A}'^{(1,1,1)} &= \{e_{2,j}'^{(1,1,1)} \mid j \equiv 0 \pmod{3}\} = (1, 2, 3, 4, 5, \dots), \\ r_{2,B}'^{(1,1,1)} &= \{e_{2,j}'^{(1,1,1)} \mid j \equiv 1 \pmod{3}\} = (-2, -4, -6, -8, \dots) \text{ and} \\ r_{2,C}'^{(1,1,1)} &= \{e_{2,j}'^{(1,1,1)} \mid j \equiv 2 \pmod{3}\} = (1, 2, 3, 4, 5, \dots). \end{aligned}$$

Then each column sum subtables with base 10^{-3} give rise to sums $S_{10^{-3}}^{r_{2,A}'^{(1,1,1)}} \approx 1.002003004005$, $S_{10^{-3}}^{r_{2,B}'^{(1,1,1)}} \approx -0.2004006008$, and $S_{10^{-3}}^{r_{2,C}'^{(1,1,1)}} \approx 0.01002003004005$. Hence we have

$$f(0.1) = S_{10^{-3}}^{r_{2,A}'^{(1,1,1)}} + S_{10^{-3}}^{r_{2,B}'^{(1,1,1)}} + S_{10^{-3}}^{r_{2,C}'^{(1,1,1)}} \approx 0.8116224332,$$

so $111^{-2} \approx (0.8116224332)10^{-4}$ is obtained from the Pascal triangle.

Similarly for 111^{-3} , consider $r_3'^{(1,1,1)} = (1, -3, 3, 2, -9, 9, 3, -18, 18, 4, -30, \dots)$ and three subsets $r_{3,A}'^{(1,1,1)} = (1, 2, 3, 4, 5, \dots)$, $r_{3,B}'^{(1,1,1)} = (-3, -9, -18, -30,$

$-45, \dots)$ and $r_{3,C}^{/(1,1,1)} = (3, 9, 18, 30, 45, \dots)$. Then the corresponding column sums $S_{10^{-3}}^{r_{3,A}^{/(1,1,1)}}$, $S_{10^{-3}}^{r_{3,B}^{/(1,1,1)}}$ and $S_{10^{-3}}^{r_{3,C}^{/(1,1,1)}}$ follow from the column sums table with base 10^{-3} that

$$\begin{array}{c|c} S_{10^{-3}}^{r_{3,A}^{/(1,1,1)}} & 1.002003004005 \\ \hline S_{10^{-3}}^{r_{3,A}^{/(1,1,1)}} & 1.002003004005 \\ \hline -S_{10^{-3}}^{r_{3,B}^{/(1,1,1)}} & \begin{array}{c} 0.3009008000005 \\ \hline 1 \quad 3 \quad 4 \\ \hline 0.3009018030045 \end{array} \end{array}$$

Hence we have

$$\begin{aligned} & S_{10^{-3}}^{r_{3,A}^{/(1,1,1)}} + S_{10^{-3}}^{r_{3,B}^{/(1,1,1)}} + S_{10^{-3}}^{r_{3,C}^{/(1,1,1)}} \\ & \approx 1.002003004005 - 0.3009018030045 + 0.03009018030045 \\ & \approx 0.7311913813. \end{aligned}$$

Since $(1.11)^{-3} = 111^{-3}10^6$, we derive $111^{-3} \approx (0.7311913813)10^{-6}$, where the calculations are basically based on Pascal triangle.

Note that, for 111^{-4} , we may consider 6 subsets $\{1, 0\}$, $\{-4, -15\}$, $\{6, 24\}$, $\{-6, -20, -45, -84\}$, $\{36, -70, -120\}$, and $\{60, 120, 210\}$ of $r_4^{/(1,1,1)}$.

The column sum tables of the first three subsets are 1 , $\frac{0.4005}{0.4015}$, and $\frac{0.06004}{2}$. So with only these values, we have $1.0 - 0.4015 + 0.06024 = 0.65874$, where it already provides $(0.65874)10^{-8} \approx 111^{-4}$. For more precise value of 111^{-4} , we may use more column sum tables from the rest subsets that

$$\begin{array}{c} 0.000006000005004 \quad 0.0000006000000 \quad 0.00000000000000 \\ \hline 2 \quad 4 \quad 8 \quad \quad \quad 3 \quad 7 \quad 120 \quad \quad \quad 6 \quad 12 \quad 210 \\ \hline 0.000006020045084 \quad 0.0000036070120 \quad 0.00000060120210 \end{array}$$

Then together with 0.65874 above, we have $0.65874 - 0.000006020045084 - 0.000003607012 + 0.0000006012021$ equals 0.6587309742 , that gives $111^{-4} \approx (0.6587309742)10^{-8}$.

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