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ON KU-ALGEBRAS CONTAINING (α, β)-US SOFT SETS

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Abstract. In this paper, we connect (α, β) union soft sets and their ideal related properties with KU-algebras. In particular, we will study (α, β)-union soft sets, (α, β)-union soft ideals, (α, β)-union soft commutative ideals and ideal relations in KU-algebras. Finally, a characterization of ideals in KU-algebras in terms of (α, β)-union soft sets have been provided.

1. Introduction

The soft set theory along with rough set theory are strong tools to work with uncertainty, vagueness and decision making problems. These two concepts are widely studied in classical as well as logical algebraic structures. Soft and rough set theory play an important role in fuzzy and neutrosophic algebraic structures.
Soft set theory was introduced by Molodtsov in [10]. Then after many researchers shown their interest in the direction of soft set theory associated to different algebraic structures. Maji et al. [9] have connected soft set theory in decision making problems. Aktas et al. [1] have defined soft groups with soft sets. Later on many researchers brought this concept with logical algebras. Jun et al. [5] studied ideal theory in BCK/BCI-algebras based on soft sets and $\mathcal{N}$-structures. Gulistan and Shahzad [3] defined soft KU-algebras. Gulistan et al. [4] defined (α, β)

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fuzzy KU ideals of KU-algebras. Xi et al. [17] defined a new type of soft ideal of KU-algebras whereas Xi et al. [18] defined a new type soft prime ideal of KU-algebras. Jana and Pal [6] studied (α, β)-US sets in BCK/BCI-algebras. Prabpayak and Leerawat [13] introduced KU-algebras and their ideals [14]. Recently Ali et al. defined and studied Pseudo-metric and n-ary block codes on KU-algebras in [7] and [8] respectively. Senapati and Shum [16], have studied Atanassov's intuitionistic fuzzy bi-normed KU-ideals of a KU-algebra. Roohallah et al. [15] showed a representation for radicals in pseudo BL-algebras. Zhang [19] et al. connected soft set theoretical approach to pseudo-BCI algebras.

2. KU-Algebras and Soft Sets

In this section, we consider those definitions and examples that are used throughout this paper. For more information regarding KU-algebras and their ideals reader can go through [13] and [14].

Definition 1. [13] By a KU-algebra we mean an algebra $(K, \circ, 1)$ of type $(2, 0)$ with a single binary operation $\circ$ that satisfies the following identities: for any $x, y, z \in K$,
(ku$_1$) $(x \circ y) \circ [(y \circ z) \circ (x \circ z)] = 1$,
(ku$_2$) $x \circ 1 = 1$,
(ku$_3$) $1 \circ x = x$,
(ku$_4$) $x \circ y = y \circ x = 1$ implies $x = y$.

In what follows, let $(K, \circ, 1)$ denote a KU-algebra unless otherwise specified. For brevity we also call K a KU-algebra. The element 1 of K is called constant which is the fixed element of K. Partial order " $\leq$ " in K is denoted by the condition $x \leq y$ if and only if $y \circ x = 1$.

Lemma 1. [13] $(K, \circ, 1)$ *is a KU-algebra if and only if it satisfies:*
(ku$_5$) $x \circ y \leq (y \circ z) \circ (x \circ z)$,
(ku$_6$) $x \leq 1$,
(ku$_7$) $x \leq y$, $y \leq x$ *implies* $x = y$,

Lemma 2. *In a KU-algebra, the following identities are true [11]:*
(ku$_8$) $z \circ z = 1$,
(ku$_9$) $z \circ (x \circ z) = 1$,
(ku$_{10}$) $x \leq y$ *imply* $y \circ z \leq x \circ z$,

(ku_{11}) $z \circ (y \circ x) = y \circ (z \circ x)$, *for all* $x, y, z \in K$,
(ku_{12}) $y \circ [(y \circ x) \circ x] = 1$.

EXAMPLE 1. [11] Let $K = \{1, 2, 3, 4, 5\}$ in which $\circ$ is defined by the following table

$\circ$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	3	4	5
3	1	2	1	4	4
4	1	1	3	1	3
5	1	1	1	1	1

It is easy to see that K is a KU-algebra.

DEFINITION 2. A non-empty subset Y of a KU-algebra K is called subalgebra of K if $x \circ y \in Y$ for all $x, y \in Y$.
A KU-algebra K is said to be commutative if $(x \circ y) \circ y = (y \circ x) \circ x$ for all $x, y \in K$. A subset I of a KU-algebra K is called an ideal of K if:
(ku_{13}) $1 \in I$,
(ku_{14}) $x \in I$ and $x \circ y \in I \Rightarrow y \in I$ for all $x, y \in K$.

A subset I of a KU-algebra K is called a commutative ideal if it satisfies (ku_{13}) and for $z \in I$ we have,
(ku_{15}) $z \circ (y \circ x) \in I \Rightarrow ((x \circ y) \circ y) \circ x \in I$.

An ideal I of a KU-algebra K is called commutative if $y \circ x \in I \Rightarrow ((x \circ y) \circ y) \circ x \in I$. By $\mathcal{X}$ we mean initial universal set, and E is the set of parameters. $\mathcal{P}(\mathcal{X})$ stands for power set of $\mathcal{X}$.

DEFINITION 3. [10] A pair $(\mathcal{K}, E)$ is called a soft set over $\mathcal{X}$, where $\mathcal{K}$ is a function given by: $\mathcal{K} : E \to \mathcal{P}(\mathcal{X})$. We say that, a soft set in the universe $\mathcal{X}$ is a parameterized family of subsets of the universal set $\mathcal{X}$. For $\varepsilon \in A, \mathcal{K}(\varepsilon)$ is the set of ε-elements of the soft $(\mathcal{K}, A)$ or can be considered to be set of ε-approximate elements of the soft set.

Here is an example for soft set in a topological spaces.

EXAMPLE 2. Let (K, τ) be a topological space, i.e., τ is a family of subsets of the set K called the open sets of K. Then, the family of open neighborhoods $N(x)$ of point x, where $N(x) = \{V \in \tau | x \in V\}$, may be considered as the soft set $(N(x), \tau)$.

DEFINITION 4. [10] Let A be a non-empty subset of E. Then a soft set $(\mathcal{K},\ E)$ over $\mathcal{X}$ satisfying the condition: $\mathcal{K}(x) = \emptyset$ for all $x \notin A$

is called the A-soft set over $\mathcal{X}$ and is denoted by $\mathcal{K}_A$, so an A-soft set $\mathcal{K}_A$ over $\mathcal{X}$ is a function $\mathcal{K}_A : E \to \mathcal{P}$ such that $\mathcal{K}_A(x) = \emptyset$ for all $x \notin A$. A soft set over U can be followed by the set of ordered pairs: $\mathcal{K}_A = \{(x, \mathcal{K}_A(x)) : x \in E, \mathcal{K}_A(x) \in \mathcal{P}(\mathcal{X})\}$.

Note that a soft set is a parameterized family of subsets of the set $\mathcal{X}$. A soft set $\mathcal{K}_A(x)$ may be an arbitrary, empty, and nonempty intersection. The set of all soft sets over $\mathcal{X}$ is denoted by $S(\mathcal{X})$.

Definition 5. [12] Let $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$. For all $x \in E$, if $\mathcal{K}_A(x) = \emptyset$, then $\mathcal{K}_A$ is said to be an empty soft set and symbolized by Φ_A. If $\mathcal{K}_A(x) = \mathcal{X}$, then $\mathcal{K}_A$ is said to be an A-universal soft set and symbolized as $\mathcal{K}_{\overline{A}}$. If $\mathcal{K}_A(x) = \mathcal{X}$ and $A = E$, then $\mathcal{K}_{\overline{A}}$ is said to be a universal soft set and is denoted by $\mathcal{K}_{\overline{\mathcal{E}}}$.

Proposition 1. [12] *Let* $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$. *Then,*
(i) $\mathcal{K}_A \overline{\cup} \mathcal{K}_A = \mathcal{K}_A,\ \mathcal{K}_A \overline{\cap} \mathcal{K}_A = \mathcal{K}_A$.
(ii) $\mathcal{K}_A \overline{\cup} \Phi_A = \mathcal{K}_A,\ \mathcal{K}_A \overline{\cap} \Phi_A = \Phi_A$.
(iii) $\mathcal{K}_A \overline{\cup} \mathcal{K}_{\mathcal{E}} = \mathcal{K}_{\mathcal{E}},\ \mathcal{K}_A \overline{\cap} \mathcal{K}_{\mathcal{E}} = \mathcal{K}_A$.
(iv) $\mathcal{K}_A \overline{\cup} \mathcal{K}_A^c = \mathcal{K}_{\mathcal{E}}, \mathcal{K}_{\tilde{A}}^c \overline{\cup} \mathcal{K}_A^c = \Phi_A$, *where* Φ_A *is an empty set.*

Definition 6. Let E be a KU-algebra and $(\mathcal{K}, A)$ be a soft set over KU-algebra $\mathcal{E}$. Then, $(\mathcal{K}, A)$ is called a soft KU-algebra over E if $\mathcal{K}(x)$ is a subalgebra of $\mathcal{E}$ for all $x \in E$.

Definition 7. Let E be a KU-algebra. Let $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$ for a given subalgebra A of E. Then, $\mathcal{K}_A$ is called a US algebra of A over $\mathcal{X}$ if, for all $x, y \in A$, it satisfies the following condition: $\mathcal{K}_A(x \circ y) \subseteq \mathcal{K}_A(x) \cup \mathcal{K}_A(y)$.

Definition 8. Let E be a KU-algebra and A be a subalgebra of $\mathcal{E}$. Let $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$. Then, $\mathcal{K}_A$ is called a US ideal over $\mathcal{X}$ if, for all $x, y \in A$, it satisfies the following condition:
(1) $\mathcal{K}_A(1) \subseteq \mathcal{K}_A(x)$
(2) $\mathcal{K}_A(y) \subseteq \mathcal{K}_A(x \circ y) \cup \mathcal{K}_A(x)$.

Definition 9. Let E be a KU-algebra. For a given subalgebras A of E, let $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$. Then, $\mathcal{K}_A$ is called a commutative ideal over $\mathcal{X}$ if, for all $x, y, z \in A$, it satisfies the following conditions:
(1) $\mathcal{K}_A(1) \subseteq \mathcal{K}_A(x)$
(2) $\mathcal{K}_A(((x \circ y) \circ y) \circ x) \subseteq \mathcal{K}_A(z \circ (y \circ x)) \cup \mathcal{K}_A(z)$.

Definition 10. Let $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$ and $\delta \subseteq \mathcal{X}$. Then, the δ-exclusion set of $\mathcal{K}_A$, denoted by $\mathcal{K}_A^\delta$, is defined by $\mathcal{K}_A^\delta(x) = \{x \in A | \mathcal{K}_A(x) \subseteq \delta\}$.

3. (α, β)-US Sets

In this section, we shall use to write $\mathcal{X}$ for initial universe, E for set of parameters, and $\diamond$ for a binary operation. Let $\mathcal{S}(\mathcal{X})$ be the set of all soft sets. We define (α, β)-US sets and illustrate them by some examples. We consider $\emptyset \subseteq \alpha \subset \beta \subseteq \mathcal{X}$.

Definition 11. For any non-empty subset A of E, consider the soft set $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$. Then, for all $x, y \in A$, the soft set $\mathcal{K}_A$ is called an (α, β)-US set over $\mathcal{X}$ if it satisfies the following condition: $\mathcal{K}_A(x \diamond y) \cap \beta \subseteq \mathcal{K}_A(x) \cup \mathcal{K}_A(y) \cup \alpha$.

Example 3. Consider universe set $\mathcal{X} = \{b_1, b_2, b_3, b_4, b_5\}$. Let the set of parameters be $E = \{\tau_1, \tau_2, \tau_3, \tau_4\}$ with the following table:

$\diamond$	τ_1	τ_2	τ_3	τ_4
τ_1	τ_1	τ_2	τ_3	τ_4
τ_2	τ_1	τ_1	τ_3	τ_4
τ_3	τ_1	τ_1	τ_1	τ_4
τ_4	τ_1	τ_2	τ_3	τ_1

We consider a soft set $\mathcal{K}_E$ over $\mathcal{X}$, which is given as $\mathcal{K}_E(\tau_1) = \{b_3, b_5\}$, $\mathcal{K}_E(\tau_2) = \{b_3, b_4, b_5\}, \mathcal{K}_E(\tau_3) = \{b_2, b_3, b_4, b_5\}$, and $\mathcal{K}_{\mathcal{E}}(\tau_4) = \{b_1, b_3, b_5\}$. Fix $\beta = \{b_1, b_2, b_3, b_5\}$ and $\alpha = \{b_2, b_3\}$. It can be seen that $\mathcal{K}_{\mathcal{E}}$ is an (α, β)-US set over $\mathcal{X}$.

Theorem 1. *Let $\mathcal{K}_A, \mathcal{K}_B \in \mathcal{S}(\mathcal{X})$ be soft sets such that $\mathcal{K}_A$ is a soft subset of $\mathcal{K}_B$. If $\mathcal{K}_B$ is an (α, β)-US set over $\mathcal{X}$, then the same holds for $\mathcal{K}_A$.*

Proof. Let $x, y \in A$ such that $x \diamond y \in A$. Then, $x \diamond y \in B$ since $A \subseteq B$. Thus, $\mathcal{K}_A(x \diamond y) \cap \beta \subseteq \mathcal{K}_B(x \diamond y) \cap \beta \subseteq \mathcal{K}_B(x) \cup \mathcal{K}_B(y) \cup \alpha = \mathcal{K}_A(x) \cup \mathcal{K}_A(y) \cup \alpha$. Therefore, $\mathcal{K}_A$ is an (α, β)-US set over $\mathcal{X}$. □

The following example shows that the converse of Theorem 1 is not true in general.

EXAMPLE 4. For the universe set $\mathcal{X} = \{b_1, b_2, b_3, b_4, b_5\}$, we consider the set of parameters $E = \{\tau_1, \tau_2, \tau_3, \tau_4\}$ with the following table:

$\diamond$	τ_1	τ_2	τ_3	τ_4
τ_1	τ_1	τ_2	τ_3	τ_4
τ_2	τ_2	τ_1	τ_4	τ_3
τ_3	τ_3	τ_3	τ_1	τ_2
τ_4	τ_4	τ_2	τ_2	τ_1

For $A = \{\tau_1, \tau_2\} \subset E$. Let $\mathcal{K}_E$ be a soft set over $\mathcal{X}$ as $\mathcal{K}_A(\tau_1) = \{b_1, b_3\}, \mathcal{K}_A(\tau_2) = \{b_1, b_3, b_4\}, \mathcal{K}_A(\tau_3) = \emptyset, \mathcal{K}_A(\tau_4) = \emptyset, \beta = \{b_1, b_3, b_4, b_5\}$, and $\alpha = \{b_3, b_4\}$. Then, it can be easily verified that $\mathcal{K}_A$ is an (α, β)-US set over $\mathcal{X}$.
Consider another soft set $\mathcal{K}_B$ as $\mathcal{K}_B(\tau_1) = \{b_1, b_3\}$, $\mathcal{K}_B(\tau_2) = \{b_1, b_3, b_4\}$, $\mathcal{K}_B(\tau_3) = \{b_2, b_4\}$, and $\mathcal{K}_B(\tau_4) = \{b_4, b_5\}$. Then, $\mathcal{K}_A$ is a soft subset of $\mathcal{K}_B$. However, for $\beta = \{b_1, b_3, b_4, b_5\}$ and $\alpha = \{b_3, b_4\}, \mathcal{K}_B$ is not an (α, β)-union soft set over $\mathcal{X}$, because $\mathcal{K}_B(\tau_3 \diamond \tau_4) \cap \beta = \{b_1, b_3, b_4\} \not\subseteq \{b_2, b_3, b_4, b_5\} = \mathcal{K}_B(\tau_3) \cup \mathcal{K}_B(\tau_4) \cup \alpha$.

4. (α, β)-US Subalgebras in KU-Algebras

In this section, we introduce the concept of the (α, β)-US subalgebra of KU-algebras and investigate some of its characterization. We shall consider E as a KU-algebras throughout this section.

DEFINITION 12. Let E be a KU-algebra. Let $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$ for a given subalgebra A of E. Then, $\mathcal{K}_A$ is called an (α, β)-US algebra of A over U if, for all $x, y \in A$, it satisfies the condition: $\mathcal{K}_A(y \circ x) \cap \beta \subseteq \mathcal{K}_A(x) \cup \mathcal{K}_A(y) \cup \alpha$.

We consider the pre-order relation $\subseteq_{(\alpha,\beta)}$ on $\mathcal{S}(\mathcal{X})$ as: for any $\mathcal{K}_E, \mathcal{L}_E \in \mathcal{S}(\mathcal{X})$ and $\emptyset \subseteq \alpha \subset \beta \subseteq \mathcal{X}$, we define $\mathcal{K}_\mathcal{E} \cap \beta \subseteq \mathcal{L}_\mathcal{E} \cup \alpha \Leftrightarrow \mathcal{K}_\mathcal{E}(x) \cap \beta \subseteq \mathcal{L}_\mathcal{E}(x) \cup \alpha$ for any $x \in E$. We define a relation $=_{(\alpha,\beta)}$ such as $\Leftrightarrow \mathcal{K}_E \cap \beta \subseteq \mathcal{L}_E \cup \alpha$ and $\mathcal{L}_E \cap \beta \subseteq \mathcal{K}_E \cup \alpha$. Using the above notion, the (α, β)-US subalgebras in KU-algebra is defined as follows:

DEFINITION 13. Let E be a KU-algebra. Let $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$ for a given subalgebra A of E. Then, $\mathcal{K}_A$ is called an (α, β)-US algebra of A over $\mathcal{X}$ if, for all $x, y \in A$, it satisfies the condition: $\mathcal{K}_A(y \circ x) \cap \beta \subseteq \mathcal{K}_A(x) \cup \mathcal{K}_A(y) \cup \alpha$.

Example 5. Let $K=\{1,a,b,c,d\}$ be a KU-algebra with the following Cayley table:

$\circ$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	c
b	1	1	1	c	c
c	1	1	1	1	a
d	1	1	1	a	1

Let $(\mathcal{K}_A, A)$ be a soft set over $\mathcal{X}=K$, where $E=A=K$ and $\mathcal{K}_A: A\longrightarrow \mathcal{S}(\mathcal{X})$ is a set-valued function defined by $\mathcal{K}_A(x)=\{y\in X|y\circ x=1\}$ for all $x\in A$. Then, $\mathcal{K}_A(1)=\{1\}, \mathcal{K}_A(a)=\{1,\ a\}, \mathcal{K}_A(b)=\{1,a,b\}$,

$\mathcal{K}_A(c)=\{1,a,b,c\}$, and $\mathcal{K}_A(d)=\{1,a,b,c,d\}$. It can be easily verified that $\mathcal{K}_A$ is an (α,β)-US algebra of A over $\mathcal{X}$, where $\beta=\{1,a,c,d\}$ and $\alpha=\{1,a,d\}$.

Theorem 2. *Let E be a KU-algebra, $\mathcal{K}_A\in\mathcal{S}(\mathcal{X})$ be a given subalgebra A of E, and $\beta\subseteq\mathcal{X}$. For $\delta\in\mathcal{X}$, $\mathcal{K}_A$ is an (α,β)-US subalgebra of A over $\mathcal{X}$ if and only if each non-empty subset $B(\mathcal{K}_A:\delta)$, which is defined by $B(\mathcal{K}_A:\delta)=\{x\in A|\mathcal{K}_A(x)\ \subseteq\delta\cup\alpha\}$ where $\delta\subseteq\beta$, is a subalgebra of A.*

Proof. Let $\mathcal{K}_A$ be an (α,β)-US algebra of A over $\mathcal{X}$ such that $\mathcal{K}_A(x)\subseteq\beta$ for every $x\in A$, and let $x,y\in B(\mathcal{K}_A:\delta)$. Then, $\mathcal{K}_A(y\circ x)\cap\beta\subseteq\mathcal{K}_A(x)\cup\mathcal{K}_A(y)\cup\alpha\subseteq\delta\cup\alpha$, which implies that $y\circ x\in B(\mathcal{K}_A:\delta)$.

Hence, $B(\mathcal{K}_A:\delta)$ is a subalgebra of A.

Conversely, let each non-empty subset $B(\mathcal{K}_A:\delta)$ be a subalgebra of A. Then, according to our assumption on $\mathcal{K}_A$, for $x,y\in A$, there are $\delta_1,\delta_2\subseteq\beta$ such that $\mathcal{K}_A(x)=\delta_1$ and $\mathcal{K}_A(y)=\delta_2$. Thus,

$\mathcal{K}_A(x)\subseteq\delta$ and $\mathcal{K}_A(y)\subseteq\delta$ for $\delta=\delta_1\cup\delta_2\subseteq\beta$. Hence, $x,y\in B(\mathcal{K}_A:\delta)$. Since $B(\mathcal{K}_A:\delta)$ is a subalgebra of A, so $y\circ x\in B(\mathcal{K}_A:\delta)$. Thus, $\mathcal{K}_A(y\circ x)\cap\beta\subseteq\delta$ and $\mathcal{K}_A(x)\cup\mathcal{K}_A(y)\cup\alpha=\delta_1\cup\delta_2\cup\alpha=\delta\cup\alpha$, which implies $\mathcal{K}_A(y\circ x)\cap\beta\subseteq\mathcal{K}_A(x)\cup\mathcal{K}_A(y)\cup\alpha$. Hence, the proof of the Theorem 2 is completed. □

Theorem 3. *Let E be a KU-algebra and $\mathcal{K}_A\in\mathcal{S}(\mathcal{X})$ be such that $A\subseteq E$. Then, $\mathcal{K}_A$ is an (α,β)-US algebra of A over $\mathcal{X}$ if, for all $x\in A$, it satisfies the condition: $\mathcal{K}_A(1)\cap\beta\subseteq\mathcal{K}_A(x)\cup\alpha$.*

Proof. If $1\notin A$, then $\mathcal{K}_A(1)\cap\beta=\emptyset\cap\beta\subseteq\mathcal{K}_A(x)\cup\alpha$ for all $x\in A$. If $1\in A$, then $\mathcal{K}_A(1)\cap\beta=\mathcal{K}_A(x\circ x)\cap\beta\subseteq\mathcal{K}_A(x)\cup\mathcal{K}_A(x)\cup\alpha=\mathcal{K}_A(x)\cup\alpha$ for all $x\in A$. Therefore, $\mathcal{K}_A(1)\cap\beta\subseteq\mathcal{K}_A(x)\cup\alpha$ holds. □

Theorem 4. *If a soft $\mathcal{K}_A$ over $\mathcal{X}$ is an (α, β)-US algebra of A, then:* $(\mathcal{K}_A(1) \cap \beta) \cup \alpha \subseteq (\mathcal{K}_A(x) \cap \beta) \cup \alpha$, *for all* $x \in A$.

Proof. Let $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$, and by using Theorem 3, we get: $(\mathcal{K}_A(1) \cap \beta) \cup \alpha = (\mathcal{K}_A(x \circ x) \cap \beta) \cup \alpha \subseteq ((\mathcal{K}_A(x) \cup \mathcal{K}_A(x) \cup \alpha) \cap \beta) \cup \alpha = ((\mathcal{K}_A(x) \cap \beta) \cup \alpha) \cup ((\mathcal{K}_A(x) \cap \beta) \cup \alpha) \subseteq (\mathcal{K}_A(x) \cap \beta) \cup \alpha$. Which completes the proof. □

Proposition 2. *Let E be a KU-algebra and $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$ for a given subalgebra A of E. Then, $\mathcal{K}_A$ is a (α, β)-US algebra of A over $\mathcal{X}$ if for all $x \in A$, it satisfies the condition:* $\mathcal{K}_A(y \circ x) \cap \beta \subseteq \mathcal{K}_A(y) \cup \alpha \Leftrightarrow \mathcal{K}_A(x) \cap \beta = \mathcal{K}_A(y) \cup \alpha$.

Proof. We assume that $\mathcal{K}_A(y \circ x) \cap \beta \subseteq \mathcal{K}_A(y) \cup \alpha$ for all $x, y \in A$. Take $y = 1$, and use (ku_6), which induces $\mathcal{K}_A(x) \cap \beta = \mathcal{K}_A(1 \circ x) \cap \beta \subseteq \mathcal{K}_A(1) \cup \alpha$. It follows from Theorem 3 that $\mathcal{K}_A(x) \cap \beta = \mathcal{K}_A(1) \cup \alpha$ for all $x \in A$.

Conversely, suppose that $\mathcal{K}_A(x) \cap \beta = \mathcal{K}_A(1) \cup \alpha$ for all $x \in A$. Then, $\mathcal{K}_A(y \circ x) \cap \beta \subseteq \mathcal{K}_A(x) \cup \mathcal{K}_A(y) \cup \alpha$
$= \mathcal{K}_A(1) \cup \mathcal{K}_A(y) \cup \alpha = \mathcal{K}_A(y) \cup \alpha$ for all $x, y \in A$. □

For a soft set $(\mathcal{K}_A, A)$ over E, we consider the set: $X_1 = \{x \in A | \mathcal{K}_A(x) = \mathcal{K}_A(1)\}$.

Theorem 5. *Let E be a KU-algebra and A a subalgebra of E. Let $(\mathcal{K}_A, A)$ be an (α, β)-US algebra over E. Then, the set $K_1^\circ = \{x \in A | (\mathcal{K}_A(x) \cap \beta) \cup \alpha = (\mathcal{K}_A(1) \cap \beta) \cup \alpha\}$ is a subalgebra of E.*

Proof. If $\mathcal{K}_A$ is an (α, β)-US algebra of A over $\mathcal{X}$, then $x, y \in X_1^\circ$; we have $(\mathcal{K}_A(x) \cap \beta) \cup \alpha = (\mathcal{K}_A(1) \cap \beta) \cup \alpha = (\mathcal{K}_A(y) \cap \beta) \cup \alpha$. Then, from Theorem 3, we have $(\mathcal{K}_A(1) \cap \beta) \cup \alpha \subseteq (\mathcal{K}_A(y \circ x) \cap \beta) \cup \alpha$ for all $x, y \in A$. This also takes the following form, $(\mathcal{K}_A(y \circ x) \cap \beta) \cup \alpha \subseteq ((\mathcal{K}_A(x) \cup \mathcal{K}_A(y) \cup \alpha) \cap \beta) \cup \alpha = ((\mathcal{K}_A(x) \cap \beta) \cup \alpha) \cup (\mathcal{K}_A(y) \cap \beta) \cup \alpha) \subseteq (\mathcal{K}_A(1) \cap \beta) \cup \alpha$. Hence, $(\mathcal{K}_A(y \circ x) \cap \beta) \cup \alpha = (\mathcal{K}_A(1) \cap \beta) \cup \alpha$, and so, $y \circ x \in X_1^\circ$. Thus, K_1° is a subalgebra of A. □

Theorem 6. *Let E be a KU-algebra and $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$. Define a soft set $\mathcal{K}_A^\circ$ over $\mathcal{X}$ by $\mathcal{K}_A^\circ : E \to \mathcal{P}(\mathcal{X})$, where $\mathcal{K}_A^\circ(x) = \mathcal{K}_A(x)$ if $x \in B(\mathcal{K}_A : \delta)$ and $\mathcal{K}_A^\circ(x) = \mathcal{X}$ if $x \notin B(\mathcal{K}_A : \delta)$. Further if $\mathcal{K}_A$ is an (α, β)-US algebra over $\mathcal{X}$, then so is $\mathcal{K}_A^\circ$.*

Proof. If $\mathcal{K}_A$ is an (α, β)-US algebra over $\mathcal{X}$, then $B(\mathcal{K}_A : \delta)$ is a subalgebra of A by Theorem 2. Let $x, y \in A$. If $x, y \in B(\mathcal{K}_A : \delta)$, then

$y \circ x \in B(\mathcal{K}_A : \delta)$, and so, $\mathcal{K}_A^{\circ}(y \circ x) \cap \beta = \mathcal{K}_A(y \circ x) \cap \beta \subseteq \mathcal{K}_A(x) \cup \mathcal{K}_A(y) \cup \alpha = \mathcal{K}_A^{\circ}(x) \cup \mathcal{K}_A^{\circ}(y) \cup \alpha$. If $x \notin B(\mathcal{K}_A : \delta)$ or $y \notin B(\mathcal{K}_A : \delta)$, then $\mathcal{K}_A^{\circ}(x) = \mathcal{X}$ or $\mathcal{K}_A^{\circ}(y) = \mathcal{X}$. Thus, we have: $\mathcal{K}_A^{\circ}(y \circ x) \cap \beta \subseteq \mathcal{X} = \mathcal{K}_A^{\circ}(x) \cup \mathcal{K}_A^{\circ}(y) \cup \alpha$. Therefore, $\mathcal{K}_A^{\circ}$ is an (α, β)-US algebra of A over $\mathcal{X}$. □

5. (α, β)-US Ideals and KU-Algebras

In this section, we define the (α, β)-US ideal and characterize their properties.

DEFINITION 14. Let A be subalgebra of a KU-algebra E. Let $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$, then $\mathcal{K}_A$ is called an (α, β)-US ideal over U if, for all $x, y \in A$, it satisfies Theorem 3 and the following condition: $\mathcal{K}_A(x) \cap \beta \subseteq \mathcal{K}_A(y \circ x) \cup \mathcal{K}_A(y) \cup \alpha$.

EXAMPLE 6. Let $\mathcal{X} = \mathbb{Z}$ be the universal set and $K = \{1, w, x, y, z\}$ be a KU-algebra with the following Cayley table:

$\circ$	1	w	x	y	z
1	1	w	x	y	z
w	1	1	x	y	z
x	1	w	1	y	x
y	1	1	1	1	x
z	1	w	1	y	1

For a subalgebra $A = \{1, x, y, z\}$ of E, define the soft set $(\mathcal{K}_A, A)$ over $\mathcal{X}$ as
$\mathcal{K}_A(1) = \{1, 3, 4, 5, 7, 9, 11, 12\}$,
$\mathcal{K}_A(x) = \{1, 2, 4, 5, 6, 7, 8, 11, 13\}$,
$\mathcal{K}_A(y) = \{2, 3, 5, 6, 8, 9, 13\}$ and
$\mathcal{K}_A(z) = \{1, 2, 3, 5, 8, 11, 13\}$.
Then, $\mathcal{K}_A$ is an (α, β)-US ideal of A over $\mathcal{X}$, where
$\alpha = \{1, 3, 6, 7, 9, 11, 11, 12\}$ and
$\beta = \{1, 2, 3, 6, 7, 8, 9, 11, 11, 12, 13\}$.

LEMMA 3. *Let A be subalgebra of a KU-algebra E. Let $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$, if $\mathcal{K}_A$ is a US ideal over $\mathcal{X}$, then for all $x, y \in A$: $x \leq y \Rightarrow \mathcal{K}_A(x) \subseteq \mathcal{K}_A(y)$.*

Proof. Let $x, y \in A$ be such that $x \leq y$. Then, $y \circ x = 1$, from which, by Definition 15 and Theorem 3, we get $\mathcal{K}_A(x) \subseteq \mathcal{K}_A(y \circ x) \cup \mathcal{K}_A(y) = \mathcal{K}_A(1) \cup \mathcal{K}_A(y) = \mathcal{K}_A(y)$. Hence, $\mathcal{K}_A(x) \subseteq \mathcal{K}_A(y)$. □

Lemma 4. *Let A be subalgebra of a KU-algebra E. Let $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$. If $\mathcal{K}_A$ is an (α,β)-US ideal over $\mathcal{X}$, then for all $x, y \in A : x \leq y \Rightarrow \mathcal{K}_A(x) \cap \beta \subseteq \mathcal{K}_A(y) \cup \alpha$.*

Proof. Let $x, y \in A$ be such that $x \leq y$. Then, $y \circ x = 1$, from which, by Definition 15 and Theorem 3 we get $\mathcal{K}_A(x) \cap \beta \subseteq \mathcal{K}_A(y \circ x) \cup \mathcal{K}_A(y) \cup \alpha = \mathcal{K}_A(1) \cup \mathcal{K}_A(y) \cup \alpha = \mathcal{K}_A(y) \cup \alpha$. Hence, $\mathcal{K}_A(x) \cap \beta \subseteq \mathcal{K}_A(y) \cup \alpha$. □

Proposition 3. *Let A be subalgebra of a KU-algebra E. Let $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$. If $\mathcal{K}_A$ is an*

(α,β)-US ideal over $\mathcal{X}$, then for all $x, y, z \in A, \mathcal{K}_A$ satisfies the following conditions:

(1) $\mathcal{K}_A(y \circ x) \cap \beta \subseteq \mathcal{K}_A(z \circ x) \cup \mathcal{K}_A(y \circ z) \cup \alpha$

(2) $\mathcal{K}_A(y \circ x) = \mathcal{K}_A(1) \Rightarrow \mathcal{K}_A(x) \cap \beta \subseteq \mathcal{K}_A(y) \cup \alpha$.

Proof. (1) Since $(z \circ x) \circ (y \circ x) \leq y \circ z$, then from Lemma 4, $\mathcal{K}_A((z \circ x) \circ (y \circ x)) \subseteq \mathcal{K}_A(y \circ z)$. Hence, $\mathcal{K}_A(y \circ x) \cap \beta \subseteq \mathcal{K}_A((z \circ x) \circ (y \circ x)) \cup \mathcal{K}_A(z \circ x) \cup \alpha \subseteq \mathcal{K}_A(z \circ x) \cup \mathcal{K}_A(y \circ z) \cup \alpha$.

(2) If $\mathcal{K}_A(y \circ x) = \mathcal{K}_A(1)$, then for all $x, y \in A$, $\mathcal{K}_A(x) \cap \beta \subseteq \mathcal{K}_A(y \circ x) \cup \mathcal{K}_A(y) \cup \alpha = \mathcal{K}_A(1) \cup \mathcal{K}_A(y) \cup \alpha = \mathcal{K}_A(y) \cup \alpha$. □

Proposition 4. *Let A be subalgebra of a KU-algebra E. If $\mathcal{K}_A$ is an (α,β)-US ideal over $\mathcal{X}$, then for all $x, y, z \in A$, the following conditions are equivalent:*

(1) $\mathcal{K}_A(x \circ y) \cap \beta \subseteq \mathcal{K}_A(x \circ (x \circ y)) \cup \alpha$.

(2) $\mathcal{K}_A((z \circ y) \circ (z \circ x)) \cap \beta \subseteq \mathcal{K}_A(z \circ (y \circ x)) \cup \alpha$.

Proof. Assume that (1) holds and $x, y, z \in A$. Since $z \circ (z \circ ((z \circ y)) \circ x) = z \circ (y \circ z) \circ (z \circ x) \leq z \circ (y \circ x)$ by (1), (ku_8), and Lemma 4, we obtain the following equality: $\mathcal{K}_A((z \circ y) \circ (z \circ x)) \cap \beta = \mathcal{K}_A(z \circ (y \circ z) \circ x) \cap \beta \subseteq \mathcal{K}_A(z \circ (z \circ (z \circ y) \circ x) \cup \alpha \subseteq \mathcal{K}_A(z \circ (y \circ x)) \cup \alpha$.

Again, assume that (2) holds. If we put $y = z$ in (2), then by (ku_3) and (ku_6), we get $\mathcal{K}_A((z \circ (z \circ x) \cup \alpha \supseteq \mathcal{K}_A((z \circ z) \circ (z \circ x)) \cap \beta = \mathcal{K}_A(1 \circ (z \circ x) \cap \beta = \mathcal{K}_A(z \circ x) \cap \beta$, which implies that (1) holds. □

Theorem 7. *Let A be subalgebra of a KU-algebra E. Then, every A-soft set is an (α,β)-US ideal over U, and an A-soft set is an (α,β)-US KU-algebra over $\mathcal{X}$.*

Proof. Let $\mathcal{K}_A$ be an (α,β)-US ideal over $\mathcal{X}$ and A a subalgebra of E. We get $y \circ x \leq x$ for all $x, y \in A$. Then, it follows from Lemma 4 that

$\mathcal{K}_A(y\circ x)\cap\beta \subseteq \mathcal{K}_A(x)\cup\alpha \subseteq \mathcal{K}_A(x\circ y)\cup\mathcal{K}_A(y)\cup\alpha \subseteq \mathcal{K}_A(x)\cup\mathcal{K}_A(y)\cup\alpha$. Hence, $\mathcal{K}_A$ is an (α,β)-US KU-algebra over $\mathcal{X}$. □

THEOREM 8. *Let E be a KU-algebra. Let $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$ and A be a subalgebra of E. If $\mathcal{K}_A$ is an (α,β)-US ideal over $\mathcal{X}$, then for all $x,y,z \in A, \mathcal{K}_A$ satisfies the following condition:*
$y\circ x \le z \Rightarrow \mathcal{K}_A(x)\cap\beta \subseteq \mathcal{K}_A(y)\cup\mathcal{K}_A(z)\cup\alpha$.

Proof. Let $x,y \in A$ be such that $y\circ x \le z$, then $z\circ(y\circ x) = 1 \Rightarrow$.
$\mathcal{K}_A(y\circ x)\cap\beta \subseteq \mathcal{K}_A(z\circ(y\circ x)\cup\mathcal{K}_A(z)\cup\alpha = \mathcal{K}_A(1)\cup\mathcal{K}_A(z)\cup\alpha = \mathcal{K}(z)\cup\alpha$. By using Definition 15 and Theorem 3, we get $\mathcal{K}_A(x)\cap\beta \subseteq \mathcal{K}_A(y\circ x)\cup \mathcal{K}_A(y) \subseteq \mathcal{K}_A(y)\cup\mathcal{K}_A(z)\cup\alpha$. □

THEOREM 9. *Let E be a KU-algebra. Given a subalgebra A of E, let $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$ and $\beta \subseteq \mathcal{X}$. Then, $\mathcal{K}_A$ is an (α,β)-US ideal over $\mathcal{X}$ if and only if the non-empty set $B(\mathcal{K}_A : \delta)$ is an ideal of A.*

Proof. The proof is same as proof of Theorem 2. □

6. (α,β)-US Commutative Ideals in KU-Algebras

DEFINITION 15. Let E be a KU-algebra. For a given subalgebras A of E, let $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$. Then, $\mathcal{K}_A$ is called an (α,β)-US commutative ideal over $\mathcal{X}$ if for all $x,y,z \in A$, it satisfies Theorem 3 and the following condition: $\mathcal{K}_A(((x\circ y)\circ y)\circ x)\cap\beta \subseteq \mathcal{K}_A(z\circ(y\circ x))\cup\mathcal{K}_A(z)\cup\alpha$.

EXAMPLE 7. Let $E = \{1,2,3,4,5\}$ be a KU-algebra with the following Cayley table:

$\circ$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	3	4	5
3	1	2	1	4	5
4	1	2	3	1	5
5	1	1	1	1	1

Let $(\mathcal{K}_A, A)$ be a soft set over $\mathcal{X}$, where $A = \{2,3,4,5\}$ and $\mathcal{K}_A : A \to P(X)$ is a set valued function defined by $\mathcal{K}_A(x) = \{y \in X | x\circ y \in \{1,3,4\}\}$. Then, $\mathcal{K}_A(1) = \emptyset$,
$\mathcal{K}_A(2) = \{y \in X|2\circ y \in \{1,3,4\}\} = \{1,2,3,4\}$,
$\mathcal{K}_A(3) = \{y \in X|3\circ y \in \{1,3,4\}\} = \{1,3,4\}$,
$\mathcal{K}_A(4) = \{y \in X|4\circ y \in \{1,3,4\}\} = \{1,3,4\}$,

$\mathcal{K}_A(5) = \{y \in X | 5 \circ y \in \{1, 3, 4\}\} = \{1, 2, 3, 4, 5\}$.
Then, $\mathcal{K}_A$ is an (α, β)-US commutative ideal of A over $\mathcal{X}$, where $\beta = \{1, 2, 4, 5\}$ and $\alpha = \{1, 2, 4\}$.

THEOREM 10. *Let E be a KU-algebra. Then, any (α, β)-US commutative ideal over $\mathcal{X}$ is an (α, β)-US ideal over $\mathcal{X}$.*

Proof. Let A be a subalgebra of E and $\mathcal{K}_A$ be an (α, β)-US commutative ideal over $\mathcal{X}$. Now, we put $y = 1$ in Definition 15 and use (ku_5) and (ku_6), then we have $\mathcal{K}_A(x) \cap \beta = \mathcal{K}_A(x \circ 1)(((x \circ 1) \circ 1) \circ x) \cap \beta \subseteq \mathcal{K}_A(z \circ (1 \circ x)) \cup \mathcal{K}_A(z) \cup \alpha = \mathcal{K}_A(z \circ x) \cup \mathcal{K}_A(z) \cup \alpha$ for all $x, z \in A$. Thus, $\mathcal{K}_A$ is an (α, β)-US ideal over $\mathcal{X}$. □

In view of the following example, we can also establish Theorem 10.

The following theorem provides the condition that an (α, β)-US ideal over $\mathcal{X}$ is an (α, β)-US commutative ideal over $\mathcal{X}$.

THEOREM 11. *Let E be a KU-algebra and A be a subalgebra of E. Let $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$, then $\mathcal{K}_A$ is an (α, β)-US. commutative ideal over $\mathcal{X}$ if and only if, for all $x, y, z \in A, \mathcal{K}_A$ is an (α, β)-$\mathcal{X}$ ideal over $\mathcal{X}$ satisfying the following condition: $\mathcal{K}_A(((x \circ y) \circ y) \circ x) \subseteq \mathcal{K}_A(y \circ x)$.*

Proof. Assume that $\mathcal{K}_A$ is an (α, β)-US ideal commutative ideal over $\mathcal{X}$. Then, $\mathcal{K}_A$ is an (α, β)-US soft ideal over $\mathcal{X}$ by Theorem 10. Now, if we take $z = 1$ in Definition 15 and use (ku$_5$), then we deduce the condition given in Theorem 11.
Conversely, if $\mathcal{K}_A$ is an (α, β)-US ideal over $\mathcal{X}$ satisfying the condition of Theorem 11, then for all $x, y, z \in A$, we have $\mathcal{K}_A(y \circ x) \cap \beta \subseteq \mathcal{K}_A(z \circ (y \circ x) \cup \mathcal{K}_A(z) \cup \alpha$ by Definition 14. Hence, from Definition 15, we conclude that $\mathcal{K}_A$ is an (α, β)-US commutative ideal over $\mathcal{X}$. □

COROLLARY 1. *Let E be a KU-algebra and $\mathcal{K}_{\mathcal{E} \in \mathcal{S}(\mathcal{X})}$. Then, $\mathcal{K}_{\mathcal{E}}$ is an (α, β)-US commutative ideal over $\mathcal{X}$ if and only if $\mathcal{K}_E$ is an (α, β)-US ideal over $\mathcal{X}$ satisfying the following condition for all $x, y \in A$: $\mathcal{K}_E(((x \circ y) \circ y) \circ x) \cap \beta \subseteq \mathcal{K}_A(y \circ x) \cup \alpha$.*

THEOREM 12. *Let E be a commutative KU-algebra. Then, every (α, β)-US ideal over $\mathcal{X}$ is an (α, β)-US commutative ideal over $\mathcal{X}$.*

Proof. Let $\mathcal{K}_A$ be an (α, β)-US ideal over $\mathcal{X}$, where A is a subalgebra of E. Then, for all $x, y, z \in A$, we get: $z \circ (((z \circ (y \circ x)) \circ (((x \circ y) \circ y) \circ x))) = (z \circ (y \circ x)) \circ (z \circ (((x \circ y) \circ y) \circ x)) \leq (y \circ x) \circ (((x \circ y) \circ y) \circ x) =$

$((x \circ y) \circ y) \circ ((y \circ x) \circ x) = 1$. Thus, $((z \circ y) \circ x) \circ (((x \circ y) \circ y) \circ x) \leq z$. Then, from Theorem 8, we get $\mathcal{K}_A(((x \circ y) \circ y) \circ x) \cap \beta \subseteq \mathcal{K}_A(z \circ (y \circ x)) \cup \mathcal{K}_A(z) \cup \alpha$. Hence, $\mathcal{K}_A$ is an (α, β)-US commutative ideal over $\mathcal{X}$. □

THEOREM 13. *Let E be a KU-algebra and A be a subalgebra of E. Let $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$. If $\mathcal{K}_A$ satisfies the following conditions:*
(1). $(y \circ x) \circ x \leq (x \circ y) \circ y$ for all $x, y \in A$;
(2) $\mathcal{K}_A$ is an (α, β)-US ideal over $\mathcal{X}$; then $\mathcal{K}_A$ is an (α, β)-US commutative ideal over $\mathcal{X}$.

Proof. For any $x, y \in A$, we have: $\mathcal{K}_A(y \circ x)(((x \circ y) \circ y) \circ x) = ((x \circ y) \circ y) \circ ((y \circ x) \circ x) = 1$ by (ku_8) and (1). Therefore, $((x \circ y) \circ y) \circ x \leq y \circ x$ for all $x, y \in A$, which indicates from Lemma 4 that $\mathcal{K}_A(((x \circ y) \circ y) \circ x) \cap \beta \subseteq \mathcal{K}_A(y \circ x) \cup \alpha$. Now, it follows from Theorem 11 that $\mathcal{K}_A$ is an (α, β)-US commutative ideal of A over $\mathcal{X}$. □

THEOREM 14. *Let E be a KU-algebra and A be a subalgebra of E. Consider $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$ and $\delta \subseteq \beta \subseteq \mathcal{X}$. Then, $\mathcal{K}_A$ is an (α, β)-US commutative ideal over $\mathcal{X}$ if and only if the non-empty set $B(\mathcal{K}_A : \delta)$ is a commutative ideal of A.*

Proof. The proof follows from Theorem 2. □

THEOREM 15. *Let E be a KU-algebra and $\mathcal{K}_A \in \mathcal{S}(\mathcal{X})$. Define a soft set $\mathcal{K}_A^\circ$ over $\mathcal{X}$ by $\mathcal{K}_A^\circ : E \to \mathcal{P}(\mathcal{X})$ such that $\mathcal{K}_A^\circ(x) = \mathcal{K}_A(x)$ if $x \in B(\mathcal{K}_A : \delta)$ and $\mathcal{K}_A^\circ(x) = \mathcal{X}$ if $x \notin B(\mathcal{K}_A : \delta)$. If $\mathcal{K}_A$ is an (α, β)-US commutative ideal over U, then so is $\mathcal{K}_A^\circ$.*

Proof. If $\mathcal{K}_A$ is an (α, β)-US commutative ideal over $\mathcal{X}$, then $B(\mathcal{K}_A : \delta)$ is a commutative ideal over $\mathcal{X}$ by Theorem 14. Hence, $1 \in B(\mathcal{K}_A : \delta)$, and so, we have $\mathcal{K}_A^\circ(1) \cap \beta = \mathcal{K}_A(1) \cap \beta \subseteq \mathcal{K}_A(x) \cup \alpha \subseteq \mathcal{K}_A^\circ(x) \cup \alpha$ for all $x \in A$. Let $x, y, z \in A$. Then, $z \circ (y \circ x) \in B(\mathcal{K}_A : \delta)$ and $z \in B(\mathcal{K}_A : \delta)$ hence, $((x \circ y) \circ y) \circ x \in B(\mathcal{K}_A : \delta)$, and so, we deduce the following equality:
$\mathcal{K}_A^\circ(((x \circ y) \circ y) \circ x) \cap \beta = \mathcal{K}_A(((x \circ y) \circ y) \circ x) \cap \beta \subseteq \mathcal{K}_A(z \circ (y \circ x)) \cup \mathcal{K}_A(z) \cup \alpha = \mathcal{K}_A^\circ(z \circ (y \circ x)) \cup \mathcal{K}_A^\circ(z) \cup \alpha$. If $z \circ (y \circ x) \notin B(\mathcal{K}_A : \delta)$ and $z \notin B(\mathcal{K}_A : \delta)$, then $\mathcal{K}_A^\circ(((x \circ y) \circ y) \circ x)$ or $\mathcal{K}_A^\circ(z) = \mathcal{X}$. Thus, we have $\mathcal{K}_A^\circ(((x \circ y) \circ y) \circ x) \cap \beta \subseteq \mathcal{X} = \mathcal{K}_A^\circ(z \circ (y \circ x)) \cup \mathcal{K}_A^\circ(z) \cup \alpha$. This shows that $\mathcal{K}_A^\circ$ is an (α, β)-US commutative ideal of A over $\mathcal{X}$. □

THEOREM 16. *Let E be a KU-algebra and A be a subset of E, which is a commutative ideal of E if and only if the soft subset $\mathcal{K}_A$ defined as,*

$\mathcal{K}_A(x) = \Omega$ if $x \in A$ and $\mathcal{K}_A(x) = \Gamma$ if $x \notin A$, where $\alpha \subseteq \Omega \subseteq \Gamma \subseteq \beta \subseteq \mathcal{X}$, is an (α, β)-US commutative ideal of A over $\mathcal{X}$.

Proof. Let A be a commutative ideal of E and if $x \in A$, then $1 \in A$. Therefore, $\mathcal{K}_A(1) = \mathcal{K}_A(x) = \Omega$, and so, $\mathcal{K}_A(1) \cap \beta = \Omega \cap \beta = \Omega$ and $\mathcal{K}_A(x) \cup \alpha = \Omega \cup \alpha = \Omega$. Thus, $\mathcal{K}_A(1) \cap \beta \subseteq \mathcal{K}_A(x) \cup \alpha$. Let for any $x, y, z \in A$ and if $z \circ (y \circ x) \in A, z \in A$, then $(((x \circ y) \circ y) \circ x) \in A$, and thus, $\mathcal{K}_A(z \circ (y \circ x) = \mathcal{K}_A(z) = \mathcal{K}_A(((x \circ y) \circ y) \circ x) = \Omega$. Then, $\mathcal{K}_A(((x \circ y) \circ y) \circ x) \cap \beta = \Omega \cap \beta = \Omega$ and $\mathcal{K}_A(z \circ (y \circ x)) \cup \mathcal{K}_A(z) \cup \alpha = \Omega \cup \alpha = \Omega$, which indicates that $\mathcal{K}_A(((x \circ y) \circ y) \circ x) \cap \beta \subseteq \mathcal{K}_A(z \circ (y \circ x)) \cup \mathcal{K}_A(z) \cup \alpha$. Now, if $x \notin A$, then $1 \in A$ or $1 \notin A$, and so, $\mathcal{K}_A(1) \cap \beta = \Omega \cap \beta = \Omega$ or $\mathcal{K}_A(1) \cap \beta = \Gamma \cap \beta = \Gamma$, but $\mathcal{K}_A(x) \cup \alpha = \Gamma \cup \alpha = \Gamma$, which implies that $\mathcal{K}_A(1) \cap \beta \subseteq \mathcal{K}_A(x) \cup \alpha$. Now, if $(z \circ (y \circ x) \notin A$ or $z \notin A$, then $(((x \circ y) \circ y) \circ x) \in A$ or $(((x \circ y) \circ y) \circ x) \notin A$, and so, $\mathcal{K}_A(((x \circ y) \circ y) \circ x) \cap \beta = \Omega \cap \beta = \Omega$ or $\mathcal{K}_A(((x \circ y) \circ y) \circ x) \cap \beta = \Gamma \cap \beta = \Gamma$, but $\mathcal{K}_A(z \circ (y \circ x) \cup \mathcal{K}_A(z) \cup \alpha = \Gamma \cup \alpha = \Gamma$, which implies that $\mathcal{K}_A(((x \circ y) \circ y) \circ x) \cap \beta \subseteq \mathcal{K}_A(z \circ (y \circ x)) \cup \mathcal{K}_A(z) \cup \alpha$. Hence, $\mathcal{K}_A$ is an (α, β)-US commutative ideal of A over $\mathcal{X}$.
Conversely, assume that $\mathcal{K}_A$ is an (α, β)-US commutative ideal of A over $\mathcal{US}$. If $x \in A$, then $\mathcal{K}_A(1) \cap \beta \subseteq \mathcal{K}_A(x) \cup \alpha = \Omega \cup \alpha = \Omega$. However, $\alpha \subseteq \Omega \subseteq \Gamma \subseteq \beta, \cdot$ hence, $\mathcal{K}_A(1) = \Omega$, and so, $1 \in A$. Again, if $z \circ (y \circ x) \in A$ and $z \in A$, then $\mathcal{K}_A(((x \circ y) \circ y) \circ x) \cap \beta \subseteq \mathcal{K}_A(z \circ (y \circ x)) \cup \mathcal{K}_A(z) \cup \alpha = \Omega \cup \alpha = \Omega$, and thus, $\mathcal{K}_A(((x \circ y) \circ y) \circ x) = \Omega$, which implies that $(((x \circ y) \circ y) \circ x) \in A$. Therefore, A is a commutative ideal of $\mathcal{E}$. □

THEOREM 17. *(Extension property). Let E be a KU-algebra. For two given subalgebras A and B of E, let $\mathcal{K}_A, \mathcal{K}_B \in \mathcal{S}(\mathcal{X})$ such that*
i. $\mathcal{K}_A \subseteq \mathcal{K}_B$.
ii. *$\mathcal{K}_B$ is an (α, β)-US ideal over $\mathcal{X}$.*
If $\mathcal{K}_A$ is an (α, β)-US commutative ideal over $\mathcal{X}$, then $\mathcal{K}_B$ is also an (α, β)-US commutative ideal over $\mathcal{X}$.

Proof. Let $\delta \in \mathcal{X}$ be such that $B(\mathcal{K}_A : \delta) \neq \emptyset$. By Condition ii and Theorem 14, we see that $B(\mathcal{K}_A : \delta)$ is an ideal. We now consider $\mathcal{K}_A$ to be an (α, β)-US commutative ideal of A over $\mathcal{X}$, then $B(\mathcal{K}_A : \delta)$ is a commutative ideal of A. Let $x, y \in A$ and $\delta \subseteq \beta$ be such that $y \circ x \in B(\mathcal{K}_A : \delta)$. Since $y \circ ((y \circ x) \circ x) = (y \circ x) \circ (y \circ x) = 1 \in B(\mathcal{K}_A : \delta)$, it follows from (ku_8) and i that $(y \circ x) \circ (((((y \circ x) \circ x) \circ y) \circ y) \circ x) = ((((y \circ x) \circ x) \circ y) \circ y) \circ ((y \circ x) \circ x) \in B(\mathcal{K}_A : \delta) \subseteq B(\mathcal{K}_B : \delta)$. We see that:

$$(((((y \circ x) \circ x) \circ y) \circ y) \circ x) \in B(\mathcal{K}_B : \delta) \ldots\ldots (1)$$

as $B(\mathcal{K}_B : \delta)$ is an ideal and $x \circ y \in B(\mathcal{K}_B : \delta)$. Furthermore, it is noted that $(y \circ x) \circ x \leq x$, and so, we have $(((y \circ x) \circ x) \circ y) \circ y \leq (x \circ y) \circ y$ by (ku_7). Thus,

$$((x \circ y) \circ y) \circ x \leq ((((y \circ x) \circ x) \circ y) \circ x) \ldots\ldots (2)$$

Hence, by using (1) and (2), we get $x \circ (y \circ (y \circ x)) \in B(\mathcal{K}_B : \delta)$. Therefore, $B(\mathcal{K}_B : \delta)$ is a commutative ideal, and so, $\mathcal{K}_B$ is an (α, β)-US commutative ideal over $\mathcal{X}$ by Theorem 14. □

7. Conclusions

In this paper, we have defined (α, β)-US sets, (α, β)-US ideals and (α, β)-US commutative ideals over KU algebras. We also investigate a characterization of (α, β)-US ideals in KU algebras. The given results over (α, β)-US sets can reflect to other branches like several algebraic structures, topologies, Vector algebras and lattices.

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