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QUASI HEMI-SLANT SUBMANIFOLDS OF COSYMPLECTIC MANIFOLDS

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Abstract. We introduce and study quasi hemi-slant submanifolds of almost contact metric manifolds (especially, cosymplectic manifolds) and validate its existence by providing some non-trivial examples. Necessary and sufficient conditions for integrability of distributions, which are involved in the definition of quasi hemi-slant submanifolds of cosymplectic manifolds, are obtained. Also, we investigate the necessary and sufficient conditions for quasi hemi-slant submanifolds of cosymplectic manifolds to be totally geodesic and study the geometry of foliations determined by the distributions.

1. Introduction

In the past two decades, almost contact geometry and related topics has been a rich research field for geometers due to its application in wide areas of physics as well as in mathematics. The notion of geometry of submanifolds begin with the idea of the extrinsic geometry of surface and it is developed for ambient space in the course of time. Nowadays this theory plays a key role in computer design, image processing, economic modeling as well as in mathematical physics and in mechanics.

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The slant submanifolds are the natural generalization of holomorphic and totally real submanifolds. B. Y. Chen defined and study the slant submanifolds in 1990 and consequent results on slant submanifolds were collected in his book [10]. Since then, this interesting subject has been studied broadly by several geometers during last two decades (for instance, [20], [21], [25]). In 1996, A. Lotta [17] introduced the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold. Further, the slant submanifolds were generalized as semi-slant submanifolds, pseudo-slant submanifolds, bi-slant submanifolds, and hemi-slant submanifolds etc. in different kinds of differentiable manifolds (see, [1], [2], [13], [16], [22], [24]- [26]).

After the very remarkable work of Chinea et al. [11], cosymplectic manifold has became of great interest in the last years. In nowadays, the importance of this manifold for the geometric description of time-dependent mechanics (see, [4], [12]) is widely recognized (especially in the formulations of time dependent mechanics cosymplectic manifold do play a major role). Recently, Ayar et al. [3] studied the properties of cosymplectic manifolds.

Motivated from above studies, we introduce the notion of quasi hemi-slant submanifolds of almost contact metric manifolds which include the classes of semi-slant and hemi-slant submanifolds as its particular cases. The present paper is organized as follows: We mention basic definitions and some properties of almost contact metric manifolds in Section 2. In Section 3, we define quasi hemi-slant submanifolds of cosymplectic manifolds and derive some basic results for these submanifolds. Section 4 deals with necessary and sufficient conditions for integrability of distributions. In Section 5, the geometry of fibers are investigated. In the last section, we provide some non-trivial examples of quasi hemi-slant submanifolds of cosymplectic manifolds.

2. Preliminaries

We consider $\widehat{M}$ is a $(2n+1)$-dimensional almost contact manifold [14] which carries a tensor field ϕ of the tangent space, 1-form η and characteristic vector field ξ satisfying

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{1}$$

where $I : T\widehat{M} \longrightarrow T\widehat{M}$ is the identity map. We have from definition, $\phi\xi = 0, \eta \circ \phi = 0$ and $rank(\phi) = 2n$. Since an almost contact manifold $(\widehat{M}, \phi, \xi, \eta)$ admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for any vector fields $X, Y \in \Gamma(T\widehat{M})$, where $\Gamma(T\widehat{M})$ represents the Lie algebra of vector fields on $\widehat{M}$. A manifold $\widehat{M}$ together with the structure (ϕ, ξ, η, g) is called an almost contact metric manifold. The immediate consequence of (1) and (2) give

$$\eta(X) = g(X, \xi) \quad \text{and} \quad g(\phi X, Y) + g(X, \phi Y) = 0$$

for all vector fields $X, Y \in \Gamma(T\widehat{M})$.

An almost contact structure (ϕ, ξ, η, g) is said to be normal [7] if the almost complex structure J on the product manifold $\widehat{M} \times R$ is given by

$$J(U,\ f\frac{d}{dt}) = (\phi U - f\xi,\ \eta(U)\frac{d}{dt}),$$

where $J^2 = -I$ and f is a differentiable function on $\widehat{M} \times R$ has no torsion, *i.e.*, J is integrable. The condition for normality in terms of ϕ, ξ and η is $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on $\widehat{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ.

An almost contact metric manifold is called a cosymplectic manifold ([8], [23]) if $(\widehat{\nabla}_X\phi)Y = 0$, $\widehat{\nabla}_X\xi = 0\ \ \forall\ X, Y \in \Gamma(T\widehat{M})$, where $\widehat{\nabla}$ represents the Levi-Civita connection of $(\widehat{M}, g)$.
The covariant derivative of ϕ is defined as

$$(\widehat{\nabla}_X\phi)Y = \widehat{\nabla}_X\phi Y - \phi\widehat{\nabla}_X Y.$$

If $\widehat{M}$ is a cosymplectic manifold, then we have

$$\phi\widehat{\nabla}_X Y = \widehat{\nabla}_X\phi Y. \tag{3}$$

Let M be a Riemannian manifold isometrically immersed in $\widehat{M}$ and the induced Riemannian metric on M is denoted by the same symbol g throughout this paper. Let A and h denote the shape operator and second fundamental form, respectively, of submanifolds of M into $\widehat{M}$. The Gauss and Weingarten formulas are given by

$$\widehat{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{4}$$

and

$$\widehat{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{5}$$

for any vector fields $X, Y \in \Gamma(TM)$ and V on $\Gamma(T^{\perp}M)$, where ∇ is the induced connection on M and $\nabla^{\perp}$ represents the connection on the normal bundle $T^{\perp}M$ of M and A_V is the shape operator of M with respect to normal vector $V \in \Gamma(T^{\perp}M)$. Moreover, A_V and the second fundamental form $h : TM \otimes TM \longrightarrow T^{\perp}M$ of M into $\widehat{M}$ are related by

$$g(h(X,Y),V) = g(A_V X, Y),$$

for any vector fields $X, Y \in \Gamma(TM)$ and V on $\Gamma(T^{\perp}M)$.

The mean curvature vector H is defined by

$$H = \frac{1}{n} trace(h) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),$$

where n denotes the dimension of submanifold M and $\{e_1, e_2, ..., e_n\}$ is the local orthonormal basis of tangent space at each point of M.

For any $X \in \Gamma(TM)$, we can write

$$\phi X = TX + NX, \tag{6}$$

where TX and NX are the tangential and normal components of ϕX on M, respectively. Similarly for any $V \in T^{\perp}M$, we have

$$\phi V = tV + nV, \tag{7}$$

where tV and nV are the tangential and normal components of ϕV on M, respectively.

A submanifold M of a cosymplectic manifold $\widehat{M}$ is said to be totally umbilical if

$$h(X,Y) = g(X,Y)H.$$

If $h(X,Y) = 0$ for all $X, Y \in \Gamma(TM)$, then M is said to be totally geodesic and if $H = 0$, then M is called a minimal submanifold.

The covariant derivative of projection morphisms in (6) and (7) are defined as

$$(\widehat{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y,$$
$$(\widehat{\nabla}_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y,$$
$$(\widehat{\nabla}_X t)V = \nabla_X tV - t\nabla^{\perp}_X V$$

and

$$(\widehat{\nabla}_X n)V = \nabla^{\perp}_X nV - n\nabla^{\perp}_X V$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Definition 2.1. Let M be a Riemannian manifold isometrically immersed in an almost contact metric manifold $\widehat{M}$. A submanifold M of an almost contact metric manifold $\widehat{M}$ is said to be invariant [6] if $\phi\left(T_xM\right) \subseteq T_xM$, for every point $x \in M$.

Definition 2.2. A submanifold M of an almost contact metric manifold $\widehat{M}$ is said to be anti-invariant [15] if $\phi\left(T_xM\right) \subseteq T_x^{\perp}M$, for every point $x \in M$.

Definition 2.3. A submanifold M of an almost contact metric manifold $\widehat{M}$ is said to be slant [9], if for each non-zero vector X tangent to M at $x \in M$, linearly independent on ξ the angle $\theta(X)$ between ϕX and T_xM is constant, i.e., it does not depend on the choice of the point $x \in M$ and $X \in T_xM$. In this case, the angle θ is called the slant angle of the submanifold. A slant submanifold M is called proper slant submanifold if neither $\theta = 0$ nor $\theta = \frac{\pi}{2}$.

We note that on a slant submanifold M if $\theta = 0$, then it is an invariant submanifold and if $\theta = \frac{\pi}{2}$, then it is an anti-invariant submanifold. This means that the slant submanifold is a generalization of invariant and anti-invariant submanifolds.

Definition 2.4. A submanifold M of an almost contact metric manifold $\widehat{M}$ is said to be semi-invariant [5], if there exist two orthogonal complementary distributions D_1 and D_2 on M such that

$$TM = D_1 \oplus D_2 \oplus < \xi >,$$

where D_1 is invariant and D_2 is anti-invariant.

Definition 2.5. A submanifold M of an almost contact metric manifold $\widehat{M}$ is said to be semi-slant [18], if there exist two orthogonal complementary distributions D and D^{θ} on M such that

$$TM = D \oplus D^{\theta} \oplus < \xi >,$$

where D is invariant and D^{θ} is slant with slant angle θ. In this case, the angle θ is called semi-slant angle.

Definition 2.6. A submanifold M of an almost contact metric manifold $\widehat{M}$ is said to be hemi-slant [22], if there exist two orthogonal complementary distributions D^{θ} and $D^{\perp}$ on M such that

$$TM = D^{\theta} \oplus D^{\perp} \oplus < \xi >,$$

where D^θ is slant with slant angle θ and $D^\perp$ is anti-invariant. In this case, the angle θ is called hemi-slant angle.

3. Quasi hemi-slant submanifolds of cosymplectic manifolds

In this section, we introduce and study quasi hemi-slant submanifolds of cosymplectic manifolds.

DEFINITION 3.1. A submanifold M of an almost contact metric manifold $\widehat{M}$ is called a quasi hemi-slant submanifold if there exist distributions D, D^θ and $D^\perp$ such that
(i) TM admits the orthogonal direct decomposition as

$$TM = D \oplus D^\theta \oplus D^\perp \oplus < \xi > .$$

(ii) The distribution D is ϕ invariant, *i.e.*, $\phi D = D$.
(iii) For any non-zero vector field $X \in (D^\theta)_p$, $p \in M$, the angle θ between JX and $(D^\theta)_p$ is constant and independent of the choice of point p and X in $(D^\theta)_p$.
(iv) The distribution $D^\perp$ is ϕ anti-invariant, i.e., $\phi D^\perp \subseteq T^\perp M$.

In this case, we call θ the quasi hemi-slant angle of M. Suppose the dimension of distributions D, D^θ and $D^\perp$ are n_1, n_2 and n_3, respectively. Then we can easily see the following particular cases:
(i) If $n_1 = 0$, then M is a hemi-slant submanifold.
(ii) If $n_2 = 0$; then M is a semi-invariant submanifold.
(iii) If $n_3 = 0$, then M is a semi-slant submanifold.
We say that a quasi hemi-slant submanifold M is proper if $D \neq \{0\}$, $D^\perp \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.
This means that the notion of quasi hemi-slant submanifold is a generalization of invariant, anti-invariant, semi-invariant, slant, hemi-slant, semi-slant submanifolds.

REMARK 3.2. The definition can be generalized by taking $TM = D \oplus D^{\theta_1} \oplus D^{\theta_2} ... \oplus D^{\theta_k} \oplus < \xi >$. Hence we can define multi-slant submanifolds, quasi multi-slant submanifolds, quasi hemi multi-slant submanifolds, etc.

Let M be a quasi hemi-slant submanifold of an almost contact metric manifold $\widehat{M}$. We denote the projections of $X \in \Gamma(TM)$ on the distributions D, D^θ and $D^\perp$ by P, Q and R, respectively. Then we can write for any $X \in \Gamma(TM)$

$$X = PX + QX + RX + \eta(X)\xi. \tag{8}$$

Now we put

$$\phi X = TX + NX, \tag{9}$$

where TX and NX are tangential and normal components of ϕX on M. Using (8) and (9), we obtain

$$\phi X = TPX + NPX + TQX + NQX + TRX + NRX.$$

Since $\phi D = D$ and $\phi D^{\perp} \subseteq T^{\perp}M$, we have $NPX = 0$ and $TRX = 0$. Therefore, we get

$$\phi X = TPX + TQX + NQX + NRX. \tag{10}$$

Then for any $X \in \Gamma(TM)$, it is easy to see that

$$TX = TPX + TQX$$

and

$$NX = NQX + NRX.$$

Thus from (10) , we have the following decomposition

$$\phi(TM) = D \oplus TD^{\theta} \oplus ND^{\theta} \oplus ND^{\perp},$$

where '$\oplus$' denotes orthogonal direct sum. Since $ND^{\theta} \subset (T^{\perp}M)$ and $ND^{\perp} \subset (T^{\perp}M)$, we have

$$T^{\perp}M = ND^{\theta} \oplus ND^{\perp} \oplus \mu,$$

where μ is the orthogonal complement of $ND^{\theta} \oplus ND^{\perp}$ in $\Gamma(T^{\perp}M)$ and it is invariant with respect to ϕ. For any non-zero vector field $V \in \Gamma(T^{\perp}M)$, we put

$$\phi V = tV + nV, \tag{11}$$

where $tV \in (D^{\theta} \oplus D^{\perp})$ and $nV \in \Gamma(\mu)$. For $X, Y \in TM$ we have

$$\nabla_X TY - A_{NY}X - T\nabla_X Y - t\,h(X,Y) = 0$$

$$h(X,TY) + \nabla^{\perp}_X NY - N(\nabla_X Y) - n\,h(X,Y) = 0.$$

and

$$TD = D,\ TD^{\theta} = D^{\theta},\ TD^{\perp} = \{0\},\ tND^{\theta} = D^{\theta},\ tND^{\perp} = D^{\perp}.$$

From equations (1), (9) and (11), we can easily observe that the endomorphism T, the projection morphisms N, t and n in the tangent bundle of M satisfy

(i) $T^2 + tN = -I + \eta \otimes \xi$ and $NT + nN = 0$ on TM,
(ii) $Nt + n^2 = -I$ and $Tt + tn = 0$ on $(T^{\perp}M)$,

$$(\widehat{\nabla}_X T)Y = A_{NY}X + th\,(X,Y)\,, \quad (\widehat{\nabla}_X N)Y = nh(X,Y) - h(X,TY),$$

and

$$(\widehat{\nabla}_X t)V = A_{nV}X - TA_V X \quad (\widehat{\nabla}_X n)V = -h(X,tV) - NA_V X$$

for any $X, Y \in \Gamma\,(TM)$ and $V \in \Gamma(T^{\perp}M)$.

LEMMA 3.3. *Let M be a quasi hemi-slant submanifold of an almost contact metric manifold $\widehat{M}$. Then*
(i) $T^2X = -(\cos^2\theta)X$,
(ii) $g(TX,TY) = (\cos^2\theta)g(X,Y)$,
(iii) $g(NX,NY) = (\sin^2\theta)g(X,Y)$
for any $X, Y \in D^{\theta}$.

Proof. The proof follows using similar steps as in Proposition 2.8 of [19]. □

LEMMA 3.4. *Let M be a quasi hemi-slant submanifold of a cosymplectic manifold $\widehat{M}$, then*

$$A_{\phi Z}W = A_{\phi W}Z - T([W,Z]) \text{ and } \nabla^{\perp}_{Z}\phi W - \nabla^{\perp}_{W}\phi Z = N([Z,W])$$

for all $Z, W \in D^{\perp}$.

Proof. Let $Z, W \in D^{\perp}$, then

$$(\widehat{\nabla}_Z\phi)W = \widehat{\nabla}_Z(\phi W) - \phi(\widehat{\nabla}_Z W)$$

$$\implies 0 = -A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W - T(\nabla_Z W) - N(\nabla_Z W) - th(Z,W) - nh(Z,W).$$

Comparing tangential and normal parts in the above equation, we get

$$-A_{\phi W}Z - T(\nabla_Z W) - th(Z,W) = 0 \tag{12}$$

$$\nabla^{\perp}_{Z}\phi W - N(\nabla_Z W) - nh(Z,W) = 0 \tag{13}$$

From equations (12) and (13), we can easily get the statement of Lemma 3.4. □

LEMMA 3.5. *Let M be a quasi hemi-slant submanifold of a cosymplectic manifold $\widehat{M}$, then*
(i) $g\left([X,Y]\,,\xi\right) = 0$,
(ii) $g\left(\widehat{\nabla}_X Y, \xi\right) = 0$
for all $X, Y \in \left(D \oplus D^{\theta} \oplus D^{\perp}\right)$.

4. Integrability of distributions

In this section, we investigate integrability conditions for the distributions involved in the definition of quasi hemi-slant submanifolds of cosymplectic manifolds.

THEOREM 4.1. *Let M be a proper quasi hemi-slant submanifold of a cosymplectic manifold $\widehat{M}$. Then the invariant distribution D is integrable if and only if*

$$g(\nabla_X TY - \nabla_Y TX, TQZ) = g(h(Y, TX) - h(X, TY), NQZ + NRZ)$$

for any X, $Y \in \Gamma(D)$ and $Z \in \Gamma(D^\theta \oplus D^\perp)$.

Proof. We know that for a cosymplectic manifold,

$$\widehat{\nabla}_X \xi = 0 \ \forall \ X \in \Gamma(D). \tag{14}$$

If $Y \in \Gamma(D)$, then $g(Y, \xi) = 0$. The covariant of this equation along X gives

$$g(\widehat{\nabla}_X Y, \xi) + g(Y, \widehat{\nabla}_X \xi) = 0. \tag{15}$$

Now, $g([X, Y], \xi) = g(\widehat{\nabla}_X Y, \xi) - g(\widehat{\nabla}_Y X, \xi) = 0$, where equations (14) and (15) are used. Next, for any X, $Y \in \Gamma(D)$ and $Z = QZ + RZ \in \Gamma(D^\theta \oplus D^\perp)$, using (2), (3) , (4) and (9), we have

$$\begin{aligned} g([X, Y], Z) &= g(\widehat{\nabla}_X \phi Y, \phi Z) - g(\widehat{\nabla}_Y \phi X, \phi Z) = g(\nabla_X TY \\ &\quad - \nabla_Y TX, TQZ) + g(h(X, TY) - h(Y, TX), NQZ + NRZ). \end{aligned}$$

This completes the proof. □

THEOREM 4.2. *Let M be a proper quasi hemi-slant submanifold of a cosymplectic manifold $(\widehat{M}, g, \phi)$. Then the slant distribution D^θ is integrable if and only if*

$$\begin{aligned} g(A_{NW} Z - A_{NZ} W, TPX) &= g(A_{NTW} Z - A_{NTZ} W, X) \\ &\quad + g(\nabla^\perp_Z NW - \nabla^\perp_W NZ, NRX) \end{aligned}$$

for any Z, $W \in \Gamma(D^\theta)$ and $X \in \Gamma(D \oplus D^\perp)$.

Proof. For any Z, $W \in \Gamma(D^\theta)$ and $X = PX + RX \in \Gamma(D \oplus D^\perp)$, using (2), (3) and (9), we obtain

$$\begin{aligned} g([Z, W], X) &= g(\widehat{\nabla}_Z NW, \phi X) - g(\widehat{\nabla}_Z \phi TW, X) \\ &\quad - g(\widehat{\nabla}_W NZ, \phi X) + g(\widehat{\nabla}_W \phi TZ, X). \end{aligned}$$

Then from (5), (9) and Lemma 3.3, we have

$$g([Z,W],X) = -g(A_{NW}Z - A_{NZ}W, \phi X) + \cos^2\theta g([Z,W],X) \\ +g(A_{NTW}Z - A_{NTZ}W, X) + g(\nabla^{\perp}_{Z}NW - \nabla^{\perp}_{W}NZ, \phi X),$$

which leads to

$$\sin^2\theta g([Z,W],X) = g(A_{NTW}Z - A_{NTZ}W, X) + g(\nabla^{\perp}_{Z}NW \\ -\nabla^{\perp}_{W}NZ, NRX) - g(A_{NW}Z - A_{NZ}W, TPX).$$

Thus the proof is completed. □

From Theorem 4.2, we have the following sufficient conditions for the slant distribution D^θ to be integrable.

THEOREM 4.3. *Let M be a proper quasi hemi-slant submanifold of a cosymplectic manifold $\widehat{M}$. If*

$$\begin{array}{rcl} \nabla^{\perp}_{Z}NW - \nabla^{\perp}_{W}NZ & \in & ND^\theta \oplus \mu, \\ A_{NTW}Z - A_{NTZ}W & \in & D^\theta, \quad \text{and} \\ A_{NW}Z - A_{NZ}W & \in & D^\perp \oplus D^\theta \end{array}$$

for any $Z, W \in \Gamma(D^\theta)$, then the slant distribution D^θ is integrable.

THEOREM 4.4. *Let M be a quasi hemi-slant submanifold of a cosymplectic manifold $\widehat{M}$. Then the anti-invariant distribution $D^\perp$ is integrable if and only if*

$$g(T([Z,W]),TX) = g(\nabla^{\perp}_{W}NZ - \nabla^{\perp}_{Z}NW, NQX)$$

for any $Z, W \in \Gamma(D^\perp)$ and $X \in \Gamma(D \oplus D^\theta)$.

Proof. For any $Z, W \in \Gamma(D^\perp)$ and $X = PX + QX \in \Gamma(D \oplus D^\theta)$, using (2), (3), (5), (9) and Lemma 3.4, we obtain

$$\begin{aligned} g([Z,W],X) &= g(\widehat{\nabla}_Z\, \phi W, \phi X) - g(\widehat{\nabla}_W\, \phi Z, \phi X) \\ &= g(A_{\phi Z}W - A_{\phi W}Z, TPX + TQX) + g(\nabla^{\perp}_{Z}\phi W - \nabla^{\perp}_{W}\phi Z, NQX) \\ &= g(T([Z,W]),TX) + g(\nabla^{\perp}_{Z}NW - \nabla^{\perp}_{W}NZ, NQX). \end{aligned}$$

The above equation together with Lemma 3.5 prove the statement of Theorem 4.4. □

5. Totally Geodesic Foliations

Geodesicness and foliations are significant geometric notions. In this section, the geometry of foliations of a quasi hemi-slant submanifold is investigated. Also, some conditions are given for the totally geodesicness.

THEOREM 5.1. *Let M be a proper quasi hemi-slant submanifold of a cosymplectic manifold $\widehat{M}$. Then M is totally geodesic if and only if*

$$g(h(X,PY)+\cos^2\theta h(X,QY),U)=g(\nabla^{\perp}_X NTQY,U)$$
$$+g(A_{NQY}X+A_{NRY}X,tU)-g(\nabla^{\perp}_X NY,nU)$$

for any $X,Y\in\Gamma(TM)$ and $U\in\Gamma(T^{\perp}M)$.

Proof. For any X, $Y\in\Gamma(TM)$, $U\in\Gamma\left(T^{\perp}M\right)$ and using (2) and (3), we have

$$\begin{aligned} g(\widehat{\nabla}_X Y,U) &= g(\widehat{\nabla}_X PY,U)+g(\widehat{\nabla}_X QY,U)+g(\widehat{\nabla}_X RY,U)\\ &= g(\widehat{\nabla}_X \phi PY,\phi U)+g(\widehat{\nabla}_X TQY,\phi U)+g(\widehat{\nabla}_X NQY,\phi U)\\ &\quad +g(\widehat{\nabla}_X \phi RY,\phi U). \end{aligned}$$

Using (2), (4), (5), (9) and Lemma 3.3, we have

$$\begin{aligned} g(\widehat{\nabla}_X Y,U) &= g(\widehat{\nabla}_X PY,U)-g(\widehat{\nabla}_X T^2QY,U)-g(\widehat{\nabla}_X NTQY,U)\\ &\quad +g(\widehat{\nabla}_X NQY,\phi U)+g(\widehat{\nabla}_X NRY,\phi U)\\ &= g(h(X,PY),U)+\cos^2\theta g(h(X,QY),U)-g(\nabla^{\perp}_X NTQY,U)\\ &\quad +g(-A_{NQY}X+\nabla^{\perp}_X NQY,\phi U)+g(-A_{NRY}X+\nabla^{\perp}_X NRY,\phi U). \end{aligned}$$

As $NY=NPY+NQY+NRY$ and $NPY=0$. Thus we have

$$\begin{aligned} g(\widehat{\nabla}_X Y,U) &= g(h(X,PY)+\cos^2\theta h(X,QY),U)-g(\nabla^{\perp}_X NTQY,U)\\ &\quad -g(A_{NQY}X+A_{NRY}X,tU)+g(\nabla^{\perp}_X NY,nU). \end{aligned}$$

Hence the proof follows. □

THEOREM 5.2. *Let M be a proper quasi hemi-slant submanifold of a cosymplectic manifold $\widehat{M}$. Then anti-invariant distribution $D^{\perp}$ defines totally geodesic foliation if and only if*

$$g(A_{\phi Y}X,TPZ+tQZ)=g(\nabla^{\perp}_X\phi Y,nQZ),\quad g(A_{\phi Y}X,tV)=g(\nabla^{\perp}_X\phi Y,nV)$$

for any X, $Y\in\Gamma(D^{\perp})$, $Z\in\Gamma(D\oplus D^{\theta})$ and $V\in\Gamma\left(T^{\perp}M\right)$.

Proof. For any X, $Y \in \Gamma(D^{\perp})$, $Z = PZ + QZ \in \Gamma(D \oplus D^{\theta})$, using (2), (3), (9) and the fact that M is cosymplectic, we have

$$\begin{aligned} g(\widehat{\nabla}_X Y, Z) &= g(\widehat{\nabla}_X \phi Y, \phi Z) = g(\widehat{\nabla}_X \phi Y, \phi PZ + \phi QZ) \\ &= g(-A_{\phi Y}X + \nabla^{\perp}_X \phi Y, TPZ + tQZ + nQZ) \\ &= -g(A_{\phi Y}X, TPZ + tQZ) + g(\nabla^{\perp}_X \phi Y, nQZ). \end{aligned} \tag{16}$$

Again, let X, $Y \in \Gamma(D^{\perp})$ and $V \in \Gamma(T^{\perp}M)$, then we have

$$\begin{aligned} g(\widehat{\nabla}_X Y, V) &= g(\widehat{\nabla}_X \phi Y, \phi V) \\ &= g(-A_{\phi Y}X + \nabla^{\perp}_X \phi Y, tV + nV) \\ &= -g(A_{\phi Y}X, tV) + g(\nabla^{\perp}_X \phi Y, nV). \end{aligned} \tag{17}$$

Also, $g(\widehat{\nabla}_X Y, \xi) = 0$. From equations (16) and (17), it is obvious that $D^{\perp}$ defines totally geodesic foliation if and only if $g(A_{\phi Y}X, TPZ + tQZ) = g(\nabla^{\perp}_X \phi Y, nQZ)$ and $g(A_{\phi Y}X, tV) = g(\nabla^{\perp}_X \phi Y, nV)$. Hence the statement of the Theorem 5.2. □

THEOREM 5.3. *Let M be a proper quasi hemi-slant submanifold of a cosymplectic manifold $\widehat{M}$. Then the slant distribution D^{θ} defines a totally geodesic foliation on M if and only if*

$$\begin{aligned} &g(\nabla^{\perp}_X NY, NRZ) = g(A_{NY}X, TPZ) - g(A_{NTY}X, Z), \quad \text{and} \\ &g(A_{NY}X, tV) = g(\nabla^{\perp}_X NY, nV) - g(\nabla^{\perp}_X NTY, V) \end{aligned}$$

for any $X, Y \in \Gamma(D^{\theta})$, $Z \in \Gamma(D \oplus D^{\perp})$ and $V \in \Gamma\left(T^{\perp}M\right)$.

Proof. For any $X, Y \in \Gamma(D^{\theta})$, $Z = PZ + RZ \in \Gamma(D \oplus D^{\perp})$ and using (2), (3) and (9), we have

$$\begin{aligned} g(\widehat{\nabla}_X Y, Z) &= g(\widehat{\nabla}_X \phi Y, \phi Z) = g(\widehat{\nabla}_X TY, \phi Z) + g(\widehat{\nabla}_X NY, \phi Z) \\ &= -g(\widehat{\nabla}_X T^2 Y, Z) - g(\widehat{\nabla}_X NTY, Z) + g(\widehat{\nabla}_X NY, TPZ + NRZ). \end{aligned}$$

Then using (5), (9) and Lemma 3.3, and the fact that $NPZ = 0$, we have

$$\begin{aligned} g(\widehat{\nabla}_X Y, Z) &= \cos^2\theta g(\widehat{\nabla}_X Y, Z) + g(A_{NTY}X, Z) \\ &\quad - g(A_{NY}X, TPZ) + g(\nabla^{\perp}_X NY, NRZ), \end{aligned}$$

$$\begin{aligned} \sin^2\theta g(\widehat{\nabla}_X Y, Z) &= g(A_{NTY}X, Z) \\ &\quad - g(A_{NY}X, TPZ) + g(\nabla^{\perp}_X NY, NRZ). \end{aligned} \tag{18}$$

Similarly, we get

(19)
$$\sin^2\theta g(\widehat{\nabla}_X Y, V) = -g(\nabla^{\perp}_X NTY, V) - g(A_{NY}X, tV) + g(\nabla^{\perp}_X NY, nV).$$

Thus from (18) and (19), we have the assertions. □

THEOREM 5.4. *Let M be a proper quasi hemi-slant submanifold of a cosymplectic manifold $\widehat{M}$. Then the invariant distribution D defines a totally geodesic foliation on M if and only if*

$$\begin{aligned} g(\nabla_X TY, TQZ) &= -g(h(X,TY), NQZ + NRZ), \quad \textit{and} \\ g(\nabla_X TY, TU) &= -g(h(X,TY), nU) \end{aligned}$$

for any $X, Y \in \Gamma(D)$, $Z \in \Gamma(D^{\theta} \oplus D^{\perp})$ and $U \in \Gamma\left(T^{\perp}M\right)$.

Proof. For any $X, Y \in \Gamma(D)$, $Z = QZ + RZ \in \Gamma(D^{\theta} \oplus D^{\perp})$ and using (2), (3) , (9) and $NY = 0$, we have

$$\begin{aligned} g(\widehat{\nabla}_X Y, Z) &= g(\widehat{\nabla}_X TY, \phi Z), \\ &= g(\nabla_X TY, TQZ) + g(h(X,TY), NQZ + NRZ). \end{aligned}$$

Now for any $U \in \Gamma\left(T^{\perp}M\right)$ and $X, Y \in \Gamma(D)$, we have

$$\begin{aligned} g(\widehat{\nabla}_X Y, U) &= g(\widehat{\nabla}_X TY, \phi U) \\ &= g(\nabla_X TY, tU) + g(h(X,TY), nU). \end{aligned}$$

Hence the proof. □

6. Examples

EXAMPLE 6.1. Consider a 15-dimensional differentiable manifold

$$\overline{M} = \{(x_i, y_i, z) = (x_1, x_2, ..., x_7, y_1, y_2, ..., y_7, z) \in \mathrm{R}^{15}\}.$$

We choose the vector fields

$$E_i = \frac{\partial}{\partial y_i}, \quad E_{7+i} = \frac{\partial}{\partial x_i}, \quad E_{15} = \xi = \frac{\partial}{\partial z}, \quad \text{for} \ \ i = 1,\ 2,\ ...,7.$$

Let g be a Riemannian metric defined by

$$g = (dx_1)^2 + (dx_2)^2 + ... + (dx_7)^2 + (dy_1)^2 + (dy_2)^2 + ... + (dy_7)^2 + (dz)^2.$$

Then we find that $g(E_i, E_i) = 1$ and $g(E_i, E_j) = 0$, for $1 \leq i \neq j \leq 15$. Hence $\{E_1, E_2, ..., E_{15}\}$ forms an orthonormal basis. Thus 1-form $\eta = dz$ is defined by $\eta(E) = g(E, \xi)$, for any $E \in \Gamma(T\overline{M})$.

We define $(1,1)$-tensor field ϕ as

$$\phi\left(\frac{\partial}{\partial x_i}\right)=\frac{\partial}{\partial y_i},\quad \phi\left(\frac{\partial}{\partial y_j}\right)=-\frac{\partial}{\partial x_j},\quad \phi\left(\frac{\partial}{\partial z}\right)=0\quad \forall\, i,j=1,2,...,7.$$

By using linearity of ϕ and g, we have

$$\phi^2=-I+\eta\otimes\xi,\quad \phi\xi=0,\quad \eta(\xi)=1,$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\quad \text{for any } X,Y\in\Gamma(T\overline{M}).$$

Hence $(\overline{M},\phi,\xi,\eta,g)$ is an almost contact metric manifold. Also, we can easily show that $(\overline{M},\phi,\xi,\eta,g)$ is a cosymplectic manifold of dimension 15.

Now, we consider a submanifold M of $\overline{M}$ defined by immersion f as follows:

$$f\,(u,v,w,r,s,t,q)=\left(u,w,0,\frac{s}{\sqrt{2}},0,\frac{t}{\sqrt{2}},0,v,r\cos\theta,r\sin\theta,0,\frac{s}{\sqrt{2}},0,\frac{t}{\sqrt{2}},q\right),$$

where $0<\theta<\frac{\pi}{2}$. By direct computation, it is easy to check that the tangent bundle of M is spanned by the set $\{Z_1,Z_2,Z_3,Z_4,Z_5,Z_6,Z_7\}$, where

$$Z_1=\frac{\partial}{\partial x_1},\quad Z_2=\frac{\partial}{\partial y_1},\quad Z_3=\frac{\partial}{\partial x_2},$$

$$Z_4=\cos\theta\frac{\partial}{\partial y_2}+\sin\theta\frac{\partial}{\partial y_3},\quad Z_5=\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_4}+\frac{\partial}{\partial y_5}\right),$$

$$Z_6=\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_6}+\frac{\partial}{\partial y_7}\right),\quad Z_7=\frac{\partial}{\partial z}.$$

Then using almost contact structure of $\overline{M}$, we have

$$\phi Z_1=\frac{\partial}{\partial y_1},\quad \phi Z_2=-\frac{\partial}{\partial x_1},\quad \phi Z_3=\frac{\partial}{\partial y_2},$$

$$\phi Z_4=-\left(\cos\theta\frac{\partial}{\partial x_2}+\sin\theta\frac{\partial}{\partial x_3}\right),\quad \phi Z_5=\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_4}-\frac{\partial}{\partial x_5}\right),$$

$$\phi Z_6=\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_6}-\frac{\partial}{\partial x_7}\right),\quad \phi Z_7=0.$$

Now, let the distributions $D=Span\{Z_1,Z_2\}$, $D^\theta=Span\{Z_3,Z_4\}$, $D^\perp=Span\{Z_5,Z_6\}$. It is easy to see that D is invariant, D^θ is slant with slant angle θ and $D^\perp$ is anti-invariant.

Example 6.2. Let $(\overline{M}, \phi, \xi, \eta, g)$ be a cosymplectic manifold of dimension 15 as defined in Example 6.1. Suppose N be a submanifold $\overline{M}$ (see, Example 6.1) defined by immersion ψ as follows:

$$\psi\,(u, v, w, r, s, t, q) = \left(\frac{u}{\sqrt{2}}, w, 0, \frac{s}{\sqrt{2}}, 0, t, \frac{u}{\sqrt{2}}, \frac{v}{\sqrt{2}}, \frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}, 0, \frac{s}{\sqrt{2}}, 0, \frac{v}{\sqrt{2}}, q\right).$$

By direct computation, it is easy to check that the tangent bundle of N is spanned by the set $\{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}$, where

$$X_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_7}\right), \quad X_2 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_7}\right), \quad X_3 = \frac{\partial}{\partial x_2},$$

$$X_4 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}\right), \quad X_5 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_5}\right),$$

$$X_6 = \frac{\partial}{\partial x_6}, \quad X_7 = \frac{\partial}{\partial z}.$$

Then using almost contact structure of $\overline{M}$, we have

$$\phi X_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_7}\right), \quad \phi X_2 = -\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_7}\right), \quad \phi X_3 = \frac{\partial}{\partial y_2},$$

$$\phi X_4 = -\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right), \quad \phi X_5 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_4} - \frac{\partial}{\partial x_5}\right),$$

$$\phi X_6 = \frac{\partial}{\partial y_6}, \quad \phi X_7 = 0.$$

Now, let the distributions $D = Span\{X_1, X_2\}$, $D^\theta = Span\{X_3, X_4\}$, $D^\perp = Span\{X_5, X_6\}$. It is easy to conclude that D is invariant, D^θ is slant with slant angle $\frac{\pi}{4}$ and $D^\perp$ is anti-invariant.

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