THE CHEREDNIK AND THE GAUSSIAN CHEREDNIK WINDOWED TRANSFORMS ON $\mathbb{R}^d$ IN THE W-INVARIANT CASE

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Abstract. In this paper we give the harmonic analysis associated with the Cherednik operators, next we define and study the Cherednik wavelets and the Cherednik windowed transforms on $\mathbb{R}^d$, in the W-invariant case, and we prove for these transforms Plancherel and inversion formulas. As application we give these results for the Gaussian Cherednik wavelets and the Gaussian Cherednik windowed transform on $\mathbb{R}^d$ in the W-invariant case.

1. Introduction

The windowed Fourier transform was introduced by the physicist Dennis Gabor in 1946. The basic idea is to replace in the usual Fourier transform, the function analysed by the product of this function by a regular function called windowed function. The classical windowed transform of a function f is given by:

$$\Psi_g(f)(\lambda, y) = \int_{\mathbb{R}^d} f(x) g_{\lambda, y}(x) dx, \quad \lambda, y \in \mathbb{R}^d,$$

where $g_{\lambda,y}$ is the classical wavelet defined by

$$g_{\lambda,y}(x) = \frac{\tau_x g(y)}{||\tau_x g||_2} e^{-i\langle\lambda,x\rangle},$$

with τ_x the classical translation operator defined for $x \in \mathbb{R}^d$, by

$$\tau_x g(y) = g(x - y), \quad y \in \mathbb{R}^d.$$

One of the aims of the windowed Fourier transform, is to provide an easily interpretable visual representation of signal. Moreover, this transform can be applied to wide scientific research areas ranging from signal analysis in geophysics and acoustics, to quantum theory and pure mathematics (see [5]).

In [1], Cherednik introduced a family of differential-difference operators that nowadays bear his name. We define and study in this paper the Cherednik wavelets and the
Cherednik windowed transform on $\mathbb{R}^d$, in the W-invariant case. To achieve this, we consider the Cherednik operators $T_j, j = 1, 2, ..., d$, on $\mathbb{R}^d$ associated to a root system $\mathcal{R}$, a reflection group W and a non negative multiplicity function k.
Next, we introduce the Heckman-Opdam hypergeometric function F_λ, $\lambda \in \mathbb{C}^d$, given by

$$\forall\, x \in \mathbb{R}^d, \quad F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(wx),$$

where G_λ, $\lambda \in \mathbb{C}^d$, is the unique solution of the differential-difference system

$$\begin{cases} T_j G_\lambda(x) & = i\lambda_j G_\lambda(x), \quad j = 1, 2, ..., d, x \in \mathbb{R}^d, \\ G_\lambda(0) & = 1. \end{cases}$$

By using the function F_λ, we define the hypergeometric Fourier transform $\mathcal{H}^W$ for regular W-invariant function f on $\mathbb{R}^d$ by

$$\mathcal{H}^W(f)(\lambda) = \int_{\mathbb{R}^d} f(x) F_{-\lambda}(x) \mathcal{A}_k(x) dx, \quad \lambda \in \mathbb{R}^d,$$

where $\mathcal{A}_k$ is a weight function, and the hypergeometric translation operator $\mathcal{T}_x^W$, $x \in \mathbb{R}^d$, by

$$\mathcal{H}^W(\mathcal{T}_x^W(f))(\lambda) = F_\lambda(x)\mathcal{H}^W(f)(\lambda), \quad \lambda \in \mathbb{R}^d.$$

With the aid of these results, we define and study the Cherednik windowed transform $\Phi_g^s(f)$ given for a regular W-invariant function f

on $\mathbb{R}^d$ by

$$\Phi_g^s(f)(\lambda, y) = \int_{\mathbb{R}^d} f(x) g_{\lambda,y}^s(x) \mathcal{A}_k(x) dx, \quad \lambda, y \in \mathbb{R}^d,$$

where $g_{\lambda,y}^s$ is the family of Cherednik wavelets associated to the Heckman-Opdam theory defined on $\mathbb{R}^d$ by

$$g_{\lambda,y}^s(x) = F_{-\lambda}(x) \frac{\mathcal{T}_y^W g(x)}{||\mathcal{T}_x^W g||_{\mathcal{A}_k,2}^s}.$$

Next, we prove Plancherel and inversion formulas for the transform Φ_g^s.

In the last section we define the heat kernel $p_t^W(x, y)$ associated to the Heckman-Opdam theory by

$$p_t^W(x, y) = \int_{\mathbb{R}^d} e^{-t(||\lambda||^2 + ||\rho||^2)} F_\lambda(x) F_\lambda(-y) \mathcal{C}_k^W(\lambda) d\lambda.$$

Then by using the relation

$$p_t^W(x, y) = \mathcal{T}_x^W(E_t^W)(y),$$

where E_t^W is the fundamental solution of the heat operator associated to the Cherednik operators on $\mathbb{R}^d$, we define and study the Gaussian Cherednik windowed transform $\Phi_G^{s,t}$ given for a regular W-invariant function f on $\mathbb{R}^d$ by

$$\Phi_G^{s,t}(f)(\lambda, y) = \int_{\mathbb{R}^d} f(x) G_{\lambda,y}^{s,t}(x) \mathcal{A}_k(x) dx,$$

where $G_{\lambda,y}^{s,t}$ is the Gaussian Cherednik wavelet given by

$$G_{\lambda,y}^{s,t}(x) = F_{-\lambda}(x) \frac{p_t^W(x, y)}{||p_t^W(x, .)||_{\mathcal{A}_k,2}^s},$$

and by taking as function g of the Section 4, the function E_t^W, we prove for the transform $\Phi_G^{s,t}$ analogous results to the results of the Section 5.

2. The Cherednik operators and their eigenfunctions (see [6][8])

We consider $\mathbb{R}^d$ with the standard basis $\{e_i, i = 1, 2, ..., d\}$ and the inner product $\langle ., . \rangle$ for which this basis is orthonormal. We extend this inner product to a complex bilinear form on $\mathbb{C}^d$.

2.1. The root system, the multiplicity function and the Cherednik operators. Let $\alpha \in \mathbb{R}^d \setminus \{0\}$ and $\check{\alpha} = \dfrac{2}{\|\alpha\|^2} \alpha$. We denote by

$$r_\alpha(x) = x - \langle \check{\alpha}, x \rangle \alpha, \quad x \in \mathbb{R}^d,$$

the reflection in the hyperplan $H_\alpha \subset \mathbb{R}^d$ orthogonal to α.

A finite set $\mathcal{R} \subset \mathbb{R}^d \setminus \{0\}$ is called a root system if $\mathcal{R} \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $r_\alpha \mathcal{R} = \mathcal{R}$, for all $\alpha \in \mathcal{R}$. For a given root system $\mathcal{R}$ the reflections $r_\alpha, \alpha \in \mathcal{R}$, generate a finite group $W \subset O(d)$, called the reflection group associated with $\mathcal{R}$. For a given $\beta \in \mathbb{R}^d \setminus \cup_{\alpha \in \mathcal{R}} H_\alpha$, we fix the positive subsystem $\mathcal{R}_+ = \{\alpha \in \mathcal{R}, \langle \alpha, \beta \rangle > 0\}$, then for each $\alpha \in \mathcal{R}$ either $\alpha \in \mathcal{R}_+$ or $-\alpha \in \mathcal{R}_+$. We denote by $\mathcal{R}_+^0$ the set of positive indivisible roots. Let

$$\mathfrak{a}^+ = \{x \in \mathbb{R}^d, \forall \, \alpha \in \mathcal{R}, \langle \alpha, x \rangle > 0\}$$

be the positive Weyl chamber. We denote by $\overline{\mathfrak{a}^+}$ its closure. Let also $\mathbb{R}_{reg}^d = \mathbb{R}^d \setminus \cup_{\alpha \in \mathcal{R}} H_\alpha$ be the set of regular elements in $\mathbb{R}^d$.

A function $k : \mathcal{R} \to [0, +\infty[$ is called a multiplicity function if it is invariant under the action of the reflection group W. We introduce the index

$$\gamma = \gamma(\mathcal{R}) = \sum_{\alpha \in \mathcal{R}_+} k(\alpha). \tag{2.1}$$

Moreover, let $\mathcal{A}_k$ be the weight function

$$\forall \, x \in \mathbb{R}^d, \ \ \mathcal{A}_k(x) = \prod_{\alpha \in \mathcal{R}_+} |2 \sinh \langle \frac{\alpha}{2}, x \rangle|^{2k(\alpha)},$$

which is W-invariant.

The Cherednik operators $T_j, j = 1, 2, ..., d$, on $\mathbb{R}^d$ associated with the reflection group W and the multiplicity function k, are defined for f of class C^1 on $\mathbb{R}^d$ and $x \in \mathbb{R}_{reg}^d$ by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathbb{R}_+} \frac{k(\alpha)\alpha^j}{1 - e^{-\langle \alpha, x \rangle}} \{f(x) - f(r_\alpha x)\} - \rho_j f(x),$$

where

$$\rho_j = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha^j, \text{ and } \alpha^j = \langle \alpha, e_j \rangle.$$

In the case $k(\alpha) = 0$, for all $\alpha \in \mathcal{R}_+$, the operators $T_j, j = 1, 2, ...d$, reduce to the corresponding partial derivatives. We suppose in the following that $k \neq 0$.

The Cherednik operators form a commutative system of differential-difference operators.

For f of class C^1 on $\mathbb{R}^d$ with compact support and g of class C^1 on $\mathbb{R}^d$, we have for $j = 1, 2, .., d$:

$$\int_{\mathbb{R}^d} T_j f(x) g(x) \mathcal{A}_k(x) dx = -\int_{\mathbb{R}^d} f(x)(T_j + S_j) g(x) \mathcal{A}_k(x) dx,$$

with

$$\forall\, x \in \mathbb{R}^d, S_j g(x) = \sum_{\alpha \in \mathcal{R}_+} k(\alpha) \alpha^j g(r_\alpha x).$$

2.2. The Opdam-Cherednik kernel and the Heckman-Opdam hypergeometric function (see [6][8]). We denote by $G_\lambda, \lambda \in \mathbb{C}^d$, the eigenfunction of the operators $T_j, j = 1, 2, .., d$. It is the unique analytic function on $\mathbb{R}^d$ which satisfies the differential-difference system

$$\begin{cases} T_j G_\lambda(x) & = i\lambda_j G_\lambda(x), \quad j = 1, 2, .., d, x \in \mathbb{R}^d, \\ G_\lambda(0) & = 1. \end{cases} \tag{2.2}$$

It is called the Opdam-Cherednik kernel.

We consider the function F_λ defined by

$$\forall\, x \in \mathbb{R}^d, \quad F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(wx). \tag{2.3}$$

The function $F_\lambda(x)$ called the Heckman-Opdam hypergeometric function, it is W-invariant both in λ and x.

The functions G_λ and F_λ possess the following properties

  i) For all $\lambda \in \mathbb{C}^d$, the functions $x \to G_\lambda(x)$ and $x \to F_\lambda(x)$ are of class C^∞ on $\mathbb{R}^d$.
 ii) For all $x \in \mathbb{R}^d$, the functions $\lambda \to G_\lambda(x)$ and $\lambda \to F_\lambda(x)$ are entire on $\mathbb{C}^d$.
iii) For all $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{C}^d$, we have

$$\overline{G_\lambda(x)} = G_{-\bar{\lambda}}(x) \quad \text{and} \quad \overline{F_\lambda(x)} = F_{-\bar{\lambda}}(x). \tag{2.4}$$

 iv) For all $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}^d$, we have

$$|G_\lambda(x)| \le |W|^{1/2} \text{ and } |F_\lambda(x)| \le |W|^{1/2}. \tag{2.5}$$

v) Let p and q be polynomials of degree m and n. Then, there exists a positive constant M such that for all $\lambda \in \mathbb{C}^d$ and $x \in \mathbb{R}^d$, we have

$$|p(\frac{\partial}{\partial\lambda})q(\frac{\partial}{\partial x})G_\lambda(x)| \leq M(1+\|x\|)^m(1+\|\lambda\|)^n F_0(x)e^{-\max_{w\in W} Im\langle w\lambda,x\rangle}.$$
(2.6)

The same inequality is also true for the function $F_\lambda(x)$.

vi) The function $F_0(x)$ satisfies the estimate

$$\forall\, x \in \bar{\mathfrak{a}}_+, F_0(x) \asymp e^{-\langle\rho,x\rangle} \prod_{a\in\mathcal{R}_+^0} (1+\langle\alpha,x\rangle).$$
(2.7)

EXAMPLE 2.1. For $d = 1$ and $W = \mathbb{Z}_2$, the root system is $\mathcal{R} = \{-2\alpha, -\alpha, \alpha, 2\alpha\}$ with $\alpha = 2$. Here $\mathcal{R}_+ = \{\alpha, 2\alpha\}$. We consider the multiplicity function k. We put $k_1 = k(\alpha) + k(2\alpha)$, $k_2 = k(2\alpha)$, and $\rho = k(\alpha) + 2k(2\alpha) = k_1 + 2k_2$.
The Cherednik operator is the following

$$T_1 f(x) = \frac{d}{dx}f(x) + \left(\frac{2k(\alpha)}{1-e^{-2x}} + \frac{4k(2\alpha)}{1-e^{-4x}}\right)(f(x) - f(-x)) - \rho f(x),$$

which can also be written in the form

$$T_1 f(x) = \frac{d}{dx}f(x) + (k_1\coth(x) + k_2\tanh(x))(f(x) - f(-x)) - \rho f(-x).$$

The Opdam-Cherednik kernel is given by

$$\forall x \in \mathbb{R}, \forall \lambda \in \mathbb{C}, \quad G_\lambda(x) = \varphi_\lambda^{(a,b)}(x) + \frac{1}{i\lambda - \rho}\frac{d}{dx}\varphi_\lambda^{(a,b)}(x),$$

where $\varphi_\lambda^{(a,b)}(x)$ is the Jacobi function (see[4]), with $a = k_1 - \frac{1}{2}$ and $b = k_2 - \frac{1}{2}$.
The Heckman-Opdam hypergeometric function has the form

$$\forall x \in \mathbb{R}, \forall \lambda \in \mathbb{C}, \quad F_\lambda(x) = \varphi_\lambda^{(a,b)}(x).$$

(See [2] p.164-165 and 167.)

3. The harmonic analysis associated to the Heckman-Opdam theory on $\mathbb{R}^d$

Notations. We denote by
 - $\mathcal{E}(\mathbb{R}^d)^W$ the space of C^∞-functions on $\mathbb{R}^d$, which are W-invariant.

- $\mathcal{D}(\mathbb{R}^d)^W$ the space of C^∞-functions on $\mathbb{R}^d$, with compact support and W-invariant.

- $\mathcal{S}(\mathbb{R}^d)^W$ the space of W-invariant functions of the classical Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

- $\mathcal{S}_2(\mathbb{R}^d)^W$ the space of C^∞-functions on $\mathbb{R}^d$, which are W-invariant, and such that for all $\ell, n \in \mathbb{N}$,

$$p_{\ell,n}(f) = \sup_{\substack{|\mu| \leq n \\ x \in \mathbb{R}^d}} (1 + \|x\|)^\ell (F_0(x))^{-1} |D^\mu f(x)| < +\infty,$$

where

$$D^\mu = \frac{\partial^{|\mu|}}{\partial x_1^{\mu_1}...\partial x_d^{\mu_d}} \ , \quad \mu = (\mu_1, ..., \mu_d) \in \mathbb{N}^d, \quad |\mu| = \sum_{i=1}^d \mu_i.$$

Its topology is defined by the semi-norms $p_{\ell,n}, \ell, n \in \mathbb{N}$.

- $PW_a(\mathbb{C}^d)^W$, $a > 0$, the space of entire functions g on $\mathbb{C}^d$, which are W-invariant and satisfying

$$\forall \, m \in \mathbb{N}, q_m(g) = \sup_{\lambda \in \mathbb{C}^d} (1 + \|\lambda\|)^m e^{-a\|Im\lambda\|} |g(\lambda)| < +\infty.$$

The topology of $PW_a(\mathbb{C}^d)$ is defined by the semi-norms $q_m, m \in \mathbb{N}$. We set

$$PW(\mathbb{C}^d)^W = \cup_{a>0} PW_a(\mathbb{C}^d)^W.$$

This space is called the Paley-Wiener space. It is equipped with the inductive limit topology.

3.1. The hypergeometric Fourier transform. The hypergeometric Fourier transform $\mathcal{H}^W$ has been defined and studied first by E.M.Opdam in [6] on the space of W-invariant C^∞-functions on $\mathbb{R}^d$.

DEFINITION 3.1. The hypergeometric Fourier transform $\mathcal{H}^W$ is defined for f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) by

$$\forall \, \lambda \in \mathbb{C}^d, \mathcal{H}^W(f)(\lambda) = \int_{\mathbb{R}^d} f(x) F_{-\lambda}(x) \mathcal{A}_k(x) dx. \tag{3.1}$$

PROPOSITION 3.1. For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) we have the following relations

$$\forall \, \lambda \in \mathbb{R}^d, \mathcal{H}^W(\bar{f})(\lambda) = \overline{\mathcal{H}^W(\check{f})(\lambda)}, \tag{3.2}$$

$$\forall \, \lambda \in \mathbb{R}^d, \mathcal{H}^W(f)(\lambda) = \mathcal{H}^W(\check{f})(-\lambda), \tag{3.3}$$

where $\check{f}$ is the function defined by

$$\forall\, x \in \mathbb{R}^d, \quad \check{f}(x) = f(-x).$$

THEOREM 3.1.

i) *The hypergeometric Fourier transform $\mathcal{H}^W$ is a topological isomorphism from*

- $\mathcal{D}(\mathbb{R}^d)^W$ *onto* $PW(\mathbb{C}^d)^W$.
- $\mathcal{S}_2(\mathbb{R}^d)^W$ *onto* $\mathcal{S}(\mathbb{R}^d)^W$.

ii) *The inverse transform $(\mathcal{H}^W)^{-1}$ is given by*

$$\forall\, x \in \mathbb{R}^d, (\mathcal{H}^W)^{-1}(h)(x) = \int_{\mathbb{R}^d} h(\lambda)F_\lambda(x)\mathcal{C}_k^W(\lambda)d\lambda, \qquad (3.4)$$

where

$$\mathcal{C}_k^W(\lambda) = c|c_k(\lambda)|^{-2}, \qquad (3.5)$$

with c a positive constant chosen in such a way that $\mathcal{C}_k^W(-\rho) = 1$, and

$$c_k(\lambda) = \prod_{\alpha \in \mathcal{R}_+} \frac{\Gamma(\langle i\lambda, \check{\alpha}\rangle + \frac{1}{2}k(\frac{\alpha}{2}))}{\Gamma(\langle i\lambda, \check{\alpha}\rangle + k(\alpha) + \frac{1}{2}k(\frac{\alpha}{2}))} \ , \qquad (3.6)$$

with the convention that $k(\frac{\alpha}{2}) = 0$ if $\frac{\alpha}{2} \notin \mathcal{R}$.
(See [10][11]).

REMARK 3.1. The function $\mathcal{C}_k^W$ is continuous on $\mathbb{R}^d$ and satisfies the estimate

$$\forall\, \lambda \in \mathbb{R}^d, |\mathcal{C}_k^W(\lambda)| \leq const.(1 + \|\lambda\|)^s, \qquad (3.7)$$

for some $s > 0$.

Notations. We denote by

- $L_{\mathcal{A}_k}^p(\mathbb{R}^d)^W$, $1 \leq p \leq +\infty$, the space of measurable functions f on $\mathbb{R}^d$ which are W-invariant and satisfying

$$\|f\|_{\mathcal{A}_k,p} = \left(\int_{\mathbb{R}^d} |f(x)|^p \mathcal{A}_k(x)dx \right)^{1/p} < +\infty, \quad 1 \leq p < +\infty,$$

$$\|f\|_{\mathcal{A}_k,\infty} = \underset{x\in\mathbb{R}^d}{ess\,sup} |f(x)| < +\infty.$$

- $L_{\mathcal{C}_k^W}^p(\mathbb{R}^d)^W$, $1 \leq p \leq +\infty$, the space of measurable functions f on $\mathbb{R}^d$, which are W-invariant and satisfying

$$\|f\|_{\mathcal{C}_k^W,p} = \left(\int_{\mathbb{R}^d} |f(\lambda)|^p \mathcal{C}_k^W(\lambda)d\lambda \right)^{1/p} < +\infty, \quad 1 \leq p < +\infty,$$

$$\|f\|_{\mathcal{C}_k^W,\infty} = \underset{\lambda\in\mathbb{R}^d}{ess\,sup} |f(\lambda)| < +\infty.$$

REMARK 3.2.

i) The space $\mathcal{D}(\mathbb{R}^d)^W$ is dense in the space $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.

ii) $S_2(\mathbb{R}^d)^W \subset L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.

DEFINITION 3.2. The hypergeometric Fourier transform $\mathcal{H}^W$ is defined for f in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$ by

$$\forall\,\lambda \in \mathbb{R}^d, \mathcal{H}^W(f)(\lambda) = \int_{\mathbb{R}^d} f(x)F_{-\lambda}(x)\mathcal{A}_k(x)dx. \qquad (3.8)$$

THEOREM 3.2.

i) **(Plancherel formulas)**. For all f, g in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ we have

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)}\mathcal{A}_k(x)dx = \int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda)\overline{\mathcal{H}^W(g)(\lambda)}\mathcal{C}^W_k(\lambda)d\lambda. \qquad (3.9)$$

and

$$||f||_{\mathcal{A}_k,2} = ||\mathcal{H}^W(f)||_{\mathcal{C}^W_k,2}. \qquad (3.10)$$

ii) **(Plancherel theorem)**. The hypergeometric Fourier transform $\mathcal{H}^W$ extends uniquely to an isometric isomorphism from $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ onto $L^2_{\mathcal{C}^W_k}(\mathbb{R}^d)^W$.

THEOREM 3.3. For all f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ such that $\mathcal{H}^W(f)$ belongs to $L^1_{\mathcal{C}^W_k}(\mathbb{R}^d)^W$, we have the following inversion formula

$$f(x) = \int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda)F_\lambda(x)\mathcal{C}^W_k(\lambda)d\lambda,\ \ a.e.\ \ x \in \mathbb{R}^d\ . \qquad (3.11)$$

REMARK 3.3. The inversion formula (3.11) is also true for all bounded function f in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$ such that $\mathcal{H}^W(f)$ belongs to $L^1_{\mathcal{C}^W_k}(\mathbb{R}^d)^W$.

3.2. The hypergeometric translation operator.

DEFINITION 3.3. The hypergeometric translation operator $\mathcal{T}^W_x, x \in \mathbb{R}^d$, is defined on $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ by

$$\mathcal{H}^W(\mathcal{T}^W_x(f))(\lambda) = F_\lambda(x)\mathcal{H}^W(f)(\lambda), \quad \lambda \in \mathbb{R}^d. \qquad (3.12)$$

PROPOSITION 3.2. For all f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, the mapping $x \to \mathcal{T}^W_x(f)$ is continuous from $\mathbb{R}^d$ into $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and we have

$$\|\mathcal{T}^W_x(f)\|_{\mathcal{A}_k,2} \le |W|^{1/2}\|f\|_{\mathcal{A}_k,2}. \qquad (3.13)$$

PROPOSITION 3.3. *The operator* $\mathcal{T}_x^W, x \in \mathbb{R}^d$, *satisfies the following properties*
i) *For all* $x \in \mathbb{R}^d$, *the operator* $\mathcal{T}_x^W$ *is continuous from* $\mathcal{E}(\mathbb{R}^d)^W$ *into itself.*
ii) *For all* f *in* $\mathcal{E}(\mathbb{R}^d)^W$ *and* $x, y \in \mathbb{R}^d$, *we have*

$$\mathcal{T}_x^W(f)(0) = f(x) \text{ and } \mathcal{T}_x^W(f)(y) = \mathcal{T}_y^W(f)(x). \qquad (3.14)$$

iii) *For all* $x, y \in \mathbb{R}^d$ *and* $\lambda \in \mathbb{C}^d$, *we have the product formula*

$$\mathcal{T}_x^W(F_\lambda)(y) = F_\lambda(x).F_\lambda(y), \qquad (3.15)$$

where F_λ *is the Heckman-Opdam hypergeometric function given by (2.3).*
iv) *For all functions* $g \in \mathcal{E}(\mathbb{R}^d)^W$, $f \in \mathcal{D}(\mathbb{R}^d)^W$ *and* $x \in \mathbb{R}^d$, *we have*

$$\int_{\mathbb{R}^d} \mathcal{T}_x^W(g)(t)f(t)\mathcal{A}_k(t)dt = \int_{\mathbb{R}^d} g(y)\mathcal{T}_{-x}^W(f)(y)\mathcal{A}_k(y)dy, \qquad (3.16)$$

PROPOSITION 3.4. *Let* g *be a function in* $(L_{\mathcal{A}_k}^1 \cap L_{\mathcal{A}_k}^2)(\mathbb{R}^d)^W$.
i) *For all* $x \in \mathbb{R}^d$, *we have*

$$||\mathcal{T}_x^W g||_{\mathcal{A}_k,2}^2 = \int_{\mathbb{R}^d} |F_\lambda(x)|^2 |\mathcal{H}^W(g)(\lambda)|^2 \mathcal{C}_k^W(\lambda)d\lambda. \qquad (3.17)$$

i) *the function* $x \longrightarrow ||\mathcal{T}_x^W g||_{\mathcal{A}_k,2}$ *is continous on* $\mathbb{R}^d$.

Proof. i) We deduce the result from Theorem 3.2 and the relation (3.12).
ii) We consider the relation (3.17).
For all $x \in \mathbb{R}^d$, the function $\lambda \longrightarrow |F_\lambda(x)|^2 |\mathcal{H}^W(g)(\lambda)|^2$ is continuous on $\mathbb{R}^d$ and from (2.5) it is bounded by $|W||\mathcal{H}^W(g)(\lambda)|^2$ which is in $L_{\mathcal{C}_k}^1(\mathbb{R}^d)^W$, then from the dominated convergence theorem, the function $x \longrightarrow ||\mathcal{T}_x^W g||_{\mathcal{A}_k,2}$ is continuous on $\mathbb{R}^d$. $\qquad \square$

3.3. Definition and properties of the heat kernels.

3.3.1. *The heat kernel* $p_t^W(x,y)$, $t > 0$.

DEFINITION 3.4. Let $t > 0$. The heat kernel $p_t^W(x,y)$ associated to the Heckman-Opdam theory, is defined for $x, y \in \mathbb{R}^d$, by

$$p_t^W(x,y) = \int_{\mathbb{R}^d} e^{-t(\|\lambda\|^2 + \|\rho\|^2)} F_\lambda(x) F_\lambda(-y) \mathcal{C}_k^W(\lambda)d\lambda. \qquad (3.18)$$

Notations We denote by

- H_k^W the heat operator associated with the Cherednik operator on $\mathbb{R}^d$ given by

$$H_k^W = \mathcal{L}_k^W - \frac{\partial}{\partial t} - \|\rho\|^2, \qquad (3.19)$$

where $\mathcal{L}_k^W$ is the Heckman-Opdam Laplacian defined for f of class C^2 on $\mathbb{R}^d$ by

$$\mathcal{L}_k^W f = \sum_{j=1}^d T_j^2 f. \qquad (3.20)$$

It has the following form : For all $x \in \mathbb{R}_{reg}^d$,

$$\mathcal{L}_k^W f(x) = \Delta f(x) + \sum_{\alpha \in \mathcal{R}_+} k(\alpha) \coth\langle\frac{\alpha}{2}, x\rangle\langle\nabla f(x), \alpha\rangle + \|\rho\|^2 f(x), \quad (3.21)$$

where Δ and ∇ are respectively the usual Laplacian and the gradient on $\mathbb{R}^d$.

- $E_t^W, t > 0$, the fundamental solution of the operator H_k^W given by

$$\forall\, x \in \mathbb{R}^d, E_t^W(x) = p_t^W(x, 0). \qquad (3.22)$$

PROPOSITION 3.5.

i) For all $t > 0$, the function E_t^W belongs to $\mathcal{S}_2(\mathbb{R}^d)$.

ii) For all $t > 0$ and $x \in \mathbb{R}^d$, we have

$$E_t^W(x) \asymp t^{-\gamma-\frac{d}{2}} \left[\prod_{\alpha \in \mathcal{R}_+^0} (1 + |\langle\alpha, x\rangle|)(1 + 2t + |\langle\alpha, x\rangle|)^{k(\alpha)+k(2\alpha)-1} \right]$$

$$\times e^{-t\|\rho\|^2 - \langle\rho, x^+\rangle - \frac{\|x\|^2}{4t}}, \qquad (3.23)$$

where x^+ denotes the unique conjugate of x in $\overline{\mathfrak{a}_+}$.

iii) For all $t > 0$, we have

$$\forall\, \lambda \in \mathbb{R}^d, \mathcal{H}^W(E_t^W)(\lambda) = e^{-t(\|\lambda\|^2 + \|\rho\|^2)}. \qquad (3.24)$$

iv) The function $(x, t) \to E_t^W(x)$ is strictly positive on $\mathbb{R}^d \times]0, +\infty[$.

v) For all $t > 0$, we have

$$\int_{\mathbb{R}^d} E_t^W(x)\mathcal{A}_k(x)dx = 1. \qquad (3.25)$$

vi) We have

$$H_k^W(E_t^W)(x) = 0, \quad \text{on } \mathbb{R}^d \times]0, +\infty[. \qquad (3.26)$$

PROPOSITION 3.6.

i) For all $t > 0$ and $x \in \mathbb{R}^d$, the function $y \to p_t^W(x, y)$ belongs to $\mathcal{S}_2(\mathbb{R}^d)$.

ii) For all $t > 0$ and $x, y \in \mathbb{R}^d$, we have

$$p_t^W(x, y) = \mathcal{T}_x^W(E_t^W)(y). \qquad (3.27)$$

iii) The function $p_t^W(x, y)$ is strictly positive on $\mathbb{R}^d \times \mathbb{R}^d \times]0, +\infty[$.

iv) For all $t > 0$ and $x \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} p_t^W(x, y) \mathcal{A}_k(y) dy = 1. \qquad (3.28)$$

v) For all $y \in \mathbb{R}^d$, the function $(x, t) \to p_t^W(x, y)$ satisfies

$$H_k^W(p_t^W)(x, y) = 0, \text{ on } \mathbb{R}^d \times]0, +\infty[. \qquad (3.29)$$

PROPOSITION 3.7.
For all $x \in \mathbb{R}^d$ and $t \in]0, +\infty[$ we have

$$||p_t^W(x, .)||_{\mathcal{A}_k, 2}^2 = \int_{\mathbb{R}^d} |F_\lambda(x)|^2 e^{-2t(||\lambda||^2 + ||\rho||^2)} \mathcal{C}_k^W(\lambda) d\lambda. \qquad (3.30)$$

Proof. We deduce (3.30) from (3.17) and (3.24). $\qquad \square$

3.3.2. *The heat kernel $p_T^W(-x, \sqrt{T}y)$, $T > 0$.*

DEFINITION 3.5. Let $T > 0$. The heat kernel $p_T^W(-x, \sqrt{T}y)$ associated to the Heckman-Opdam theory, is defined for $x, y \in \mathbb{R}^d$, by

$$p_T^W(-x, \sqrt{T}y) = \int_{\mathbb{R}^d} e^{-\frac{T}{2}(||\lambda||^2 + ||\rho||^2)} F_\lambda(x) F_\lambda(\sqrt{T}y) \mathcal{C}_k^W(\lambda) d\lambda. \qquad (3.31)$$

(See [7],[8]).

The function $p_T^W(-x, \sqrt{T}y)$ satisfies the following property.

PROPOSITION 3.8. There exists a constant $K > 0$, such that for any $x \in \overline{\mathfrak{a}^+}$ and any $y \in \mathfrak{a}^+$, we have

$$p_T^W(-x, \sqrt{T}y) \sim K e^{-\frac{1}{2}(||y||^2 + ||\rho||^2)} T^{-(\frac{x}{2} + |\mathcal{R}_0^+|)} F_0(x) F_0(\sqrt{T}y). \qquad (3.32)$$

(See [7] p.56)

REMARK 3.4. We denote by

$$\check{E}_T^W(x) = p_T^W(-x, 0).$$

Thus

$$p_T^W(-x, \sqrt{T}y) = \tau_x^W(\check{E}_T^W)(\sqrt{T}y). \qquad (3.33)$$

PROPOSITION 3.9. Foa all $\epsilon > 0$, there exist $T_0 > 0$ such that for all $T \geq T_0$, we have

$$\forall x, y \in \mathfrak{a}^+, \quad 0 \leq p_T^W(-x, \sqrt{T}y) \leq D_\epsilon(T)e^{-\frac{1}{2}||y||^2}F_0(x)F_0(\sqrt{T}y), \quad (3.34)$$

with

$$D_\epsilon(T) = (1 + \epsilon)Ke^{-\frac{1}{2}||\rho||^2}T^{-(\frac{x}{2} + |\mathcal{R}_0^+|)}. \qquad (3.35)$$

Proof. Let $C(x, y, T)$ the second member of the relation (3.32). For all $\epsilon > 0$, there exists $T_0 > 0$ such that for all $T \geq T_0$, we have

$$|\frac{p_T^W(-x, \sqrt{T}y)}{C(x, y, T)} - 1| \leq \epsilon.$$

Thus

$$p_T^W(-x, \sqrt{T}y) \leq (1 + \epsilon)C(x, y, T). \qquad (3.36)$$

The relation (3.36) imply that

$$p_T^W(-x, \sqrt{T}y) \leq D_\epsilon(T)e^{-\frac{1}{2}||y||^2}F_0(x)F_0(\sqrt{T}y). \qquad (3.37)$$

with $D_\epsilon(T)$ given by the relation (3.34). $\qquad\qquad\square$

COROLLARY 3.1. For all $\epsilon > 0$, there exist $T_0 > 0$ such that for all $T \geq T_0$, we have

$$\forall x, y \in \mathfrak{a}^+, \quad |\tau_x^W(\check{E}_T^W)(\sqrt{T}y)|^2 \leq C_\epsilon(y, T)|F_0(x)|^2, \qquad (3.38)$$

with

$$C_\epsilon(y, T) = D_\epsilon^2(T)e^{-||y||^2}|F_0(\sqrt{T}y)|^2. \qquad (3.39)$$

PROPOSITION 3.10. For all $\epsilon > 0$, there exists $T_0 > 0$ such that for all $T \geq T_0$, we have

$$\forall x \in \mathbb{R}^d, \quad ||\tau_x^W(\check{E}_T)||_{\mathcal{A}_k,2}^2 \leq const\, D_\epsilon^2(T)|F_0(x)|^2. \qquad (3.40)$$

Proof. The functions $\tau_x^W(\check{E}_T^W)(\sqrt{T}y)$ and $\mathcal{A}_k(\sqrt{T}y)$ are W-invariant with respect to the variables x, y on $\mathbb{R}^d$. Then we have

$$\forall x \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} |\tau_x^W(\check{E}_T^W)(z)|^2 \mathcal{A}_k(z)dz \leq \text{const} \int_{\mathfrak{a}^+} |\tau_x^W(\check{E}_T^W)(\sqrt{T}y)|^2 \mathcal{A}_k(\sqrt{T}y)dy.$$
(3.41)

By using the relations (3.41),(3.38),(3.39) we obtain

$$\forall x \in \mathbb{R}^d, \quad ||\tau_x^W(\check{E}_T)||_{\mathcal{A}_k,2}^2 \leq \text{const } D_\epsilon^2(T)|F_0(x)|^2 \int_{\mathfrak{a}^+} e^{-||y||^2} |F_0(\sqrt{T}y)|^2 \mathcal{A}_k(\sqrt{T}y)dy.$$
(3.42)

From the relation (2.7) and the fact that from [12] p. 237, we have

$$\forall y \in \mathbb{R}^d, \quad \mathcal{A}_k(\sqrt{T}y) \leq 2^{2\gamma} e^{2\sqrt{T}\langle \rho, y \rangle}.$$
(3.43)

Then the integral of the second member of the relation (3.42) is equal to the following integral which converge

$$2^{2\gamma} \int_{\mathfrak{a}^+} e^{-||y||^2} \prod_{\alpha \in \mathcal{R}_+^0} (1 + \sqrt{T}\langle \alpha, y \rangle)^2 dy.$$

Thus the relation (3.42) can be written in the form given by the relation (3.40). $\square$

PROPOSITION 3.11. *Let $y \in \mathfrak{a}^+$ and $s \in]-\infty, 0[$, then for all $T \geq T_0$, We have*

$$\int_{\mathbb{R}^d} \frac{|\tau_x^W(\check{E}_T^W)(\sqrt{T}y)|^2}{||\tau_x^W(E_T^W)||_{\mathcal{A}_k,2}^{2s}} \mathcal{A}_k(x)dx \leq \text{const } C_0(y,T) \int_{\mathfrak{a}^+} e^{2s\langle \rho, x \rangle} [\prod_{\alpha \in \mathcal{R}_+^0} (1 + \langle \alpha, x \rangle)]^{2(1-s)} dx.$$
(3.44)

Proof. As the functions $x \longmapsto \dfrac{|\tau_x^W(\check{E}_T^W)(\sqrt{T}y)|^2}{||\tau_x^W(E_T^W)||_{\mathcal{A}_k,2}^{2s}}$ and $\mathcal{A}_k$, are W-invariant on $\mathbb{R}^d$, then we have

$$\int_{\mathbb{R}^d} \frac{|\tau_x^W(\check{E}_T^W)(\sqrt{T}y)|^2}{||\tau_x^W(\check{E}_T^W)||_{\mathcal{A}_k,2}^{2s}} \mathcal{A}_k(x)dx \leq \text{const} \int_{\mathfrak{a}^+} \frac{|\tau_x^W(\check{E}_T^W)(\sqrt{T}y)|^2}{||\tau_x^W(\check{E}_T^W)||_{\mathcal{A}_k,2}^{2s}} \mathcal{A}_k(x)dx.$$
(3.45)

By using the relations (3.40),(3.38), (3.39), we obtain

$$\int_{\mathbb{R}^d} \frac{|\tau_x^W(\check{E}_T^W)(\sqrt{T}y)|^2}{||\tau_x^W(\check{E}_T^W)||_{\mathcal{A}_k,2}^{2s}} \mathcal{A}_k(x)dx \leq \text{const } C_\epsilon(y,T) \int_{\mathfrak{a}^+} |F_0(x)|^{2(1-s)} \mathcal{A}_k(x)dx.$$
(3.46)

From the relations (2.7) and (3.43), we obtain

$$\int_{\mathfrak{a}^+} |F_0(x)|^{2(1-s)} \mathcal{A}_k(x) dx \leq \text{const} \int_{\mathfrak{a}^+} e^{2s\langle \rho, x \rangle} \Big[\prod_{\alpha \in \mathcal{R}_+^0} (1 + \langle \alpha, x \rangle) \Big]^{2(1-s)} dx.$$

(3.47)

Then we deduce the relation (3.44) from the relations (3.46),(3.47).
□

The relation (3.44) implies the following Corollary.

COROLLARY 3.2. *The function* $x \longmapsto \dfrac{\tau_x^W(\check{E}_T^W)(\sqrt{T}y)}{||\tau_x^W(\check{E}_T^W)||_{\mathcal{A}_k, 2}^s}$ *belongs to*
$L_{\mathcal{A}_k}^2(\mathbb{R}^d)^W$ *for* $s \in]-\infty, 0[$.

PROPOSITION 3.12. *The function* $x \longmapsto \dfrac{\tau_x^W(\check{E}_T^W)(\sqrt{T}y)}{||\tau_x^W(\check{E}_T^W)||_{\mathcal{A}_k, 2}^s}$ *belongs to*
$L_{\mathcal{A}_k}^\infty(\mathbb{R}^d)^W$ *for*
$s \in]-\infty, 0[$.

Proof. From the relations (3.33), (3.34), (3.40) we have

$$\forall x \in \mathbb{R}^d, \quad \frac{\tau_x^W(\check{E}_T^W)(\sqrt{T}y)}{||\tau_x^W(\check{E}_T^W)||_{\mathcal{A}_k, 2}^s} \leq \text{const } D_\epsilon^{(1-s)}(T) e^{-\frac{1}{2}||y||^2} (F_0(x))^{(1-s)} F_0(\sqrt{T}y).$$

□

Thus this inequality gives the result of the Proposition.

COROLLARY 3.3. *The function* $x \longmapsto \dfrac{\tau_x^W(\check{E}_T^W)(\sqrt{T}y)}{||\tau_x^W(\check{E}_T^W)||_{\mathcal{A}_k, 2}^s}$ *belongs to*
$(L_{\mathcal{A}_k}^2 \cap L_{\mathcal{A}_k}^\infty)(\mathbb{R}^d)^W$ *for* $s \in]-\infty, 0[$.

3.4. The hypergeometric convolution product.

DEFINITION 3.6. The hypergeometric convolution product $f *_{\mathcal{H}^W} g$ of the functions f, g in $\mathcal{D}(\mathbb{R}^d)^W$ (*resp.* $\mathcal{S}_2(\mathbb{R}^d)^W$) is defined by

$$\forall x \in \mathbb{R}^d, f *_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(-y) g(y) \mathcal{A}_k(y) dy. \qquad (3.48)$$

PROPOSITION 3.13. *Let f be in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and g in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$, then the function $f *_{\mathcal{H}^W} g$ defined all most everywhere on $\mathbb{R}^d$ by*

$$f *_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(-y)g(y)\mathcal{A}_k(y)dy, \qquad (3.49)$$

belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, and we have

$$\|f *_{\mathcal{H}^W} g\|_{\mathcal{A}_k,2} \le |W|^{1/2}\|f\|_{\mathcal{A}_k,2}\|g\|_{\mathcal{A}_k,1}, \qquad (3.50)$$

and

$$\mathcal{H}^W(f *_{\mathcal{H}^W} g) = \mathcal{H}^W(f).\mathcal{H}^W(g). \qquad (3.51)$$

PROPOSITION 3.14. *Let f and g be in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. Then the function $f *_{\mathcal{H}^W} g$ defined on $\mathbb{R}^d$ by*

$$f *_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(-y)g(y)\mathcal{A}_k(y)dy, \qquad (3.52)$$

is continuous on $\mathbb{R}^d$, tends to zero at the infinity and we have

$$\sup_{x\in\mathbb{R}^d} |f *_{\mathcal{H}^W} g(x)| \le |W|^{1/2}\|f\|_{\mathcal{A}_k,2}\|g\|_{\mathcal{A}_k,2}. \qquad (3.53)$$

4. The Cherednik wavelets on $\mathbb{R}^d$ in the W-invariant case

We consider in this section a non negligible function g in $(L^1_{\mathcal{A}_k} \cap L^2_{\mathcal{A}_k})(\mathbb{R}^d)^W$.

Notation. We denote by $\mathcal{N}^2_{g,s}(\mathbb{R}^d)^W$, $s \in]-\infty, 0[$, the space of measurable functions on $\mathbb{R}^d$, which are W-invariant and satisfying

$$\|f\|^2_{\mathcal{N}^2_{g,s}} = \int_{\mathbb{R}^d} |f(x)|^2 \frac{\mathcal{A}_k(x)dx}{\|\mathcal{T}_x^W g\|^{2(s-1)}_{\mathcal{A}_k,2}} < +\infty.$$

PROPOSITION 4.1. *We have*

$$L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W \subset \mathcal{N}^2_{g,s}(\mathbb{R}^d)^W.$$

Proof. Let f be in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)$. As the function g belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)$, then from the Proposition 3.2, we have

$$\|\tau_x^W(g)\|_{\mathcal{A}_k,2} \le |W|^{\frac{1}{2}}\|g\|_{\mathcal{A}_k,2}. \qquad (4.2)$$

By using the relations (4.1),(4.2) we obtain

$$\|f\|^2_{\mathcal{N}^2_{g,s}} \le |W|^{(1-s)}\|g\|^{2(1-s)}_{\mathcal{A}_k,2}\|f\|_{\mathcal{A}_k,2} < \infty.$$

Thus f belongs to $\mathcal{N}^2_{g,s}(\mathbb{R}^d)^W$ $\qquad\qquad\qquad\square$

DEFINITION 4.1. Let $\lambda, y \in \mathbb{R}^d$. The family $\{g^s_{\lambda,y}\}_{s\in]-\infty,0[}$ of Cherednik wavelets on $\mathbb{R}^d$ in the W-invariant case, is defined on $\mathbb{R}^d$ by

$$g^s_{\lambda,y}(x) = F_{-\lambda}(x)\frac{\mathcal{T}^W_x g(y)}{||\mathcal{T}^W_x g||^s_{\mathcal{A}_k,2}}. \qquad (4.2)$$

PROPOSITION 4.2. We suppose that the function g is such that, for all $y \in \mathbb{R}^d$ and
$s \in]-\infty, 0[$, the function $x \longrightarrow \frac{\mathcal{T}^W_y g(x)}{||\mathcal{T}^W_x g||^s_{\mathcal{A}_k,2}}$ belongs to $(L^\infty_{\mathcal{A}_k} \cap L^2_{\mathcal{A}_k})(\mathbb{R}^d)^W$,
then the function $g^s_{\lambda,y}$ belongs to $(L^\infty_{\mathcal{A}_k} \cap L^2_{\mathcal{A}_k})(\mathbb{R}^d)^W$.

Proof. We deduce the results from the relations (4.2),(2.5). $\qquad\square$

REMARK 4.1. Let $p^W_T(-x, \sqrt{T}y)$ the function given by the relation (3.40). Then $\{(p^W_T)^s_{\lambda,y}\}_{s\in]-\infty,0[}$ is a family of Cherednik wavelets on $\mathbb{R}^d$.

PROPOSITION 4.3. Under the hypothesis of Proposition 4.2 and if moreover the function $y \longrightarrow \frac{\mathcal{T}^W_x g(y)}{||\mathcal{T}^W_x g||^s_{\mathcal{A}_k,2}}$ is continuous from $\mathbb{R}^d$ into $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, then the function $(\lambda, y) \longrightarrow g^s_{\lambda,y}$ is continuous from $\mathbb{R}^d \times \mathbb{R}^d$ into $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.

Proof. Let $(\lambda_0, y_0) \in \mathbb{R}^d \times \mathbb{R}^d$. Using (4.2) we obtain

$$\left\|g^s_{\lambda,y} - g^s_{\lambda_0,y_0}\right\|_{\mathcal{A}_k,2} \leq \left\|F_{-\lambda_0}(\xi)\Big(\frac{\mathcal{T}^W_\xi g(y)}{||\mathcal{T}^W_\xi g||^s_{\mathcal{A}_k,2}} - \frac{\mathcal{T}^W_\xi g(y_0)}{||\mathcal{T}^W_\xi g||^s_{\mathcal{A}_k,2}}\Big)\right\|_{\mathcal{A}_k,2}$$

$$+ \left\|(F_{-\lambda}(\xi) - F_{-\lambda_0}(\xi)).\frac{\mathcal{T}^W_\xi g(y_0)}{||\mathcal{T}^W_\xi g||^s_{\mathcal{A}_k,2}}\right\|_{\mathcal{A}_k,2}$$

$$+ \left\|(F_{-\lambda}(\xi) - F_{-\lambda_0}(\xi))\Big(\frac{\mathcal{T}^W_\xi g(y)}{||\mathcal{T}^W_\xi g||^s_{\mathcal{A}_k,2}} - \frac{\mathcal{T}^W_\xi g(y_0)}{||\mathcal{T}^W_\xi g||^s_{\mathcal{A}_k,2}}\Big)\right\|_{\mathcal{A}_k,2}$$

Using (2.5), we get

$$\left\|g^s_{\lambda,y} - g^s_{\lambda_0,y_0}\right\|_{\mathcal{A}_k,2} \leq 3|W|^{\frac{1}{2}}\left\|\frac{\mathcal{T}^W_\xi g(y)}{||\mathcal{T}^W_\xi g||^s_{\mathcal{A}_k,2}} - \frac{\mathcal{T}^W_\xi g(y_0)}{||\mathcal{T}^W_\xi g||^s_{\mathcal{A}_k,2}}\right\|_{\mathcal{A}_k,2}$$

$$+ \left\|(F_{-\lambda}(\xi) - F_{-\lambda_0}(\xi)).\frac{\mathcal{T}^W_\xi g(y_0)}{||\mathcal{T}^W_\xi g||^s_{\mathcal{A}_k,2}}\right\|_{\mathcal{A}_k,2}$$

From hypothesis i) we obtain

$$\lim_{y \to y_0} \left\|\frac{\mathcal{T}_\xi^W g(y)}{||\mathcal{T}_\xi^W g||_{k,\beta,2}^s} - \frac{\mathcal{T}_\xi^W g(y_0)}{||\mathcal{T}_\xi^W g||_{\mathcal{A}_k,2}^s}\right\|_{\mathcal{A}_k,2} = 0, \qquad (4.3)$$

and from Proposition 4.2, the ralation (2.5) and the dominated convergence theorem, we get

$$\lim_{\lambda \to \lambda_0} \left\|(F_{-\lambda}(\xi) - F_{-\lambda_0}(\xi)).\frac{\mathcal{T}_\xi^W g(y_0)}{||\mathcal{T}_\xi^W g||_{\mathcal{A}_k,2}^s}\right\|_{\mathcal{A}_k,2} = 0. \qquad (4.4)$$

Using (4.3),(4.4) we deduce that

$$\lim_{(\lambda,y) \to (\lambda_0,y_0)} ||g_{\lambda,y}^s - g_{\lambda_0,y_0}^s||_{\mathcal{A}_k,2} = 0.$$

$$\square$$

5. The Cherednik windowed transform on $\mathbb{R}^d$ in the W-invariant case

In this section, we take a non negligible function g in $(L_{\mathcal{A}_k}^1 \cap L_{\mathcal{A}_k}^2)(\mathbb{R}^d)^W$ satisfying the hypothesis of Propositions 4.2, 4.3.

DEFINITION 5.1. Let $s \in]-\infty, 0[$. The Cherednik windowed transform Φ_g^s is defined for a regular W-invariant function f on $\mathbb{R}^d$ by

$$\Phi_g^s(f)(\lambda, y) = \int_{\mathbb{R}^d} f(x)g_{\lambda,y}^s(x)\mathcal{A}_k(x)dx, \quad \lambda, y \in \mathbb{R}^d. \qquad (5.1)$$

REMARK 5.1. By using the relation (4.2) the relation (5.1) can also be written in the following form

$$\Phi_g^s(f)(\lambda, y) = \mathcal{H}^W(f.\frac{\mathcal{T}_y^W(g)}{||\mathcal{T}_\xi^W g||_{\mathcal{A}_k,2}^s})(\lambda), \quad \lambda, y \in \mathbb{R}^d \qquad (5.2)$$

where $\mathcal{H}^W$ is the hypergeometric Fourier transform given by (3.8).

5.1. Plancherel formula for the Cherednik windowed transform in the W-invariant case.

THEOREM 5.1. *For all $s \in]-\infty, 0[$ and $f \in L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, we have for the transform Φ^s_g the following Plancherel formula*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\Phi^s_g(f)(\lambda, y)|^2 \mathcal{A}_k(y) \mathcal{C}^W_k(\lambda) dy d\lambda = ||f||^2_{\mathcal{N}^2_{g,s}}.$$

Proof. For all $y \in \mathbb{R}^d$, the function $\frac{\mathcal{T}^W_x(g)(y)}{||\mathcal{T}^W_x g||^s_{\mathcal{A}_k,2}}$ is in $L^\infty_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and as f is in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)$, then the function $x \longrightarrow f(x)\frac{\mathcal{T}^W_x(g)(y)}{||\mathcal{T}^W_x g||^s_{\mathcal{A}_k,2}}$ belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. Thus, from (5.2) we deduce that

$$\int_{\mathbb{R}^d} (\int_{\mathbb{R}^d} |\Phi^s_g(f)(\lambda, y)|^2 \mathcal{C}^W_k(\lambda) d\lambda) \mathcal{A}_k(y) dy$$

$$= \int_{\mathbb{R}^d} (\int_{\mathbb{R}^d} |\mathcal{H}^W(f.\frac{\mathcal{T}^W_y(g)}{||\mathcal{T}^W_\xi g||^s_{\mathcal{A}_k,2}})(\lambda)|^2 \mathcal{C}^W_k(\lambda) d\lambda) \mathcal{A}_k(y) dy.$$

From Theorem 3.2 and Fubini-Tonnelli's theorem we obtain

$$\int_{\mathbb{R}^d} (\int_{\mathbb{R}^d} |\Phi^s_g(f)(\lambda, y)|^2 \mathcal{C}^W_k(\lambda) d\lambda) \mathcal{A}_k(y) dy$$

$$= \int_{\mathbb{R}^d} \frac{|f(x)|^2}{||\mathcal{T}^W_x(g)||^{2s}_{\mathcal{A}_k,2}} (\int_{\mathbb{R}^d} |\mathcal{T}^W_y g(x)|^2 \mathcal{A}_k(y) dy) \mathcal{A}_k(x) dx$$

$$= \int_{\mathbb{R}^d} \frac{|f(x)|^2}{||\mathcal{T}^W_x(g)||^{2(s-1)}_{\mathcal{A}_k,2}} \mathcal{A}_k(x) dx$$

$$= ||f||^2_{\mathcal{N}^2_{g,s}}.$$

$\square$

5.2. Inversion formula for the Cherednik windowed transform in the W-invariant case.

THEOREM 5.2. *For all $s \in]-\infty, 0[$, the transform Φ^s_g admits the following inversion formula. Let $B(0, n)$ be the closed ball of center 0 and radius $n \in \mathbb{N} \backslash \{0\}$.*
Then we have

$$f(z) = \lim_{n \to +\infty} \int_{B(0,n)} (\int_{\mathbb{R}^d} \Phi^s_g(f)(\lambda, y) \tilde{g}^{(2-s)}_{\lambda, y, z}(x) \mathcal{A}_k(y) dy) \mathcal{C}^W_k(\lambda) d\lambda, \ a.e \ x \in \mathbb{R}^d$$

$$(5.3)$$

with

$$\tilde{g}^{2-s}_{\lambda,y,z}(x) = F_\lambda(z)\frac{\overline{\mathcal{T}^W_y g(x)}}{||\mathcal{T}^W_x g||^{2-s}_{\mathcal{A}_k,2}}.$$

This formula is true for the functions f of the space $(L^1_{\mathcal{A}_k} \cap L^2_{\mathcal{A}_k})(\mathbb{R}^d)^W$.

Proof. For all function f in $(L^1_{\mathcal{A}_k} \cap L^2_{\mathcal{A}_k})(\mathbb{R}^d)^W$. We have from Fubini-Tonnelli's theorem

$$\int_{\mathbb{R}^d} \Phi^s_g(f)(\lambda,y)\tilde{g}^{2-s}_{\lambda,y,z}(x)\mathcal{A}_k(y)dy = F_\lambda(z)\mathcal{H}^W(f)(\lambda).$$

Then from Theorem 3.3 we obtain

$$\lim_{n\to+\infty}\int_{B(0,n)}(\int_{\mathbb{R}^d}\Phi^s_g(f)(\lambda,y)\tilde{g}^{2-s}_{\lambda,y,z}(x)\mathcal{A}_k(y)dy)\mathcal{C}^W_k(\lambda)d\lambda) = f(z), \ a.e \ x \in \mathbb{R}^d.$$

$\square$

THEOREM 5.3. *For all $s \in]-\infty,0[$. We suppose that the function g is in $S_2(\mathbb{R}^d)^W$. Then for all f in $S_2(\mathbb{R}^d)^W$, we have the following inversion formula*

$$\forall x \in \mathbb{R}^d, \quad f(z) = \int_{\mathbb{R}^d}(\int_{\mathbb{R}^d}\Phi^s_g(f)(\lambda,y)\tilde{g}^{2-s}_{\lambda,y,z}(x)\mathcal{A}_k(y)dy)\mathcal{C}^W_k(\lambda)d\lambda. \quad (5.4)$$

Proof. We deduce the relation (5.4) from (5.3) and Remark 3.3. $\square$

6. The Gaussian Cherednik windowed transform in the W-invariant case

DEFINITION 6.1. The Gaussian Cherednik windowed transform $\Phi^{s,T}_G$, $s \in]-\infty,0[$,
$T \in]0,+\infty[$, associated to the Cherednik operators, is defined for a regular W-invariant function f by

$$\Phi^{s,T}_G(f)(\lambda,y) = \int_{\mathbb{R}^d} f(x)G^{s,T}_{\lambda,y}(x)\mathcal{A}_k(x)dx, \quad (6.1)$$

where $G^{s,T}_{\lambda,y}$ is the Gaussian Cherednik wavelet given by

$$G^{s,T}_{\lambda,y}(x) = F_{-\lambda}(x)\frac{\tau^W_x(\breve{E}^W_T)(\sqrt{T}y)}{||\tau^W_x(\breve{E}^W_T)||^s_{\mathcal{A}_k,2}}.$$

As from the relation (3.33) we have

$$G_{\lambda,y}^{s,T}(x) = F_{-\lambda}(x) \frac{p_T^W(-x, \sqrt{T}y)}{||p_T^W(-x,.)||_{\mathcal{A}_k,2}^s}. \tag{6.2}$$

REMARK 6.1. By using the relation (6.2) the relation (6.1) can also be written in the following form

$$\Phi_G^{s,T}(f)(\lambda, y) = \mathcal{H}^W(f. \frac{p_T^W(-\xi, \sqrt{T}y)}{||p_T^W(-\xi,.)||_{\mathcal{A}_k,2}^s})(\lambda), \ \lambda, y \in \mathbb{R}^d \tag{6.3}$$

where $\mathcal{H}^W$ is the hypergeometric Fourier transform given by (3.8).

Then by applying the results of the previous sections we obtain for the transform $\Phi_G^{s,T}$, $s \in]-\infty, 0[$, $T \in]0, +\infty[$, the following Plancherel and inversion formulas.

6.1. Plancherel formula for the Gaussian Cherednik windowed transform. Notation. We denote by $\mathcal{N}_{T,s}^2(\mathbb{R}^d)^W$, $s \in]-\infty, 0[$, $T \in]0, +\infty[$, the space of measurable functions f on $\mathbb{R}^d$, which are W-invariant and satisfying

$$||f||_{\mathcal{N}_{T,s}^2}^2 = \int_{\mathbb{R}^d} |f(x)|^2 \frac{\mathcal{A}_k(x)dx}{||p_T^W(-x,.)||_{\mathcal{A}_k,2}^{2(s-1)}} < +\infty.$$

THEOREM 6.1. *For all f in $L_{\mathcal{A}_k}^2(\mathbb{R}^d)^W$, we have for the transform $\Phi_G^{s,T}$ the following*
Plancherel formula

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\Phi_G^{s,T}(f)(\lambda, y)|^2 \mathcal{A}_k(y)\mathcal{C}_k^W(\lambda)dyd\lambda = ||f||_{\mathcal{N}_{T,s}^2}^2. \tag{6.3}$$

Proof. From Corollary 3.3 and relation (3.33), for all $y \in \mathbb{R}^d$, the function $\frac{p_T^W(-x, \sqrt{T}y)}{||p_T^W(-x,.)||_{\mathcal{A}_k,2}^s}$ is in $L_{\mathcal{A}_k}^\infty(\mathbb{R}^d)^W$. Then we obtain the result by using the same proof as of Theorem 5.1. $\qquad\square$

6.2. Inversion formula for the Gaussian Cherednik windowed transform.

THEOREM 6.2. *The transform* $\Phi_G^{s,T}$ *admits the following inversion formula. Let* $B(0,n)$ *be the closed ball of center* 0 *and radius* $n \in \mathbb{N}\backslash\{0\}$. *Then we have*

$$f(z) = \lim_{n\to+\infty} \int_{B(0,n)} (\int_{\mathbb{R}^d} \Phi_G^{s,T}(f)(\lambda,y)\tilde{G}_{\lambda,y,z}^{2-s,T}(x)\mathcal{A}_k(y)dy)\mathcal{C}_k^W(\lambda)d\lambda, \ a.e \ x \in \mathbb{R}^d,$$
(6.4)

with

$$\tilde{G}_{\lambda,y,z}^{2-s,T}(x) = F_\lambda(z)\frac{\overline{p_T^W(-x,\sqrt{T}y)}}{\|p_T^W(-x,.)\|_{\mathcal{A}_k,2}^{2-s}}.$$

This formula is true for the functions f *of the space* $(L_{\mathcal{A}_k}^1 \cap L_{\mathcal{A}_k}^2)(\mathbb{R}^d)^W$.

Proof. We obtain the result by making the same proof as for Theorem 5.2. □

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