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COUPLED FIXED POINT RESULTS IN G-FUZZY METRIC SPACES FOR WEAKLY COMPATIBLE MAPPINGS

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Abstract. Coupled fixed point results have attracted much attention among the researchers in recent times specially in the field of fuzzy metric spaces. In this paper we established a coupled fixed point result for weakly compatible mappings in G-fuzzy metric spaces. We have deduced a corollary to our main theorem. Our result also supported by examples.

1. Introduction

The notion of coupled fixed point was introduced by Guo and Lakshmikantham in [18]. Recently in [3] Bhaskar and Lakshmikantham established some coupled fixed point result in partially ordered metric spaces. Today this line of research is getting more attention among the researchers of fixed point theory. One of the reasons for this attention is the application of this results, specially to the field of boundary value problems. The result of Bhaskar and Lakshmikantham [3] was generalized to coupled coincidence point results some of which may be seen in [24, 26, 32, 39]. Coupled fixed point results have also been studied to find fixed point results in probabilistic metric spaces, in cone metric spaces and G-metric spaces. Some more results of coupled fixed points may be noted in [4, 8, 9, 11, 15–17, 24, 35, 40].

Mustafa and Sims [28] introduced the notion of G-metric spaces in 2006. After this introduction, lots of results came out into the literature of fixed point theory. Some of the results may be seen in [2, 10, 16, 27, 31]. A generalization of G-metric space to G-Menger space was proposed by K. Das in his Ph. D. Thesis [12] in the year 2010. Also in 2010 Sun and Yang [38] introduced the notion of G-fuzzy metric spaces. This is a generalization of G-metric spaces introduced by Mustafa and Sims and fuzzy metric spaces also.

Jungck [21] introduced the notion of compatible mappings in 1986. Compatible mappings are generalization of commuting mappings and this is further generalized to weakly compatible mappings. Compatible and weakly compatible mappings have been used to find fixed point and coincidence point results in various spaces. Some of the references of use commuting mappings, compatible mappings and weakly compatible

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mappings in various spaces may be noted in [29, 30, 34]. In [9] compatible mappings also used to find coupled coincidence point results.

The concept of altering distance function came in front of us in 1984 due to Khan, Swaleh and Sasas [22] in the context of metric spaces. After introduction of altering distance function there is a fundamental change in the research of fixed point theory. Choudhury and Das [5] introduced the concept of altering distance function in Menger spaces with the help of Φ-function. This Φ-function have been used to find the fixed point results in probabilistic metric spaces, Menger spaces and fuzzy metric spaces also. This is a generalization of contraction mappings in probabilistic metric spaces which was introduced by Sehgal and Bharucha-Reid [33]. Today Φ-function have been used by lots of researchers, a few may be seen in [1, 6, 7, 13, 14, 25, 39].

There are so many inequivalent definitions of fuzzy metric spaces in the literature. The definition of G-fuzzy metric came to us due to Sun and Yang [38] in the year 2010. This is generalization of G-metric spaces. A reference of generalized fuzzy metric spaces also be seen in [36]. In the present paper we have proved a coupled fixed point result for weakly compatible mappings in G-fuzzy metric spaces. We have used Φ-function to prove our fixed point result. We have deduced a corollary to our result and examples are given to validate our main result. If we take $\phi(t) = t$ then the result of Gupta and Kanwar [19] may be a consequence of our result.

2. Definitions and Mathematical Preliminaries

In this section we give some definitions and theorems which are needed for our results.

Definition: 2.1. (G-metric space [28]) Let X be a nonempty set and let $G : X \times X \times X \longrightarrow [0, \infty)$ be a function satisfying the following:

(i) $G(x, y, z) = 0$ if x=y=z,
(ii) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,
(iii) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $y \neq z$,
(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$ (symmetry in all the three variables),
(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$(rectangle inequality), then the function is called a generalized metric or more specially a G-metric on X and the pair (X, G) is a G-metric space.

Definition: 2.2. (t-norm [20, 37]) A t-norm is a function $\star : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions :

(i) $\star(1, a) = a$,
(ii) $\star(a, b) = \star(b, a)$,
(iii) $\star(c, d) \geq \star(a, b)$ whenever $c \geq a$ and $d \geq b$,
(iv) $\star(\star(a, b), c) = \star(a, \star(b, c))$.

Definition: 2.3. (G-fuzzy metric space [38]) The 3-tuple $(X, G, \star)$ is said to be G-fuzzy metric space if X is arbitrary set, $\star$ is a continuous t-norm and G is a fuzzy set on $X \times X \times X \times [0, \infty)$ satisfying the following conditions:

(i) $G(x, x, y, t) > 0$ with $x \neq y$,
(ii) $G(x, x, y, t) \geq G(x, y, z, t)$ with $y \neq z$,
(iii) $G(x, y, z, t) = 1$ if and only if $x = y = z$,

(iv) $G(x, y, z, t) = G(p(x, z, y, t))$, where p is permutation function on x, y, z,
(v) $G(x, y, z, t + s) \geq G(x, a, a, t) \star G(a, y, z, s)$,
(vi) $G(x, y, z, .) : [0, \infty) \to [0, 1]$ is left continuous.

DEFINITION: 2.4. (Φ-FUNCTION [5]) A function $\phi : [0, \infty) \to [0, \infty)$ is said to be a Φ-function if it satisfies the following conditions:
(i) $\phi(t) = 0$ if and only if $t = 0$,
(ii) $\phi(t)$ is strictly increasing and $\phi(t) \to \infty$ as $t \to \infty$,
(iii) ϕ is left continuous in $(0, \infty)$ and
(iv) ϕ is continuous at 0.

By the property of Φ-function we can prove the following lemma.

LEMMA: 2.5. Let $(X, G, \star)$ be a G-fuzzy metric spaces. If there exist $k \in (0, 1)$ such that $G(x, y, z, \phi(t)) \geq G(x, y, z, \phi(\frac{t}{k}))$, for all $x, y, z \in X$ and $t > 0$, then $x = y = z$.

Proof. By the given condition we have,
$$\begin{aligned} G(x, y, z, \phi(t)) &\geq G(x, y, z, \phi(\tfrac{t}{k})) \\ &\geq G(x, y, z, \phi(\tfrac{t}{k^2})) \\ &\cdots\cdots\cdots\cdots \\ &\geq G(x, y, z, \phi(\tfrac{t}{k^n})). \end{aligned}$$
Taking limit as $n \to \infty$ we have,
$$\lim_{n\to\infty} G(x, y, z, \phi(t)) \geq \lim_{n\to\infty} G(x, y, z, \phi(\frac{t}{k^n})).$$
As $n \to \infty$ we have $\frac{t}{k^n} \to \infty$ and hence by the property of Φ-function we have $\phi(\frac{t}{k^n}) \to \infty$ as $n \to \infty$. Hence we have $G(x, y, z, \phi(t)) = \lim_{n\to\infty} G(x, y, z, \phi(\frac{t}{k^n})) = 1$. From the properties of Φ-function it is immediate that given $\epsilon > 0$ we can find a $t > 0$ such that $\epsilon > \phi(t)$. Hence we have $G(x, y, z, \epsilon) \geq G(x, y, z, \phi(t)) = \lim_{n\to\infty} G(x, y, z, \phi(\frac{t}{k^n})) = 1$, which gives us that $x = y = z$. This completes the proof of the lemma. □

DEFINITION: 2.6. [3] An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $P : X \times X \to X$ if $P(x, y) = x$ and $P(y, x) = y$.

DEFINITION: 2.7. [23] An element $(x, y) \in X \times X \to X$ is called a common coupled fixed point of the mappings $P : X \times X \to X$ and $Q : X \to X$ if $P(x, y) = Q(x) = x$ and $P(y, x) = Q(y) = y$.

DEFINITION: 2.8. (BHASKAR AND LAKSHMIKANTHAM [3]) Let $(X, \preceq)$ be a partially ordered set and $P : X \times X \to X$ be a mapping. The mapping P is said to have the mixed monotone property if P is non-decreasing in its first argument and is non-increasing in its second argument, that is, if, for all
$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow P(x_1, y) \preceq P(x_2, y), \text{ for fixed } y \in X$$
and for all
$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow P(x, y_1) \succeq P(x, y_2), \text{ for fixed } x \in X.$$

DEFINITION: 2.9. (LAKSHMIKANTHAM AND CIRIC [23]) Let $(X, \preceq)$ be a partially ordered set, and $P : X \times X \to X$ and $Q : X \to X$ be two mappings. The mapping

P is said to have the mixed Q-monotone property if P is monotone Q-non-decreasing in its first argument and is monotone Q-non-increasing in its second argument, that is, if, for all

$$x_1, x_2 \in X,\ Q(x_1) \preceq Q(x_2) \Rightarrow P(x_1, y) \preceq P(x_2, y) \text{ for any } y \in X$$

and for all

$$y_1, y_2 \in X,\ Q(y_1) \preceq Q(y_2) \Rightarrow P(x, y_1) \succeq P(x, y_2) \text{ for any } x \in X.$$

Definition: 2.10. [9] The mappings $P : X \times X \to X$ and $Q : X \to X$ are said to be weakly compatible if

$$Q(P(x, y)) = P(Q(x), Q(y))$$

and

$$Q(P(y, x)) = P(Q(y), Q(x)),$$

whenever $P(x, y) = Q(x)$ and $P(y, x) = Q(y)$ for some $(x, y) \in X \times X$.

Definition: 2.11. [38] A sequence $\{x_n\}$ in G-fuzzy metric space X is said to be G-convergence to $x \in X$ if $G(x_n, x_n, x, t) \to 1$ as $n \to \infty$ for each $t > 0$.

Definition: 2.12. [38] A sequence $\{x_n\}$ in G-fuzzy metric space X is said to be G-Cauchy sequence if $G(x_n, x_n, x_m, t) \to 1$ as $n, m \to \infty$ for each $t > 0$. X is called G-complete if every G-Cauchy sequence in X is G-convergent in X.

3. Main Results

Theorem : 3.1. Let $(X, G, \star)$ be a G-fuzzy metric spaces with $a \star b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $P : X \times X \to X$ let $Q : X \to X$ be satisfies the following:

C1. $(X, \leq)$ be a partially ordered set,
C2. $P(X \times X) \subseteq Q(X)$,
C3. $$\begin{aligned} G(P(x, y), P(u, v), P(u, v), \phi(t)) \geq \min\{&G(Q(x), Q(u), Q(u), \phi(\tfrac{t}{k})),\\ &G(Q(x), P(x, y), P(x, y), \phi(\tfrac{t}{k})),\\ &G(Q(u), P(u, v), P(u, v), \phi(\tfrac{t}{k}))\},\end{aligned}$$
for all $x, y, u, v \in X, t > 0$ and $0 < k < 1$,
C4. P has the mixed Q-monotone property,
C5. Q is continuous and $Q(X)$ is a G-complete subset of X,
C6. the mappings P and Q are weakly compatible.

If there exists $x_0, y_0 \in X$ such that $Q(x_0) \leq P(x_0, y_0)$ and $Q(y_0) \geq P(y_0, x_0)$ then P and Q have a unique common coupled fixed point in $X \times X$.

Proof. Let $(x_0, y_0) \in X \times X$ such that $Q(x_0) \leq P(x_0, y_0)$ and $Q(y_0) \geq P(y_0, x_0)$. From (C2) we choose x_1, y_1 such that,

$$P(x_0, y_0) = Q(x_1) \text{ and } P(y_0, x_0) = Q(y_1). \tag{3.1}$$

Now we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$P(x_n, y_n) = Q(x_{n+1}) \text{ and } P(y_n, x_n) = Q(y_{n+1}), \tag{3.2}$$

for all $n \geq 0$ where n is a natural number.
Now we want to prove, $Q(x_n) \leq Q(x_{n+1})$ and $(Qy_n) \geq Q(y_{n+1})$. (3.3)

By mathematical induction for $n = 0$,
as $Q(x_0) \leq P(x_0, y_0)$ and $Q(y_0) \geq P(y_0, x_0)$ with help of (3.1) we can write,
$Q(x_0) \leq Q(x_1)$ and $Q(y_0) \geq Q(y_1)$.
So inequality (3.3) hold for $n = 0$.
Let us now suppose that (3.3) hold for $n = m$, with $m \geq 0$.
So, $Q(x_m) \leq Q(x_{m+1})$ and $Q(y_m) \geq Q(y_{m+1})$.
Since P has the mixed Q-monotone property, using (3.2) we have,

$$Q(x_{m+1}) = P(x_m, y_m) \leq P(x_{m+1}, y_m) \text{ and}$$
$$Q(y_{m+1}) = P(y_m, x_m) \geq P(y_{m+1}, x_m). \tag{3.4}$$

Also, $Q(x_{m+2}) = P(x_{m+1}, y_{m+1}) \geq P(x_{m+1}, y_m)$ and

$$Q(y_{m+2}) = P(y_{m+1}, x_{m+1}) \leq P(y_{m+1}, x_m). \tag{3.5}$$

From (3.4) and (3.5), we get.
$Q(x_{m+1}) \leq Q(x_{m+2})$ and $Q(y_{m+1}) \geq Q(y_{m+2})$.
Therefore in general we can write for $n = m + 1$,

$$Q(x_n) \leq Q(x_{n+1}) \text{ and } Q(y_n) \geq Q(y_{n+1}). \tag{3.6}$$

Putting $x = x_{n-1}, y = y_{n-1}, u = x_n, v = y_n$ in (C3) we get

$$\begin{aligned} G(P(x_{n-1}, y_{n-1}), P(x_n, y_n), P(x_n, y_n), \phi(t)) \geq \min\{&G(Q(x_{n-1}), Q(x_n), Q(x_n), \phi(\tfrac{t}{k})),\\ &G(Q(x_{n-1}), P(x_{n-1}, y_{n-1}), P(x_{n-1}, y_{n-1}), \phi(\tfrac{t}{k})),\\ &G(Q(x_n), P(x_n, y_n), P(x_n, y_n), \phi(\tfrac{t}{k}))\}. \end{aligned}$$

Therefore

$$\begin{aligned} G(Q(x_n), Q(x_{n+1}), Q(x_{n+1}), \phi(t)) \geq \min\{&(G(Q(x_{n-1}), Q(x_n), Q(x_n), \phi(\tfrac{t}{k})),\\ &G(Q(x_{n-1}), Q(x_n), Q(x_n), \phi(\tfrac{t}{k})),\\ &G(Q(x_n), Q(x_{n+1}), Q(x_{n+1}), \phi(\tfrac{t}{k}))\}. \end{aligned} \tag{3.7}$$

If $G(Q(x_{n-1}), Q(x_n), Q(x_n), \phi(\frac{t}{k})) \geq G(Q(x_n), Q(x_{n+1}), Q(x_{n+1}), \phi(\frac{t}{k}))$, for some n then from inequality (3.7) we have

$$G(Q(x_n), Q(x_{n+1}), Q(x_{n+1}), \phi(t)) \geq G(Q(x_n), Q(x_{n+1}), Q(x_{n+1}), \phi(\tfrac{t}{k})). \tag{3.8}$$

By Lemma 2.5, we can write $Q(x_n) = Q(x_{n+1})$.
So we can claim that for all $n > 0$,
$G(Q(x_n), Q(x_{n+1}), Q(x_{n+1}), \phi(\frac{t}{k})) > G(Q(x_{n-1}), Q(x_n), Q(x_n), \phi(\frac{t}{k}))$ and by the inequality (3.7), we have

$$\begin{aligned} G(Q(x_n), Q(x_{n+1}), Q(x_{n+1}), \phi(t)) &\geq G(Q(x_{n-1}), Q(x_n), Q(x_n), \phi(\tfrac{t}{k}))\\ &\geq G(Q(x_{n-2}), Q(x_{n-1}), Q(x_{n-1}), \phi(\tfrac{t}{k^2}))\\ &\geq G(Q(x_{n-3}), Q(x_{n-2}), Q(x_{n-2}), \phi(\tfrac{t}{k^3}))\\ &\dots\dots\dots\dots\dots\dots\\ &\geq G(Q(x_0), Q(x_1), Q(x_1), \phi(\tfrac{t}{k^n})). \end{aligned}$$

Taking limit as $n \to \infty$ on both sides of above inequality, for all $t > 0$, and $0 < k < 1$, we obtain

$$\lim_{n\to\infty} G(Q(x_n), Q(x_{n+1}), Q(x_{n+1}), \phi(t)) = 1. \tag{3.9}$$

Again by virtue of a property of ϕ, given $\epsilon > 0$, we can find $t > 0$ such that $\epsilon > \phi(t)$. Thus the above limit implies that

$$\lim_{n\to\infty} G(Q(x_n), Q(x_{n+1}), Q(x_{n+1}), \epsilon) = 1 \ . \tag{3.10}$$

We next prove that $\{Q(x_n)\}$ is a Cauchy sequence. If possible, let $\{Q(x_n)\}$ be not a Cauchy sequence. Then there exist $\epsilon > 0$ and $0 < \gamma < 1$ for which we can find subsequences $\{Q(x_{m(r)})\}$ and $\{Q(x_{n(r)})\}$ of $\{x_n\}$ with $n(r) > m(r) > r$ such that

$$G(Q(x_{n(r)}), Q(x_{m(r)}), Q(x_{m(r)}), \epsilon) < 1 - \gamma. \tag{3.11}$$

We take $n(r)$ corresponding to $m(r)$ to be the smallest integer satisfying (3.11) so that,

$$G(Q(x_{n(r)-1}), Q(x_{m(r)}), Q(x_{m(r)}), \epsilon) \geq 1 - \gamma. \tag{3.12}$$

If $\epsilon_1 < \epsilon$, then we have

$$G(Q(x_{n(r)}), Q(x_{m(r)}), Q(x_{m(r)}), \epsilon_1) \leq G(Q(x_{n(r)}), Q(x_{m(r)}), Q(x_{m(r)}), \epsilon).$$

We conclude that it is possible to construct $\{Q(x_{m(r)})\}$ and $\{Q(x_{n(r)}\})$ with $n(r) > m(r) > r$ and satisfying (3.11) and (3.12) whenever ϵ is replaced by a smaller positive value. As ϕ is continuous at 0 and strictly monotone increasing with $\phi(0) = 0$, it is possible to obtain $\epsilon_2 > 0$ such that $\phi(\epsilon_2) < \epsilon$.
Then by the above argument, it is possible to obtain an increasing sequence of integers $\{Q(x_{m(r)})\}$ and $\{Q(x_{n(r)})\}$ with $n(r) > m(r) > r$ such that

$$G(Q(x_{n(r)}), Q(x_{m(r)}), Q(x_{m(r)}), \phi(\epsilon_2)) < 1 - \gamma, \tag{3.13}$$

and

$$G(Q(x_{n(r)-1}), Q(x_{m(r)}), Q(x_{m(r)}), \phi(\epsilon_2)) \geq 1 - \gamma. \tag{3.14}$$

Putting $x = x_{n(r)-1}, y = y_{n(r)-1}, u = x_{m(r)-1}, v = y_{m(r)-1}$ in (C3) we get

$$\begin{aligned} &G(P(x_{n(r)-1}, y_{n(r)-1}), P(x_{m(r)-1}, y_{m(r)-1}), P(x_{m(r)-1}, y_{m(r)-1}), \phi(t)) \\ &\quad \geq \min\ \{G(Q(x_{n(r)-1}), Q(x_{m(r)-1}), Q(x_{m(r)-1}), \phi(\tfrac{t}{k})), \\ &\qquad G(Q(x_{n(r)-1}), P(x_{n(r)-1}, y_{n(r)-1}), P(x_{n(r)-1}, y_{n(r)-1}), \phi(\tfrac{t}{k})), \\ &\qquad G(Q(x_{m(r)-1}), P(x_{m(r)-1}, y_{m(r)-1}), P(x_{m(r)-1}, y_{m(r)-1}), \phi(\tfrac{t}{k}))\}. \end{aligned}$$

That is

$$\begin{aligned} &G(Q(x_{n(r)}), Q(x_{m(r)}), Q(x_{m(r)}), \phi(t)) \\ &\quad \geq \min\{G(Q(x_{n(r)-1}), Q(x_{m(r)-1}), Q(x_{m(r)-1}), \phi(\tfrac{t}{k})), \\ &\qquad G(Q(x_{n(r)-1}), Q(x_{n(r)}), Q(x_{n(r)}), \phi(\tfrac{t}{k})), \\ &\qquad G(Q(x_{m(r)-1}), Q(x_{m(r)}), Q(x_{m(r)}), \phi(\tfrac{t}{k}))\}. \end{aligned} \tag{3.15}$$

From (3.13) and (3.15), we can write.

$$\begin{aligned} 1 - \gamma &> G(Q(x_{n(r)}), Q(x_{m(r)}), Q(x_{m(r)}), \phi(t)) \\ &\geq \min\{G(Q(x_{n(r)-1}), Q(x_{m(r)-1}), Q(x_{m(r)-1}), \phi(\tfrac{t}{k})), \\ &\qquad G(Q(x_{n(r)-1}), Q(x_{n(r)}), Q(x_{n(r)}), \phi(\tfrac{t}{k})), \\ &\qquad G(Q(x_{m(r)-1}), Q(x_{m(r)}), Q(x_{m(r)}), \phi(\tfrac{t}{k}))\}. \end{aligned} \tag{3.16}$$

We have $t > 0$ and $0 < k < 1$ and ϕ is strictly monotone increasing function,

therefore we can write $\phi(\frac{t}{k}) > \phi(t)$.

We make a choice of the positive number η such that $\eta < \phi(\frac{t}{k}) - \phi(t)$,
that is, $\phi(\frac{t}{k}) - \eta > \phi(t)$.

In view of (3.9) we may choose r large enough so that
$G(Q(x_{m(r)}), Q(x_{m(r)-1}), Q(x_{m(r)-1}), \eta) > 1 - \gamma_1$ for some $0 < \gamma_1 < \gamma$.

With the above choice of η and r, we can write

$$\begin{aligned} G(Q(x_{n(r)-1}), Q(x_{m(r)-1}), Q(x_{m(r)-1}), \phi(\tfrac{t}{k})) \\ \geq \min\{&G(Q(x_{n(r)-1}), Q(x_{m(r)}), Q(x_{m(r)}), \phi(\tfrac{t}{k}) - \eta), \\ &G(Q(x_{m(r)}), Q(x_{m(r)-1}), Q(x_{m(r)-1}), \eta)\} \\ \geq \min\{&G(Q(x_{n(r)-1}), Q(x_{m(r)}), Q(x_{m(r)}), (\phi(t))), \\ &G(Q(x_{m(r)}), Q(x_{m(r)-1}), Q(x_{m(r)-1}), \eta)\}. \end{aligned}$$

Therefore
$G(Q(x_{n(r)-1}), Q(x_{m(r)-1}), Q(x_{m(r)-1}), \phi(\frac{t}{k})) \geq \min\ \{1 - \gamma, 1 - \gamma_1\} = 1 - \gamma$. [as $0 < \gamma_1 < \gamma$]

So $G(Q(x_{n(r)-1}), Q(x_{m(r)-1}), Q(x_{m(r)-1}), \phi(\frac{t}{k})) \geq 1 - \gamma$. (3.17)

Using (3.14) and (3.17) in (3.16), we can write.

$$1 - \gamma > G(Q(x_{n(r)}), Q(x_{m(r)}), Q(x_{m(r)}), \phi(t)) \geq \min(1-\gamma, 1-\gamma, 1-\gamma, 1-\gamma) = 1-\gamma.$$

Therefore
$1 - \gamma > 1 - \gamma$, which is a contradiction.

Thus $\{Q(x_n)\}$ is a Cauchy sequence. (3.18)

As $(X, \leq)$ be a partially ordered set. Interchanging x and y in (C3) we have

$$\begin{aligned} G(P(y,x), P(v,u), P(v,u), \phi(t)) \geq \min\ \{&G(Q(y), Q(v), Q(v), \phi(\tfrac{t}{k})), \\ &G(Q(y), P(y,x), P(y,x), \phi(\tfrac{t}{k})), \\ &G(Q(v), P(v,u), P(v,u), \phi(\tfrac{t}{k}))\}, \end{aligned} \tag{3.19}$$

for all $x, y, u, v \in X, t > 0$ and $0 < k < 1$.
Now, from (3.19) we can write

$$\begin{aligned} G(P(y_n, x_n), P(y_{n-1}, x_{n-1}, P(y_{n-1}, x_{n-1}), \phi(t)) \\ \geq\min\ \{&G(Q(y_n), Q(y_{n-1}), Q(y_{n-1}), \phi(\tfrac{t}{k})), \\ &G(Q(y_n), P(y_n, x_n), P(y_n, x_n), \phi(\tfrac{t}{k})), \\ &G(Q(y_{n-1}), P(y_{n-1}, x_{n-1}), P(y_{n-1}, x_{n-1}), \phi(\tfrac{t}{k}))\}. \end{aligned}$$

Therefore

$$\begin{aligned} G(Q(y_{n+1}), Q(y_n), Q(y_n), \phi(t)) \geq \min\ \{&G(Q(y_n), Q(y_{n-1}), Q(y_{n-1}), \phi(\tfrac{t}{k})), \\ &G(Q(y_n), Q(y_{n+1}), Q(y_{n+1}), \phi(\tfrac{t}{k})), \\ &G(Q(y_{n-1}), Q(y_n), Q(y_n), \phi(\tfrac{t}{k}))\}. \end{aligned} \tag{3.20}$$

If $G(Q(y_{n-1}), Q(y_n), Q(y_n), \phi(\frac{t}{k})) \geq G(Q(y_n), Q(y_{n+1}), Q(y_{n+1}), \phi(\frac{t}{k}))$, for some $n > 0$, then inequality (3.20) become

$G(Q(y_{n+1}), Q(y_n), Q(y_n), \phi(t)) \geq G(Q(y_{n+1}), Q(y_n), Q(y_n), \phi(\frac{t}{k}))$ and by the Lemma 2.5, we can write. $Q(y_n) = Q(y_{n+1})$.

So we claim that for all $n > 0$,

$G(Q(y_{n+1}), Q(y_n), Q(y_n), \phi(t)) > G(Q(y_n), Q(y_{n-1}), Q(y_{n-1})), \phi(\frac{t}{k}))$ and by the inequality (3.20) we get

$G(Q(y_{n+1}), Q(y_n), Q(y_n), \phi(t)) \geq G(Q(y_n), Q(y_{n-1}), Q(y_{n-1}), \phi(\frac{t}{k}))$.

By the similar argument (as above) we can prove that $\{Q(y_n)\}$ is a Cauchy sequence.

Since $Q(X)$ is G-complete, $\{Q(x_n)\}$ and $\{Q(y_n)\}$ converge to some α and β in $Q(X)$ respectively. Therefore, $\lim_{n\to\infty} Q(x_n) = \alpha$ and $\lim_{n\to\infty} Q(y_n) = \beta$. As $\alpha, \beta \in Q(X)$ there exist some x and y in X such that, $\lim_{n\to\infty} Q(x_n) = Q(x)$ and $\lim_{n\to\infty} Q(y_n) = Q(y)$.

Now we want to show that $P(x, y) = Q(x)$ and $P(y, x) = Q(y)$.
For this, putting $u = x_n, v = y_n$ in (C3) we have

$$\begin{aligned} G(P(x,y), P(x_n,y_n), P(x_n,y_n), \phi(t)) \geq \min\{&G(Q(x), Q(x_n), Q(x_n), \phi(\tfrac{t}{k})),\\ &G(Q(x), P(x,y), P(x,y), \phi(\tfrac{t}{k})),\\ &G(Q(x)_n, P(x_n,y_n), P(x_n,y_n), \phi(\tfrac{t}{k}))\}. \end{aligned}$$

Therefore

$$\begin{aligned} G(P(x,y), Q(x_{n+1}), Q(x_{n+1}), \phi(t)) \geq \min\{&G(Q(x), Q(x_n), Q(x_n), \phi(\tfrac{t}{k})),\\ &G(Q(x), P(x,y), P(x,y), \phi(\tfrac{t}{k})),\\ &G(Q(x_n), Q(x_{n+1}), Q(x_{n+1}), \phi(\tfrac{t}{k}))\}. \end{aligned}$$

Taking limit as $n\to\infty$ on both sides of above inequality, for all $t > 0$, we obtain

$$\begin{aligned} G(P(x,y), Q(x), Q(x), \phi(t)) \geq \min\{&G(Q(x), Q(x), Q(x), \phi(\tfrac{t}{k})),\\ &G(Q(x), P(x,y), P(x,y), \phi(\tfrac{t}{k})),\\ &G(Q(x), Q(x), Q(x), \phi(\tfrac{t}{k}))\}. \end{aligned}$$

Therefore

$G(P(x,y), Q(x), Q(x), \phi(t)) \geq G(Q(x), P(x,y), P(x,y), \phi(\frac{t}{k}))$.
By Lemma 2.5, we can write $P(x, y) = Q(x)$.
Similarly we can prove $P(y, x) = Q(y)$.
That is $P(x, y) = Q(x)$ and $P(y, x) = Q(y)$. (3.21)
Since the pair (P, Q) is weakly-compatible by (3.21), we have
$Q(Q(x)) = Q(P(x, y)) = P(Q(x), Q(y))$
and $Q(Q(y)) = Q(P(y, x)) = P(Q(y), Q(x))$. (3.22)

We now want to show that $Q(Q(x)) = Q(x)$ and $Q(Q(y)) = (Q)y$.

With the help of (C6), we can write.

$\lim_{n\to\infty} P(Q(x_n), Q(y_n)) = \lim_{n\to\infty} Q(P(x_n, y_n)) = \lim_{n\to\infty} Q(Q(x_{n+1})) = Q(Q(x))$,
and

$$\lim_{n\to\infty} P(Q(y_n), Q(x_n)) = \lim_{n\to\infty} Q(P(y_n, x_n)) = \lim_{n\to\infty} Q(Q(y_{n+1})) = Q(Q(y)).$$

Now taking, $u = Q(x_n), v = Q(y_n)$ we have from (C3)

$$\begin{aligned} G(P(x,y), P(Q(x_n), Q(y_n), P(Q(x_n), Q(y_n)), \phi(t)) \\ \geq \min \{&G(Q(x), Q(Q(x_n)), Q(Q(x_n)), \phi(\tfrac{t}{k})), \\ &G(Q(x), P(x,y), P(x,y), \phi(\tfrac{t}{k})), \\ &G(Q(Q(x_n)), P(Q(x_n), Q(y_n)), P(Q(x_n), Q(y_n)), \phi(\tfrac{t}{k}))\}. \end{aligned}$$

Therefore

$$\begin{aligned} G(Q(x), Q(Q(x_{n+1})), Q(Q(x_{n+1})), \phi(t)) \geq \min \{&G(Q(x), Q(Q(x_n)), Q(Q(x_n)), \phi(\tfrac{t}{k})), \\ &G(Q(x), Q(x), Q(x), \phi(\tfrac{t}{k})), \\ &G(Q(Q(x_n)), Q(Q(x_{n+1})), Q(Q(x_{n+1})), \phi(\tfrac{t}{k}))\}. \end{aligned}$$

Taking limit as $n\to\infty$ on both sides of above inequality, for all $t > 0$, we obtain

$$\begin{aligned} G(Q(x), Q(Q(x)), Q(Q(x)), \phi(t)) \geq \min \{&G(Q(x), Q(Q(x)), Q(Q(x)), \phi(\tfrac{t}{k})), \\ &G(Q(x), Q(x), Q(x), \phi(\tfrac{t}{k})), \\ &G(Q(Q(x)), Q(Q(x)), Q(Q(x)), \phi(\tfrac{t}{k}))\}. \end{aligned}$$

Therefore

$G(Q(x), Q(Q(x)), Q(Q(x)), \phi(t)) \geq G(Q(x), Q(Q(x)), Q(Q(x)), \phi(\frac{t}{k}))$ and by the Lemma 2.5, we can write $Q(Q(x)) = Q(x)$.

Similarly we can prove $Q(Q(y)) = Q(y)$.

That is $Q(Q(x)) = Q(x)$ and $Q(Q(y)) = Q(y)$ (3.23)

and hence $Q(x)$ and $Q(y)$ are fixed points of Q.

By (3.22) and (3.23), we can write.

$$Q(x) = Q(Q(x)) = Q(P(x,y)) = P(Q(x), Q(y))$$

and

$$Q(y) = Q(Q(y)) = Q(P(y,x)) = P(Q(y), Q(x)). \tag{3.24}$$

Therefore $(Q(x), Q(y))$ is a common coupled fixed point of P and Q.

Suppose that $(Q(x^1), Q(y^1))$ is another common coupled fixed point of P and Q. Then by (C3) we have

$$\begin{aligned} G(P(x,y), P(x^1,y^1), P(x^1,y^1), (\phi(t))) \geq \min \{&G(Q(x), Q(x^1), Q(x^1), \phi(\tfrac{t}{k})), \\ &G(Q(x), P(x,y), P(x,y)), \phi(\tfrac{t}{k})), \\ &G(Q(x^1), P(x^1,y^1), P(x^1,y^1), \phi(\tfrac{t}{k}))\}. \end{aligned}$$

So,

$$\begin{aligned} G(Q(x), Q(x^1), Q(x^1), \phi(t)) \geq \min \{&G(Q(x), Q(x^1), Q(x^1), \phi(\tfrac{t}{k})), \\ &G(Q(x), Q(x), Q(x), \phi(\tfrac{t}{k})), \\ &G(Q(x^1), Q(x^1), Q(x^1), \phi(\tfrac{t}{k}))\}. \end{aligned}$$

Therefore

$G(Q(x), Q(x^1), Q(x^1), \phi(t)) \geq G(Q(x), Q(x^1), Q(x^1), \phi(\frac{t}{k}))$ and by Lemma 2.5, we can write $Q(x^1) = Q(x)$.

Similarly we can prove $Q(y^1) = Q(y)$.

Hence, $(Q(x), Q(y))$ is unique common coupled fixed point of P and Q. □

Now putting $\phi(t) = t$ in Theorem 3.1 we get the following corollary,

Corollary : 3.2. Let $(X, G, \star)$ be a G-fuzzy metric spaces with $a \star b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $P : X \times X \to X$ let $Q : X \to X$ be satisfies the following:

C1. $(X, \leq)$ be a partially ordered set,
C2. $P(X \times X) \subseteq Q(X)$,
C3. $$G(P(x,y), P(u,v), P(u,v), t) \geq \min\{G(Q(x), Q(u), Q(u), (\tfrac{t}{k})), G(Q(x), P(x,y), P(x,y), (\tfrac{t}{k})), G(Q(u), P(u,v), P(u,v), (\tfrac{t}{k}))\},$$
for all $x, y, u, v \in X, t > 0$ and $0 < k < 1$,
C4. P has the mixed Q-monotone property,
C5. Q is continuous and $Q(X)$ is a G-complete subset of X,
C6. the mappings P and Q are weakly compatible.

If there exists $x_0, y_0 \in X$ such that $Q(x_0) \leq P(x_0, y_0)$ and $Q(y_0) \geq P(y_0, x_0)$ then P and Q have a unique common coupled fixed point in $X \times X$.
Now we give an example in support of our main result.

Example : 3.3. Let $(X, \leq)$ be a partially ordered set with $X = [0, 1]$ and G-fuzzy metric spaces is defined as $G(x, y, z, t) = \frac{t}{t+|x-y|+|y-z|+|z-x|}$ and $a \star b = \min\{a, b\}$.
Let $P : X \times X \to X$ and $Q : X \to X$ be satisfy the followings:
$$P(x,y) = \begin{cases} \frac{x-y}{2} & \text{if } x \geq y, \\ 0 & \text{if } x < y, \end{cases} \qquad \text{and } Q(x) = x.$$
If we take $\phi(t) = \frac{t}{2}$ and $k = \frac{1}{2}$ then P and Q satisfy all the conditions of main Theorem 3.1 and we have $(0, 0)$ is unique common coupled fixed point of P and Q.

Example : 3.4. Let $(X, \leq)$ be a partially ordered set with $X = [0, 1]$ and G-fuzzy metric spaces is defined as $G(x, y, z, t) = \frac{t}{t+|x-y|+|y-z|+|z-x|}$ and $a \star b = \min\{a, b\}$.
Let $P : X \times X \to X$ and $Q : X \to X$ be satisfy the followings:
$$P(x,y) = \begin{cases} \frac{x^2-y^2}{64} & \text{if } x \geq y, \\ 0 & \text{if } x < y, \end{cases} \qquad \text{and } Q(x) = x^2.$$
If we take $\phi(t) = \frac{t}{8}$ and $\frac{1}{32} \leq k < 1$ then P and Q satisfy all the conditions of main Theorem 3.1 and we have $(0, 0)$ is unique common coupled fixed point of P and Q.

4. Conclusion

Fixed point theory has many applications in different branches of sciences such as nonlinear programming, economics, game theory, theory of differential equations and many more. In this paper we have proved a coupled fixed point result in G-fuzzy metric spaces for a pair of weakly compatible mappings. We have used minimum t-norm to establish our result. This result leads us to further investigate which type of t-norm may be replaced *min* t-norm in our main theorem . This theorem is also an instance of use of Φ-function to find fixed point result.

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