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A NEW PARANORMED SERIES SPACE USING EULER TOTIENT MEANS AND SOME MATRIX TRANSFORMATIONS

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Abstract. Paranormed spaces are important as a generalization of the normed spaces in terms of having more general properties. The aim of this study is to introduce a new paranormed space $|\phi_z|(p)$ over the paranormed space $\ell(p)$ using Euler totient means, where $p = (p_k)$ is a bounded sequence of positive real numbers. Besides this, we investigate topological properties and compute the $\alpha-, \beta-$, and γ duals of this paranormed space. Finally, we characterize the classes of infinite matrices $(|\phi_z|(p), \lambda)$ and $(\lambda, |\phi_z|(p))$, where λ is any given sequence space.

1. Introduction, Definitions and Notation

The study of sequence spaces is one of the important research areas in several branches of analysis, namely, theory of topological vector spaces, summability theory, Schauder basis theory.

Especially, constructing new sequence spaces by means of the matrix domain of an infinite matrix has been widely studied by many authors. Besides this, some new series spaces by using absolute summability methods have been introduced. In this topic, recently İlkhan and Hazar [15] have introduced the space $|\phi_z|_p$ using matrix domain over a

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normed space and also characterized certain matrix operators on that space. However, paranormed spaces are important as a generalization of the normed spaces in terms of having more general properties. In this study, we introduce a new series space over the paranormed space which extends the results of İlkhan and Hazar in [15] to paranormed space.

Let ω be the space of real sequences. Any vector subspace of ω is called as a sequence space. By ℓ_∞, c and c_0, we denote the sequence spaces of all bounded, convergent and null sequences, respectively. We write $\ell_p = \left\{ x = (x_k) \in w : \sum_k |x_k|^p < \infty \right\}$ for $1 \leq p < \infty$. Also, by bs and cs, we denote the spaces of all bounded and convergent series, respectively.

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g : X \to \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous, i.e.,

$$|\alpha_n - \alpha| \to 0 \text{ and } g(x_n - x) \to 0 \text{ imply } g(\alpha_n x_n - \alpha x) \to 0$$

for all $\alpha' s$ in $\mathbb{R}$ and all $x' s$ in X, where θ is the zero vector of X.

Throughout the text, we assume that (p_k) is a bounded sequence of strictly positive real numbers such that $H = \sup_k p_k$ and $M = \max\{1, H\}$. The linear space $\ell(p)$ was defined by Maddox [20, 21] (see also Nakano [25] and Simons [29]) as follows.

$$\ell(p) = \left\{ x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty \right\}, (0 < p_k \leq H < \infty)$$

which is a complete paranormed space by

$$g(x) = \left(\sum_k |x_k|^{p_k} \right)^{1/M}.$$

Also, we shall assume throughout that $p_k^{-1} + \left(p_k'\right)^{-1} = 1$ provided $1 < \inf p_k \leq H < \infty$. We denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

For the sequence spaces X and Y define the set $M(X, Y)$ by

$$M(X, Y) = \{u = (u_k) \in w : xu = (x_k u_k) \in Y \text{ for all } x \in X\}.$$

Then, the sets

$$X^\alpha = M(X, \ell_1), X^\beta = M(X, cs) \text{ and } X^\gamma = M(X, bs)$$

are called the $\alpha-, \beta-$ and $\gamma-$ duals of the sequence space X, respectively.

Let X and Y be subspaces of ω and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $k, n \in \mathbb{N}$. Then we say that A defines a matrix transformation from X into Y, and we denote it by writing $A \in (X, Y)$, if for every sequence $x = (x_k) \in X$, the sequence $Ax = (A_n(x))$, the A-transform of x, is in Y, where

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$$

provided that the series is convergent for every $n \in \mathbb{N}$. Also, we write A_n for the sequence in the n-th row of A, that is, $A_n = (a_{nk})_{k=1}^{\infty}$ for every $n \in \mathbb{N}$.

An infinite matrix $A = (a_{nv})$ is called a triangle if $a_{nn} \neq 0$ and $a_{nv} = 0$ for all n, v with $v > n$ ([31]).

A sequence (b_k) of the elements of X is called a basis for a sequence space X paranormed by g if and only if, for every $x \in X$, there exists a unique sequence (α_n) of scalars such that

$$g\left(x - \sum_{k=1}^{n} \alpha_k b_k\right) \to 0 \text{ as } n \to \infty,$$

and in this case we write $x = \sum_{k=1}^{\infty} \alpha_k b_k$.

Let $\sum x_n$ be infinite series, $s = (s_n)$ be the sequence of partial sums of the series and (z_n) be a sequence of non-negative terms. The series $\sum x_n$ is said to be summable $|A, z_n|_p$, $p \geq 1$, if

$$\sum_{n=1}^{\infty} z_n^{p-1} |\Delta A_n(s)|^p < \infty,$$

where $\Delta A_n(s) = A_n(s) - A_{n-1}(s)$, for $n \geq 1$ ([28]).

For an infinite matrix A and a sequence space X, the matrix domain X_A is introduced by

$$X_A = \{x \in w : Ax \in X\}. \tag{1}$$

Recently, several authors have introduced new sequence spaces by using matrix domain over the sequence spaces ℓ_p and $\ell(p)$ [3–5, 8, 14, 17, 18, 32, 33]. For example, the authors have defined the sequence spaces $X = (\ell_p)_{C_1}$ in [26], $r^t(p) = (\ell(p))_{R^t}$ in [1], $e_p^r = (\ell_p)_{E^r}$ and $e^r(p) =$

$(\ell(p))_{E^r}$ in [3,16,24], $Z(u,v,\ell_p) = (\ell_p)_{G(u,v)}$ and $\ell(u,v,p) = (\ell(p))_{G(u,v)}$ in [2,23], $a_p^r = (\ell_p)_{A^r}$ and $a^r(u,p) = (\ell(p))_{A_u^r}$ in [4,5], $bv_p = (\ell_p)_\Delta$ and $bv(u,p) = (\ell(p))_{A^u}$ in [6,22] $\overline{\ell(p)} = (\ell(p))_S$ in [7], where C_1, R^t, E^r and $G(u,v)$ are the Cesàro, Riesz, Euler and generalized means, respectively, S is the summation matrix, Δ is the difference matrix, and A^r, A^u and A_u^r are defined in [4–6].

On the other hand, some new series spaces have been introduced by using matrix domain over normed spaces in [10–12, 15, 27]. In this context, using Euler totient matrix $\Phi = (\phi_{nk})$, series space $|\phi_z|_p$, as the set of all series summable by the method $|\Phi, z_n|_p$, has been introduced and studied in [15] for $1 \le p < \infty$. In this study, we introduce a new series space by using matrix domain over the paranormed space $\ell(p)$, which extends the results of İlkhan and Hazar in [15] to paranormed space.

Now, we give some notations and basic concepts including Euler totient matrix.

Euler totient function φ counts the positive integer up to a given $m \in \mathbb{N}$ with $m > 1$ which are coprime with m and $\varphi(1) = 1$. For a prime m, $\varphi(m) = m-1$, since all numbers less than m are coprime with m.

If two numbers m and n are coprime, then $\varphi(mn) = \varphi(m)\varphi(n)$ and also another identity relates the divisors d of m such that $m = \sum_{d|m} \varphi(d)$ holds.

Now, consider the infinite matrix $\Phi = (\phi_{nk})$ such that

$$\phi_{nk} = \begin{cases} \frac{\varphi(k)}{n} & , \quad \text{if } k \mid n \\ 0 & , \quad \text{if } k \nmid n. \end{cases}$$

Schoenberg [30] has proved that this matrix is regular and defined that a sequence (x_n) of real numbers is $\varphi-$convergent to $\xi \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{d|n} \varphi(d)\, x_d = \xi.$$

This regular matrix is called as Euler totient matrix operator and the new sequence space has been defined by using this matrix by İlkhan and Kara in [13].

For any given $m \in \mathbb{N}$ with $m > 1$, Möbius function μ is defined as

$$\mu(m) = \begin{cases} (-1)^r & \text{if } r \text{ is the number of prime factors of } m \\ \\ 0 & \text{if } m \text{ is divisible by the square of a prime number} \end{cases}$$

and $\mu(1) = 1$. The Möbius function is multiplicative, that is, if two numbers m and n are coprime, then $\mu(mn) = \mu(m)\mu(n)$ and satisfies $\sum_{d|m} \mu(d) = 0$ except for $m = 1$.

2. A New Paranormed Space $|\phi_z|(p)$

In this section, we define a new space $|\phi_z|(p)$ and prove that this space is complete paranormed linear space according to its paranorm. Moreover, we show that the spaces $|\phi_z|(p)$ and $\ell(p)$ are linearly isomorphic and give the basis for the space $|\phi_z|(p)$. Now, we define the space $|\phi_z|(p)$, as the set of all series summable by the method $|\Phi, z_n|(p)$,

$$|\phi_z|(p) = \left\{ x = (x_n) \in w : \sum_{n=1}^{\infty} z_n^{p_n - 1} |\Delta\Phi_n(s)|^{p_n} < \infty \right\}.$$

Also since $s = (s_n)$ is the sequence of partial sums of infinite series $\sum x_n$, we obtain that

$$|\phi_z|(p) = \left\{ x \in \omega : \sum_{n=2}^{\infty} z_n^{p_n - 1} \left| \sum_{j=1}^{n-1} x_j \left(\sum_{\substack{k=j \\ k|n}}^{n} \frac{\varphi(k)}{n} - \sum_{\substack{k=j \\ k|n-1}}^{n-1} \frac{\varphi(k)}{n-1} \right) + x_n \frac{\varphi(n)}{n} \right|^{p_n} + z_1^{p_1 - 1} |x_1|^{p_1} < \infty \right\}.$$

If $p_n = p$ for every $n \in \mathbb{N}$, the space $|\phi_z|(p)$ is reduced to the space $|\phi_z|_p$ for $1 \leq p < \infty$.

Here, if we define the matrices $E(p) = (e_{nk}(p))$ and $F = (f_{nk})$ by

$$e_{nk}(p) = \begin{cases} -z_n^{1/p_n'} & , \quad k = n-1 \\ z_n^{1/p_n'} & , \quad k = n \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2}$$

and

$$f_{nk}=\begin{cases}\dfrac{1}{n}\sum\limits_{\substack{j=k\\ j|n}}^{n}\varphi(j) & , \quad 1\leq k\leq n\\ 0 & , \quad k>n,\end{cases} \tag{3}$$

then, we can write that $x=(x_n)\in|\phi_z|(p)$ if and only if $(E(p)\circ F)$ transform of the sequence $x=(x_n)$ is in the space $\ell(p)$, where the sequence $y=(y_n)$, the $E(p)\circ F$-transform of the sequence $x=(x_k)$, is defined by

$$y_1=z_1^{1/p_1'}x_1,\, y_n=z_n^{1/p_n'}\left(\sum_{j=1}^{n-1}x_j\left(\sum_{\substack{k=j\\ k|n}}^{n}\frac{\varphi(k)}{n}-\sum_{\substack{k=j\\ k|n-1}}^{n-1}\frac{\varphi(k)}{n-1}\right)+x_n\frac{\varphi(n)}{n}\right), \tag{4}$$

$n\geq 2$.

By the notation of matrix domain, we can redefine the space $|\phi_z|(p)$ as

$$|\phi_z|(p)=(\ell(p))_{E(p)\circ F}\ .$$

Now, we give some interesting results of the newly defined space $|\phi_z|(p)$ concerning its topological structures.

Theorem 2.1. *The space $|\phi_z|(p)$ is a complete paranormed space with the paranorm given by*

$$h(x)=\left(\sum_n\left|(E(p)\circ F)_n(x)\right|^{p_n}\right)^{1/M}, \tag{5}$$

where $0<p_n\leq H<\infty$ for all $n\in\mathbb{N}$, $H=\sup_n p_n$ and $M=\max\{1,H\}$.

Proof. It is clear that $h(\theta)=0$ and $h(x)=h(-x)$ for all $x\in|\phi_z|(p)$. For linearity of $|\phi_z|(p)$ with respect to coordinate wise addition and scalar multiplication, take any $x,y\in|\phi_z|(p)$ and $\alpha\in\mathbb{R}$. By using

Minkowski's inequality, we have

(6)

$$\left(\sum_n \left|\left(E(p)\circ F\right)_n(x+y)\right|^{p_n}\right)^{1/M} \le \left(\sum_n \left|\left(E(p)\circ F\right)_n(x)\right|^{p_n}\right)^{1/M} + \left(\sum_n \left|\left(E(p)\circ F\right)_n(y)\right|^{p_n}\right)^{1/M} < \infty$$

which means $x+y \in |\phi_z|(p)$.

Also, since $|\alpha|^{p_n} \le \max\left\{1, |\alpha|^M\right\}$, we get $h(\alpha x) \le \max\{1,|\alpha|\}\, h(x)$. Thus, $\alpha x \in |\phi_z|(p)$.

Subadditivity of h is seen from (6), i.e.,

$$h(x+y) \le h(x) + h(y). \tag{7}$$

Let (x^n) be any sequence in $|\phi_z|(p)$ with $h(x^n - x) \to 0$ as $n \to \infty$ and also (α_n) be any sequence of real scalars such that $\alpha_n \to \alpha$ as $n \to \infty$. Then, by using inequality (7), we have that $\{h(x^n)\}$ is bounded, since

$$h(x^n) \le h(x) + h(x - x^n).$$

So we have,

$$\begin{aligned} h\left(\alpha_k x^k - \alpha x\right) &= \left(\sum_n \left|\left(E(p)\circ F\right)_n \left(\alpha_k x^k - \alpha x\right)\right|^{p_n}\right)^{1/M} \\ &= \left(\sum_n \left|\left(E(p)\circ F\right)_n \left(\alpha_k x^k - \alpha x + \alpha x^k - \alpha x^k\right)\right|^{p_n}\right)^{1/M} \\ &\le |\alpha_k - \alpha|\, h\left(x^k\right) + |\alpha|\, h\left(x^k - x\right) \to 0 \text{ as } k \to \infty, \end{aligned}$$

which implies that scalar multiplication is continuous. Hence, h is a paranorm on the space $|\phi_z|(p)$.

Now, we prove the completeness of the space $|\phi_z|(p)$ with respect to the paranorm h. Suppose that (x^n) is any Cauchy sequence in the space $|\phi_z|(p)$. Given any $\varepsilon > 0$, there exists a positive integer n_0 such that

$$h\left(x^i - x^j\right) < \varepsilon$$

for all $i,j\geq n_0$. By using the definition of h for each fixed $n\in\mathbb{N}$, we have

$$\begin{aligned}(8)\qquad &\left|(E(p)\circ F)_n\left(x^i\right)-(E(p)\circ F)_n\left(x^j\right)\right| \\ &\leq \left(\sum_n \left|(E(p)\circ F)_n\left(x^i\right)-(E(p)\circ F)_n\left(x^j\right)\right|^{p_n}\right)^{1/M} < \varepsilon\end{aligned}$$

for all $i,j\geq n_0$. This implies that $\{(E(p)\circ F)_n(x^i)\}$ is a Cauchy sequence in $\mathbb{R}$ for every fixed $n\in\mathbb{N}$ and so we have

$$\lim_{i\to\infty}(E(p)\circ F)_n\left(x^i\right)=(E(p)\circ F)_n(x).$$

Using these infinitely many limits, we define a sequence

$$\{(E(p)\circ F)_1(x),(E(p)\circ F)_2(x),...\}.$$

We have from (8) that

$$(9)\quad \sum_{n=1}^{m}\left|(E(p)\circ F)_n\left(x^i\right)-(E(p)\circ F)_n\left(x^j\right)\right|^{p_n}<h\left(x^i-x^j\right)^M<\varepsilon^M$$

for each $m\in\mathbb{N}$ and $i,j\geq n_0$. Let $j,m\to\infty$ in (9), then we obtain $h(x^i-x)<\varepsilon$ for $i\geq n_0$. Thus, (x^i) converges to x in $|\phi_z|(p)$. Also, from Minkowski's inequality, we have

$$\begin{aligned}\left(\sum_n |(E(p)\circ F)_n(x)|^{p_n}\right)^{1/M} &\leq \left(\sum_n \left|(E(p)\circ F)_n\left(x-x^i\right)\right|^{p_n}\right)^{1/M} \\ &+ \left(\sum_n \left|(E(p)\circ F)_n\left(x^i\right)\right|^{p_n}\right)^{1/M} \\ &= h\left(x-x^i\right)+h\left(x^i\right)<\infty\end{aligned}$$

which implies that $x\in|\phi_z|(p)$. Therefore we have shown that $|\phi_z|(p)$ is complete and this concludes the proof. □

THEOREM 2.2. *The space $|\phi_z|(p)$ is linearly isomorphic to the space $\ell(p)$, i.e.,*

$$|\phi_z|(p)\cong\ell(p),$$

where $1<p_n\leq H<\infty$ for all $n\in\mathbb{N}$.

Proof. To prove the theorem, we should show that there exists a bijective linear mapping from $|\phi_z|(p)$ to $\ell(p)$. With (4), we define a mapping

$$E(p)\circ F : |\phi_z|(p) \to \ell(p) \tag{10}$$

by $(E(p)\circ F)(x) = y$. Then, the composite function $E(p)\circ F$ is a linear operator, since $E(p)$ and F are linear operators. Further, it is obvious that $x=\theta$ whenever $(E(p)\circ F)(x)=\theta$ and hence $(E(p)\circ F)$ is injective. For any $y\in\ell(p)$, consider the sequence $x=(x_n)$ by

$$x_n = \sum_{j=1}^{n}\left(\sum_{r=j}^{n}\hat{f}_{nr}z_j^{-1/p_j'}\right)y_j, \tag{11}$$

where $\hat{F}=\left(\hat{f}_{nk}\right)$ is inverse of the matrix $F=(f_{nk})$ defined by

$$\hat{f}_{jr} = \begin{cases} \dfrac{\mu\left(\frac{j}{r}\right)r}{\varphi(j)} & , \quad r \mid j \\ -\dfrac{\mu\left(\frac{j-1}{r}\right)r}{\varphi(j-1)} & , \quad r \mid j-1 \\ \dfrac{\mu(j)}{\varphi(j)} - \dfrac{\mu(j-1)}{\varphi(j-1)} & , \quad r=1 \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{12}$$

Then,

$$h(x) = \left(\sum_n \left|(E(p)\circ F)_n(x)\right|^{p_n}\right)^{1/M} = h_1(y) < \infty,$$

where h_1 is the usual paranorm on $\ell(p)$. Thus, we deduce that $x\in|\phi_z|(p)$. As a result, $E(p)\circ F$ is surjective and is paranorm preserving. Hence, we conclude that the spaces $|\phi_z|(p)$ and $\ell(p)$ are linearly isomorphic. □

Since $|\phi_z|(p)\cong\ell(p)$, the Schauder basis of the space $|\phi_z|(p)$ is the inverse image of the basis of $\ell(p)$. So we give the following theorem without proof.

Theorem 2.3. *Let* $y_k = ((E(p) \circ F)(x))_k$, *for all* $k \in \mathbb{N}$ *and* $1 < p_n \leq H < \infty$. *Define the sequence* $b^{(j)} = \left(b_n^{(j)}\right)$ *as*

$$b_n^{(j)} = \begin{cases} z_j^{-1/p_j'} \sum\limits_{r=j}^{n} \hat{f}_{nr} & , \quad 1 \leq j \leq n \\ 0 & , \quad j > n. \end{cases}$$

The sequence $b^{(j)}$ *is a basis for the space* $|\phi_z|(p)$ *and any* $x \in |\phi_z|(p)$ *has a unique representation of the form*

$$x = \sum_{j=1}^{\infty} y_j b^{(j)}.$$

3. α, β, γ-Duals of the Space $|\phi_z|(p)$ and Matrix Mappings

In this section, we determine the $\alpha-, \beta-$, and γ duals of the space $|\phi_z|(p)$. In addition to that, we also characterize the classes of infinite matrices $(|\phi_z|(p), \lambda)$ and $(\lambda, |\phi_z|(p))$, where λ is any given sequence space.

Now, we give following lemmas which are required to prove duals spaces.

Lemma 3.1. *[9]*

(i) Let $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A = (a_{nk}) \in (\ell(p), \ell_1)$ *if and only if there exists an integer* $B > 1$ *such that*

$$\sup_{N \in \mathcal{F}} \sum_k \left| \sum_{n \in N} a_{nk} B^{-1} \right|^{p_k'} < \infty.$$

(ii) Let $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$. *Then,* $A = (a_{nk}) \in (\ell(p), \ell_1)$ *if and only if*

$$\sup_{N \in \mathcal{F}} \sup_{k \in \mathbb{N}} \left| \sum_{n \in N} a_{nk} \right|^{p_k} < \infty.$$

Lemma 3.2. *[19]*

(i)Let $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A = (a_{nk}) \in (\ell(p), \ell_\infty)$ *if and only if there exists an integer* $B > 1$ *such that*

$$\sup_{n \in \mathbb{N}} \sum_k \left| a_{nk} B^{-1} \right|^{p_k'} < \infty. \tag{13}$$

(ii) Let $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$. *Then,* $A = (a_{nk}) \in (\ell(p), \ell_\infty)$ *if and only if*

$$\sup_{n,k \in \mathbb{N}} |a_{nk}|^{p_k} < \infty. \tag{14}$$

Lemma 3.3. *[19] Let* $0 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A = (a_{nk}) \in (\ell(p), c)$ *if and only if* (13), (14) *hold, and*

$$\lim_{n \to \infty} a_{nk} = \beta_k, (k \in \mathbb{N})$$

also holds.

Let $B \in \{n \in \mathbb{N} : n \geq 2\}$, $\hat{F} = \left(\hat{f}_{nk}\right)$ be given in (12), and define the sets $D_1(p), D_2(p), D_3(p)$, $D_4(p)$ and $D_5(p)$ as follows:

$$D_1(p) = \left\{ a = (a_k) \in w : \sup_{N \in \mathcal{F}} \sup_{j \in \mathbb{N}} \left| \sum_{n \in N} \sum_{r=j}^{n} a_n \hat{f}_{nr} z_j^{-1/p_j'} \right|^{p_j} < \infty \right\},$$

$$D_2(p) = \cup_{B>1} \left\{ a = (a_k) \in w : \sup_{N \in \mathcal{F}} \sum_j \left| \sum_{n \in N} \sum_{r=j}^{n} a_n \hat{f}_{nr} z_j^{-1/p_j'} B^{-1} \right|^{p_j'} < \infty \right\},$$

$$D_3(p) = \left\{ a = (a_k) \in w : \sup_{m,j \in \mathbb{N}} \left| \sum_{k=j}^{m} a_k \left(\sum_{r=j}^{k} \hat{f}_{kr} z_j^{-1/p_j'} \right) \right|^{p_j} < \infty \right\},$$

$$D_4(p) = \cup_{B>1} \left\{ a = (a_k) \in w : \sup_{m \in \mathbb{N}} \sum_{j=1}^{m} \left| \sum_{k=j}^{m} a_k \left(\sum_{r=j}^{k} \hat{f}_{kr} z_j^{-1/p_j'} \right) B^{-1} \right|^{p_j'} < \infty \right\},$$

$$D_5(p) = \left\{ a = (a_k) \in w : \lim_{m \to \infty} \sum_{k=j}^{m} a_k \left(\sum_{r=j}^{k} \hat{f}_{kr} z_j^{-1/p_j'} \right) \text{ exists for all } j \in \mathbb{N} \right\}.$$

Theorem 3.4.
(i) Let $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$. *Then,*

$$\{|\phi_z|(p)\}^\alpha = D_1(p).$$

(ii) Let $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,*

$$\{|\phi_z|(p)\}^\alpha = D_2(p).$$

Proof. We only give the proof for the case (ii). Since the proof may be obtained by the same way for the case (i), so we omit it.

Let $p_k > 1$, $a = (a_n) \in w$ and $x = (x_k) \in |\phi_z|(p)$. Then, we immediately derive with (11)

$$a_n x_n = \sum_{j=1}^{n} \left(\sum_{r=j}^{n} a_n \hat{f}_{nr} z_j^{-1/p_j'} \right) y_j = \delta_n(y), \quad (n \in \mathbb{N}) \tag{15}$$

where $\delta_n = (\delta_{nj})$ is defined by

$$\delta_{nj} = \begin{cases} \sum_{r=j}^{n} a_n \hat{f}_{nr} z_j^{-1/p_j'} & , \quad 1 \leq j \leq n, \\ 0 & , \quad j > n \end{cases}$$

and $\hat{F} = \left(\hat{f}_{nk} \right)$ is as in (12). Thus, we observe by (15) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in |\phi_z|(p)$ if and only if $\delta y \in \ell_1$whenever $y = (y_k) \in \ell(p)$. This means that the sequence $a = (a_n)$ is in the $\alpha-$dual of the space $|\phi_z|(p)$ if and only if $\delta \in (\ell(p), \ell_1)$. By using Lemma 3.1 (i), we have $\{|\phi_z|(p)\}^{\alpha} = D_2(p)$. □

THEOREM 3.5.
(i) Let $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$. *Then,*

$$\{|\phi_z|(p)\}^{\gamma} = D_3(p).$$

(ii) Let $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,*

$$\{|\phi_z|(p)\}^{\gamma} = D_4(p).$$

Proof. As in the proof of Theorem 3.4, we prove only case (ii). Let $p_k > 1$, $a = (a_n) \in w$ and $x = (x_k) \in |\phi_z|(p)$. We consider the equation

$$\begin{aligned} \sum_{k=1}^{m} a_k x_k &= \sum_{k=1}^{m} a_k \sum_{j=1}^{k} \left(\sum_{r=j}^{k} \hat{f}_{kr} z_j^{-1/p_j'} \right) y_j \\ &= \sum_{j=1}^{m} \sum_{k=j}^{m} a_k \left(\sum_{r=j}^{k} \hat{f}_{kr} z_j^{-1/p_j'} \right) y_j = G_m(y), \end{aligned} \tag{16}$$

where $G = (g_{mj})$ is defined by

$$g_{mj} = \begin{cases} \sum_{k=j}^{m} a_k \left(\sum_{r=j}^{k} \hat{f}_{kr} z_j^{-1/p_j'} \right) & , \quad 1 \leq j \leq m, \\ 0 & , \quad j > m. \end{cases} \tag{17}$$

Therefore, we deduce with (16) that $ax = (a_k x_k) \in bs$ whenever $x \in |\phi_z|(p)$ if and only if $Gy \in \ell_\infty$ whenever $y \in \ell(p)$. This shows that $a = (a_n) \in \{|\phi_z|(p)\}^\gamma$ if and only if $G \in (\ell(p), \ell_\infty)$. By Lemma 3.2 (i) we obtain that $\{|\phi_z|(p)\}^\gamma = D_4(p)$. □

Theorem 3.6.
(i) Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then,

$$\{|\phi_z|(p)\}^\beta = D_3(p) \cap D_5(p).$$

(ii) Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then,

$$\{|\phi_z|(p)\}^\beta = D_4(p) \cap D_5(p).$$

Proof. By the same way in the proof of Theorem 3.5, we can show with (16) that $ax = (a_k x_k) \in cs$ whenever $x \in |\phi_z|(p)$ if and only if $Gy \in c$ whenever $y \in \ell(p)$, where $G = (g_{mj})$ is defined in (17). That is to say that $a = (a_n) \in \{|\phi_z|(p)\}^\beta$ if and only if $G \in (\ell(p), c)$. Therefore we deduce from Lemma 3.3 that

$$\sup_{m\in\mathbb{N}} \sum_{j=1}^{m} \left| \sum_{k=j}^{m} a_k \left(\sum_{r=j}^{k} \hat{f}_{kr} z_j^{-1/p_j'} \right) B^{-1} \right|^{p_j'} < \infty$$

and

$$\lim_{m\to\infty} \sum_{k=j}^{m} a_k \left(\sum_{r=j}^{k} \hat{f}_{kr} z_j^{-1/p_j'} \right) \text{ exists for all } j \in \mathbb{N},$$

which tells us that $\{|\phi_z|(p)\}^\beta = D_4(p) \cap D_5(p)$. □

Now, we give the characterization of the classes of infinite matrices $(|\phi_z|(p), \lambda)$ and $(\lambda, |\phi_z|(p))$, where λ is any given sequence space.

Theorem 3.7. *Let λ be any given sequence space. Define the infinite matrix $D = (d_{nj})$ via an infinite matrix $A = (a_{nk})$ by*

$$d_{nj} = \sum_{k=j}^{\infty} a_{nk} \left(\sum_{r=j}^{k} \hat{f}_{kr} z_j^{-1/p_j'} \right).$$

Then, $A = (a_{nk}) \in (|\phi_z|(p), \lambda)$ if and only if $A_n \in \{|\phi_z|(p)\}^\beta$ for all $n \in \mathbb{N}$ and $D \in (\ell(p), \lambda)$.

Proof. In proving the theorem, we apply the technique used by Yeşilkayagil and Başar in [32]. Suppose that λ be any given sequence space and take into account that the spaces $|\phi_z|(p)$ and $\ell(p)$ are linearly isomorphic. Let $A \in (|\phi_z|(p), \lambda)$ and $y \in \ell(p)$. For brevity we denote $E(p) \circ F = T(p)$ by

$$t_{nj}(p) = \sum_{k=1}^{n} e_{nk}(p) f_{kj} = e_{n,n-1}(p) f_{n-1,j} + e_{nn}(p) f_{nj} = z_n^{1/p_n'} (f_{nj} - f_{n-1,j}).$$

Then,

$$(DT(p))_{nk} = \sum_{j=k}^{\infty} d_{nj} t_{jk}(p) = \sum_{j=k}^{\infty} \sum_{i=j}^{\infty} a_{ni} \left(\sum_{r=j}^{i} \hat{f}_{ir} z_j^{-1/p_j'} \right) z_j^{1/p_j'} (f_{jk} - f_{j-1,k})$$
$$= a_{nk}.$$

Then, $DT(p)$ exists and $A_n \in \{|\phi_z|(p)\}^{\beta}$, which gives us that $D_n \in \{\ell(p)\}^{\beta}$ for each $n \in \mathbb{N}$. Thus, Dy exists for each $y \in \ell(p)$ and

$$\begin{aligned} \sum_k d_{nk} y_k &= \sum_k \sum_{j=k}^{\infty} a_{nj} \left(\sum_{r=k}^{j} \hat{f}_{jr} z_k^{-1/p_k'} \right) \times \left(\sum_{j=1}^{k} z_k^{1/p_k'} (f_{kj} - f_{k-1,j}) x_j \right) \\ &= \sum_k \sum_{j=k}^{\infty} a_{nj} (t_{jk}(p))^{-1} \times \left(\sum_{j=1}^{k} t_{kj}(p) x_j \right) \\ &= \sum_{k=1} a_{nk} x_k \end{aligned}$$

for all $n \in \mathbb{N}$. Hence, we have $Dy = Ax$ and this leads us to $D \in (\ell(p), \lambda)$.

Conversely, let $A_n \in \{|\phi_z|(p)\}^{\beta}$ for all $n \in \mathbb{N}$, $D \in (\ell(p), \lambda)$ and $x \in |\phi_z|(p)$. Then Ax exists. So we derive from the equality

$$\sum_{k=1}^{m} a_{nk} x_k = \sum_{k=1}^{m} a_{nk} \sum_{j=1}^{k} \left(\sum_{r=j}^{k} \hat{f}_{kr} z_j^{-1/p_j'} \right) y_j = \sum_{k=1}^{m} d_{nk}^{(m)} y_k,$$

where $d_{nj}^{(m)} = \sum_{k=j}^{m} a_{nk} \left(\sum_{r=j}^{k} \hat{f}_{kr} z_j^{-1/p_j'} \right)$, as $m \to \infty$ that $Ax = Dy$. Then, we have $A \in (|\phi_z|(p), \lambda)$. □

THEOREM 3.8. *Let λ be any given sequence space. Then, $A \in (\lambda, |\phi_z|(p))$ if and only if $B \in (\lambda, \ell(p))$, where $B = (b_{nk})$ is defined by*

$$b_{nk} = \sum_{j=1}^{n} z_n^{1/p_n'} (f_{nj} - f_{n-1,j}) a_{jk}.$$

Proof. Let $u \in \lambda$ and consider

$$\sum_{k=1}^{m} b_{nk} u_k = \sum_{j=1}^{n} z_n^{1/p_n'} (f_{nj} - f_{n-1,j}) \sum_{k=1}^{m} a_{jk} u_k. \tag{18}$$

Then, as $m \to \infty$ in (18), we have $(Bu)_n = (T(p)(Au))_n$, where $T(p) = E(p) \circ F$. So, we can see that $Au \in |\phi_z|(p)$ whenever $u \in \lambda$ if and only if $Bu \in \ell(p)$ whenever $u \in \lambda$. This completes the proof. □

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