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SOME SEQUENCE SPACES OVER n-NORMED SPACES DEFINED BY FRACTIONAL DIFFERENCE OPERATOR AND MUSIELAK-ORLICZ FUNCTION

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Abstract. In the present paper we introduce some sequence spaces over n-normed spaces defined by fractional difference operator and Musielak-Orlicz function $\mathcal{M} = (\mathfrak{F}_i)$. We also study some topological properties and prove some inclusion relations between these spaces.

1. Introduction and Preliminaries

A function $\mathfrak{F}$ which is continuous, non-decreasing and convex with $\mathfrak{F}(0) = 0$, $\mathfrak{F}(x) > 0$ for $x > 0$ and $\mathfrak{F}(x) \longrightarrow \infty$ as $x \longrightarrow \infty$, is called an Orlicz function (see [12]); and a sequence $\mathcal{M} = (\mathfrak{F}_i)$ of Orlicz function is called a Musielak-Orlicz function (see [15], [26]). By a lacunary sequence $\theta = (\theta_r)$, we mean a sequence of positive integers such that $\theta_0 = 0$, $0 < \theta_r < \theta_{r+1}$ and $\phi_r = \theta_r - \theta_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $J_r = (\theta_{r-1}, \theta_r)$ and $t_r = \frac{\theta_r}{\theta_{r-1}}$. The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [5] as:

$$N_\theta = \Big\{ \xi = (\xi_k) : \lim_{r\to\infty} \frac{1}{\phi_r} \sum_{k\in J_r} |\xi_k - l| = 0, \text{ for some } \ l \Big\}.$$

Parashar and Choudhary [26] defined and studied some sequence spaces by using an Orlicz function $\mathfrak{F}$, which generalized the well-known Orlicz sequence spaces $[C, 1, p]$, $[C, 1, p]_0$ and $[C, 1, p]_\infty$ (see [13], [14]).

The basic definition of of 2-normed space was given by Gähler [6], and for n-normed space one can see Misiak [19]. A sequence (ξ_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||\xi_k - L, x_1, \cdots, x_{n-1}|| = 0 \text{ for every } x_1, \cdots, x_{n-1} \in X.$$

A sequence (ξ_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,\, p\to\infty} ||\xi_k - \xi_p, x_1, \cdots, x_{n-1}|| = 0 \text{ for every } x_1, \cdots, x_{n-1} \in X.$$

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If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.
Lateron this concept was studied by several authors, e.g. see Gunawan ([7], [8]) and Gunawan and Mashadi [9].
The notion of difference sequence spaces was introduced by Kızmaz [10], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [4] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_o(\Delta^n)$. Subsequently, difference sequence spaces have been discussed by several authors (see [1], [13], [11], [16], [17], [18], [20], [21], [22], [23], [24], [27], [28]).
In [2] Baliarsingh defined the fractional difference operator as follows:
Let $x = (\xi_i) \in w$ and α be a real number, then the fractional difference operator $\Delta^{(\alpha)}$ is defined by

$$\Delta_i^{(\alpha)}\xi = \sum_{k=0}^{i} \frac{(-\alpha)_k}{k!}\xi_{i-k},$$

where $(-\alpha)_k$ denotes the Pochhammer symbol defined as:

$$(-\alpha)_k = \begin{cases} 1, & \text{if } \alpha = 0 \text{ or } k = 0, \\ \alpha(\alpha+1)(\alpha+2)...(\alpha+k-1), & \text{otherwise.} \end{cases}$$

Let $\mathcal{M} = (\mathfrak{F}_i)$ be a Musielak-Orlicz function, $q = (q_i)$ be a bounded sequence of positive real numbers. Then we define the following sequence spaces in the present paper:

$$w_0^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|) = \Big\{\xi = (\xi_i) \in w : \lim_{r\to\infty} \frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}\xi_i}{\rho},\ x_1, \cdots, x_{n-1}\|\Big]^{q_i} = 0, \quad \rho > 0,\ s \geq 0\Big\},$$

$$w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|) = \Big\{\xi = (\xi_i) \in w : \lim_{r\to\infty} \frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}\xi_i - L}{\rho},\ x_1, \cdots, x_{n-1}\|\Big]^{q_i} = 0, \quad \text{for some } L,\ \ \rho > 0,\ s \geq 0\Big\}$$

and

$$w_\infty^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|) = \Big\{\xi = (\xi_i) \in w : \sup_r \frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta_i^{(\alpha)}\xi}{\rho},\ x_1, \cdots, x_{n-1}\|\Big]^{q_i} < \infty, \quad \rho > 0,\ s \geq 0\Big\}.$$

If we take $\mathcal{M}(\xi) = \xi$, we get

$w_0^\theta(s,\Delta^{(\alpha)},q,\|\cdot,\cdots,\cdot\|)=$

$$\Big\{\xi=(\xi_i)\in w:\lim_{r\to\infty}\frac{1}{\phi_r}\sum_{i\in J_r}i^{-s}\Big[\|\frac{\Delta_i^{(\alpha)}\xi}{\rho},\ x_1,\cdots,x_{n-1}\|\Big]^{q_i}=0,$$
$$\rho>0,\ s\geq 0\Big\},$$

$w^\theta(s,\Delta^{(\alpha)},q,\|\cdot,\cdots,\cdot\|)=$

$$\Big\{x=(x_i)\in w:\lim_{r\to\infty}\frac{1}{\phi_r}\sum_{i\in J_r}i^{-s}\Big[\|\frac{\Delta_i^{(\alpha)}\xi-L}{\rho},\ x_1,\cdots,x_{n-1}\|\Big]^{q_i}=0,$$
$$\text{for some } L,\ \ \rho>0,\ s\geq 0\Big\}$$

and

$w_\infty^\theta(s,\Delta^{(\alpha)},q,\|\cdot,\cdots,\cdot\|)=$

$$\Big\{x=(x_i)\in w:\sup_r\frac{1}{\phi_r}\sum_{i\in J_r}i^{-s}\Big[\|\frac{\Delta_i^{(\alpha)}\xi}{\rho},\ x_1,\cdots,x_{n-1}\|\Big]^{q_i}<\infty,$$
$$\rho>0,\ s\geq 0\Big\}.$$

If we take $q=(q_i)=1$ for all $i\in\mathbb{N}$, we have

$w_0^\theta(\mathcal{M},s,\Delta^{(\alpha)},\|\cdot,\cdots,\cdot\|)=$

$$\Big\{x=(x_i)\in w:\lim_{r\to\infty}\frac{1}{\phi_r}\sum_{i\in J_r}i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta_i^{(\alpha)}\xi}{\rho},\ x_1,\cdots,x_{n-1}\|\Big]=0,$$
$$\rho>0,\ s\geq 0\Big\},$$

$w^\theta(\mathcal{M},s,\Delta^{(\alpha)},\|\cdot,\cdots,\cdot\|)=$

$$\Big\{x=(x_i)\in w:\lim_{r\to\infty}\frac{1}{\phi_r}\sum_{i\in J_r}i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta_i^{(\alpha)}\xi-L}{\rho},\ x_1,\cdots,x_{n-1}\|\Big]=0,$$
$$\text{for some } L,\ \ \rho>0,\ s\geq 0\Big\}$$

and

$w_\infty^\theta(\mathcal{M},s,\Delta^{(\alpha)},\|\cdot,\cdots,\cdot\|)=$

$$\Big\{x=(x_i)\in w:\sup_r\frac{1}{\phi_r}\sum_{i\in J_r}i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}\xi}{\rho},\ x_1,\cdots,x_{n-1}\|\Big]<\infty,$$
$$\rho>0,\ s\geq 0\Big\}.$$

If we take $\mathcal{M}(\xi)=\xi, s=0$, then these spaces reduces to

$w_0^\theta(\Delta^{(\alpha)},q,\|\cdot,\cdots,\cdot\|)=$

$$\Big\{x=(x_i)\in w:\lim_{r\to\infty}\frac{1}{\phi_r}\sum_{i\in J_r}\Big[\|\frac{\Delta_i^{(\alpha)}\xi}{\rho},\ x_1,\cdots,x_{n-1}\|\Big]^{q_i}=0,\ \ \rho>0\Big\},$$

$w^{\theta}(\Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|) =$

$$\Big\{x = (x_i) \in w : \lim_{r\to\infty} \frac{1}{\phi_r} \sum_{i\in J_r} \Big[\|\frac{\Delta_i^{(\alpha)}\xi - L}{\rho},\ x_1, \cdots, x_{n-1}\|\Big]^{q_i} = 0,$$

$$\text{for some } L,\quad \rho > 0\Big\}$$

and

$w_{\infty}^{\theta}(\Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|) =$

$$\Big\{x = (x_i) \in w : \sup_r \frac{1}{\phi_r} \sum_{i\in J_r} \Big[\|\frac{\Delta^{(\alpha)}\xi_i}{\rho},\ x_1, \cdots, x_{n-1}\|\Big]^{q_i} < \infty,\ \rho > 0\Big\}.$$

The following inequality will be used throughout the paper. If $0 \le q_i \le \sup q_i = H$, $K = \max(1, 2^{H-1})$ then

$$|a_i + b_i|^{q_i} \le K\{|a_i|^{q_i} + |b_i|^{q_i}\} \tag{1.1}$$

for all i and $a_i, b_i \in \mathbb{C}$. Also $|a|^{q_i} \le \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

In this paper we study some topological properties and prove some inclusion relations between the above defined sequence spaces.

2. Basic properties

THEOREM 2.1. *Let $\mathcal{M} = (\mathfrak{F}_i)$ be a Musielak-Orlicz function and $q = (q_i)$ be a bounded sequence of positive real numbers. Then the sequences $w_0^{\theta}(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$, $w^{\theta}(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$ and $w_{\infty}^{\theta}(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$ are linear spaces over the field of complex number $\mathbb{C}$.*

Proof. Let $x = (x_i), y = (y_i) \in w_0^{\theta}(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$ and $\beta, \gamma \in \mathbb{C}$. In order to prove the result we need to find some ρ_3 such that

$$\lim_{r\to\infty} \frac{1}{\phi_r} \sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}(\beta x_i + \gamma y_i)}{\rho_3}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i} = 0.$$

Since $x = (x_i), y = (y_i) \in w_0^{\theta}(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$, there exist positive numbers $\rho_1, \rho_2 > 0$ such that

$$\lim_{r\to\infty} \frac{1}{\phi_r} \sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}x_i}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i} = 0$$

and

$$\lim_{r\to\infty} \frac{1}{\phi_r} \sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}y_i}{\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i} = 0.$$

Define $\rho_3 = \max(2|\beta|\rho_1, 2|\gamma|\rho_2)$. Since $\mathfrak{F}_i$ is non-decreasing, convex function and so by using inequality (1.1), we have

$$\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}(\beta x_i+\gamma y_i)}{\rho_3}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}$$

$$\begin{aligned}
&\leq \frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\beta\Delta^{(\alpha)}x_i}{\rho_3}, z_1, z_2, \cdots, z_{n-1}\| + \|\frac{\gamma\Delta^{(\alpha)}y_i}{\rho_3}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\\
&\leq K\frac{1}{\phi_r}\sum_{i\in J_r}\frac{1}{2^{q_i}} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}x_i}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\\
&+ K\frac{1}{\phi_r}\sum_{i\in J_r}\frac{1}{2^{q_i}} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}y_i}{\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\\
&\leq K\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}x_i}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\\
&+ K\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}y_i}{\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\\
&\to 0 \text{ as } r\to\infty.
\end{aligned}$$

Thus we have $\beta x+\gamma y \in w_0^{\theta}(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$. Hence $w_0^{\theta}(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$ is a linear space. On the similar lines, we can prove that $w^{\theta}(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$ and
$w_\infty^{\theta}(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$ are linear spaces. □

THEOREM 2.2. *Let $\mathcal{M} = (\mathfrak{F}_i)$ be a Musielak-Orlicz function and $q = (q_i)$ be a bounded sequence of positive real numbers. Then $w_0^{\theta}(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$ is a topological linear space paranormed by*

$$g(x) = \inf\Big\{\rho^{\frac{q_r}{H}} : \Big(\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}x_i}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\Big)^{\frac{1}{H}} \leq 1\Big\},$$

where $H = \max(1, \sup_i q_i) < \infty$.

Proof. Clearly $g(x)\geq 0$ for $x=(x_i)\in w_0^{\theta}(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$. Since $M_i(0) = 0$ we get $g(0)=0$. Again if $g(x)=0$ then

$$\inf\Big\{\rho^{\frac{q_r}{H}} : \Big(\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}x_i}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\Big)^{\frac{1}{H}} \leq 1\Big\} = 0.$$

This implies that for a given $\epsilon > 0$ there exist some $\rho_\epsilon (0<\rho_\epsilon<\epsilon)$ such that

$$\Big(\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}x_i}{\rho_\epsilon}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\Big)^{\frac{1}{H}} \leq 1.$$

Thus

$$\Big(\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}x_i}{\epsilon}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\Big)^{\frac{1}{H}}$$

$$\leq \Big(\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}x_i}{\rho_\epsilon}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\Big)^{\frac{1}{H}}.$$

Suppose $(x_i) \neq 0$ for each $i \in \mathbb{N}$. This implies that $\Delta^{(\alpha)}(x_i) \neq 0$ for each $i \in \mathbb{N}$. Let $\epsilon \to 0$ then

$$\|\frac{\Delta^{(\alpha)} x_i}{\epsilon}, z_1, z_2, \cdots, z_{n-1}\| \to \infty.$$

It follows that

$$\Big(\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)} x_i}{\epsilon}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\Big)^{\frac{1}{H}} \to \infty,$$

which is a contradiction. Therefore $\Delta^{(\alpha)}(x_i) = 0$ for each i and thus $(x_i) = 0$ for each $i \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\Big(\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)} x_i}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\Big)^{\frac{1}{H}} \leq 1$$

and

$$\Big(\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)} y_i}{\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\Big)^{\frac{1}{H}} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$, then by using Minkowski's inequality, we have

$$\begin{aligned}
&\Big(\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)} (x_i+y_i)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\Big)^{\frac{1}{H}} \\
&\leq \Big(\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)} x_i + \Delta^{(\alpha)} y_i}{\rho_1+\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\Big)^{\frac{1}{H}} \\
&\leq \Big(\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big(\frac{\rho_1}{\rho_1+\rho_2}\Big[\|\frac{\Delta^{(\alpha)} x_i}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big] \\
&+ \frac{\rho_2}{\rho_1+\rho_2}\Big[\|\frac{\Delta^{(\alpha)} y_i}{\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big]\Big)^{q_i}\Big)^{\frac{1}{H}} \\
&\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)} x_i}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\Big)^{\frac{1}{H}} \\
&+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)} y_i}{\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\Big)^{\frac{1}{H}} \\
&\leq 1.
\end{aligned}$$

Since ρ, ρ_1 and ρ_2 are non-negative, so we have
$g(x+y)$

$$\begin{aligned}
&= \inf\Big\{\rho^{\frac{q_r}{H}} : \Big(\frac{1}{h_r}\sum_{i\in I_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)} (x_i+y_i)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\Big)^{\frac{1}{H}} \leq 1\Big\} \\
&\leq \inf\Big\{(\rho_1)^{\frac{q_r}{H}} : \Big(\frac{1}{h_r}\sum_{i\in I_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)} x_i}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\Big)^{\frac{1}{H}} \leq 1\Big\} \\
&+ \inf\Big\{(\rho_2)^{\frac{q_r}{H}} : \Big(\frac{1}{h_r}\sum_{i\in I_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)} y_i}{\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\Big)^{\frac{1}{H}} \leq 1\Big\}.
\end{aligned}$$

Therefore $g(x+y) \leq g(x)+g(y)$. Finally we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition

$$g(\lambda x) = \inf\Big\{\rho^{\frac{q_r}{H}} : \Big(\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}\lambda x_i}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\Big)^{\frac{1}{H}} \leq 1\Big\}.$$

Thus

$$g(\lambda x) = \inf\Big\{(|\lambda| t)^{\frac{q_r}{H}} : \Big(\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)} x_i}{t}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\Big)^{\frac{1}{H}} \leq 1\Big\},$$

where $\frac{1}{t} = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{q_r} \leq \max(1, |\lambda|^{\sup q_r})$, we have

$$g(\lambda x) \leq \max(1, |\lambda|^{\sup q_r}) \inf\Big\{t^{\frac{q_r}{H}} : \Big(\frac{1}{h_r}\sum_{i\in I_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)} x_i}{t},$$

$$z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\Big)^{\frac{1}{H}} \leq 1\Big\}.$$

From above inequality it follows that scalar multiplication is continuous. This completes the proof of the theorem. □

3. Inclusion relations

THEOREM 3.1. *Let $\mathcal{M} = (\mathfrak{F}_i)$ be a Musielak-Orlicz function. If $\sup_i [\mathfrak{F}_i(x)]^{q_i} < \infty$ for all fixed $x > 0$, then*

$$w_0^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|) \subseteq w_\infty^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|).$$

Proof. Let $x = (x_i) \in w_0^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$, then there exists positive number ρ_1 such that

$$\lim_{r\to\infty} \frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)} x_i}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i} = 0.$$

Define $\rho = 2\rho_1$. Since $\mathfrak{F}_i$ is non-decreasing, convex and so by using inequality (1.1), we have

$$
\begin{aligned}
\sup_r \frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}x_i}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}
&= \sup_r \frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}x_i + L - L}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\\
&\le K\sup_r \frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\frac{1}{2^{q_i}}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}x_i - L}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\\
&+ K\sup_r \frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\frac{1}{2^{q_i}}\mathfrak{F}_i\Big[\|\frac{L}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\\
&\le K\sup_r \frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}x_i - L}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\\
&+ K\sup_r \frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{L}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}\\
&< \infty.
\end{aligned}
$$

Hence $x = (x_i) \in w^\theta_\infty(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$. □

THEOREM 3.2. *Let* $0 < \inf q_i = h \le q_i \le \sup q_i = H < \infty$ *and* $\mathcal{M} = (\mathfrak{F}_i), \mathcal{M}' = (\mathfrak{F}'_i)$ *be Musielak-Orlicz functions satisfying* Δ_2*-condition, then we have*

(i) $w^\theta_0(\mathcal{M}', s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|) \subset w^\theta_0(\mathcal{M}\circ\mathcal{M}', s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$;

(ii) $w^\theta(\mathcal{M}', s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|) \subset w^\theta(\mathcal{M}\circ\mathcal{M}', s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$;

(iii) $w^\theta_\infty(\mathcal{M}', s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|) \subset w^\theta_\infty(\mathcal{M}\circ\mathcal{M}', s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$.

Proof. Let $x = (x_i) \in w^\theta_0(\mathcal{M}', s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$ then we have

$$\lim_{r\to\infty} \frac{1}{\phi_r}\sum_{i\in J_r} i_i^{-s}\mathfrak{F}'_i\Big[\|\frac{\Delta^{(\alpha)}x_i}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i} = 0.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_i(t) < \epsilon$ for $0 \le t \le \delta$. Let $(y_i)^{q_i} = M'_i\Big[\|\frac{\Delta^{(\alpha)}x_i}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i}$ for all $i \in \mathbb{N}$. We can write

$$\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i[y_i]^{q_i} = \frac{1}{\phi_r}\sum_{i\in J_r, y_i\le\delta} i^{-s}\mathfrak{F}_i[y_i]^{q_i} + \frac{1}{\phi_r}\sum_{i\in J_r, y_i\ge\delta} i^{-s}\mathfrak{F}_i[y_i]^{q_i}.$$

So we have

$$
\begin{aligned}
\frac{1}{\phi_r}\sum_{i\in J_r, y_i\le\delta} i^{-s}\mathfrak{F}_i[y_i]^{q_i} &\le [\mathfrak{F}_i(1)]^H \frac{1}{\phi_r}\sum_{i\in J_r, y_i\le\delta} i^{-s}\mathfrak{F}_i[y_i]^{q_i}\\
&\le [\mathfrak{F}_i(2)]^H \frac{1}{\phi_r}\sum_{i\in J_r, y_i\le\delta} i^{-s}\mathfrak{F}_i[y_i]^{q_i} \qquad (3.1)
\end{aligned}
$$

For $y_i > \delta, y_i < \frac{y_i}{\delta} < 1 + \frac{y_i}{\delta}$. Since $\mathfrak{F}'_i s$ are non-decreasing and convex, it follows that

$$\mathfrak{F}_k(y_i) < \mathfrak{F}_i(1 + \frac{y_i}{\delta}) < \frac{1}{2}\mathfrak{F}_i(2) + \frac{1}{2}\mathfrak{F}_i(\frac{2y_i}{\delta}).$$

Since $\mathcal{M} = (\mathfrak{F}_i)$ satisfies Δ_2-condition, we can write

$$\mathfrak{F}_i(y_i) < \frac{1}{2}T\frac{y_i}{\delta}\mathfrak{F}_i(2) + \frac{1}{2}T\frac{y_i}{\delta}\mathfrak{F}_i(2) = T\frac{y_i}{\delta}\mathfrak{F}_i(2).$$

Hence,

$$\frac{1}{\phi_r}\sum_{i\in J_r, y_i\geq\delta} i^{-s}\mathfrak{F}_i[y_i]^{q_i} \leq \max\Big(1, (T\frac{\mathfrak{F}_i(2)}{\delta})^H\Big)\frac{1}{\phi_r}\sum_{i\in J_r, y_i\leq\delta} i^{-s}[y_i]^{q_i} \tag{3.2}$$

From equation (3.1) and (3.2), we have $x = (x_i) \in w_0^\theta(\mathcal{M}\circ\mathcal{M}', s, \Delta^{(\alpha)}, q, \|\cdot,\cdots,\cdot\|)$. This completes the proof of (i). Similarly we can prove that

$$w^\theta(\mathcal{M}', s, \Delta^{(\alpha)}, q, \|\cdot,\cdots,\cdot\|) \subset w^\theta(\mathcal{M}\circ\mathcal{M}', s, \Delta^{(\alpha)}, q, \|\cdot,\cdots,\cdot\|)$$

and

$$w_\infty^\theta(\mathcal{M}', s, \Delta^{(\alpha)}, q, \|\cdot,\cdots,\cdot\|) \subset w_\infty^\theta(\mathcal{M}\circ\mathcal{M}', s, \Delta^{(\alpha)}, q, \|\cdot,\cdots,\cdot\|).$$

□

THEOREM 3.3. *Let* $0 < h = \inf q_i = q_i < \sup q_i = H < \infty$. *Then for a Musielak-Orlicz function* $\mathcal{M} = (\mathfrak{F}_i)$ *which satisfies* Δ_2*-condition, we have*

(i) $w_0^\theta(s, \Delta^{(\alpha)}, q, \|\cdot,\cdots,\cdot\|) \subset w_0^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot,\cdots,\cdot\|)$;

(ii) $w^\theta(s, \Delta^{(\alpha)}, q, \|\cdot,\cdots,\cdot\|) \subset w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot,\cdots,\cdot\|)$;

(iii) $w_\infty^\theta(s, \Delta^{(\alpha)}, q, \|\cdot,\cdots,\cdot\|) \subset w_\infty^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot,\cdots,\cdot\|)$.

Proof. It is easy to prove so we omit the details. □

THEOREM 3.4. *Let* $\mathcal{M} = (\mathfrak{F}_i)$ *be a Musielak-Orlicz function and* $0 < h = \inf q_i$. *Then* $w_\infty^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot,\cdots,\cdot\|) \subset w_0^\theta(s, \Delta^{(\alpha)}, q, \|\cdot,\cdots,\cdot\|)$ *if and only if*

$$\lim_{r\to\infty}\frac{1}{\phi_r}\sum_{i\in J_r} i^{-s}\mathfrak{F}_i(t)^{q_i} = \infty \tag{3.3}$$

for some $t > 0$.

Proof. Let $w_\infty^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot,\cdots,\cdot\|) \subset w_0^\theta(s, \Delta^{(\alpha)}, q, \|\cdot,\cdots,\cdot\|)$. Suppose that equation (2.3) does not hold. Therefore there are subinterval $I_{r(j)}$ of the set of interval I_r and a number $t_0 > 0$, where

$$t_0 = \|\frac{\Delta^{(\alpha)}x_i}{\rho}, z_1, z_2, \cdots, z_{n-1}\| \text{ for all } i,$$

such that

$$\frac{1}{\phi_{r(j)}} = \sum_{i\in I_{r(j)}} i^{-s}\mathfrak{F}_i(t_0)^{q_i} \leq K < \infty, m = 1, 2, 3, \cdots \tag{3.4}$$

let us define $x = (x_i)$ as follows :

$$\Delta^{(\alpha)}x_i = \begin{cases} \rho t_0, & i \in I_{r(j)} \\ 0, & i \notin I_{r(j)} \end{cases}.$$

Thus, by equation (3.4), $x \in w_\infty^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot,\cdots,\cdot\|)$. But $x \notin w_\infty^0(s, \Delta^{(\alpha)}, q, \|\cdot,\cdots,\cdot\|)$. Hence equation (2.3) must hold.

Conversely, suppose that equation (3.3) holds and that $x \in w_\infty^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$. Then for each r,

$$\frac{1}{\phi_r}\sum_{i \in J_r} i^{-s} M_i\Big[\|\frac{\Delta^{(\alpha)} x_i}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i} \leq K < \infty. \tag{3.5}$$

Suppose that $x \notin w_0^\theta(s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$. Then for some number $\epsilon > 0$, there is a number i_0 such that for a subinterval $J_{r(j)}$, of the set of interval J_r,

$$\|\frac{\Delta^{(\alpha)} x_i}{\rho}, z_1, z_2, \cdots, z_{n-1}\| \ > \epsilon \text{ for } i \geq i_0.$$

From properties of sequence of Orlicz function, we obtain

$$\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)} x_i}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i} \geq M_i(\epsilon)^{q_i}$$

which contradicts equation (3.3), by using equation (2.5). Hence we get

$$w_\infty^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|) \subset w_0^\theta(s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|).$$

This completes the proof. □

THEOREM 3.5. *Let $\mathcal{M} = (\mathfrak{F}_i)$ be a Musielak-Orlicz function. Then the following statements are equivalent :*

(i) $w_\infty^\theta(s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|) \subset w_\infty^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$;

(ii) $w_0^\theta(s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|) \subset w_\infty^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$;

(iii) $\sup_r \frac{1}{\phi_r}\sum_{i \in J_r} i^{-s}\mathfrak{F}_i(t)^{q_i} < \infty$ *for all* $t > 0$.

Proof. (i) ⇒ (ii). Let (i) holds. To verify (ii), it is enough to prove

$$w_0^\theta(s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|) \subset w_\infty^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|).$$

Let $x = (x_i) \in w_0^\theta(s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$. Then for $\epsilon > 0$ there exists $r \geq 0$, such that

$$\frac{1}{\phi_r}\sum_{i \in J_r} i^{-s}\Big[\|\frac{\Delta^{(\alpha)} x_i}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i} < \epsilon.$$

Hence there exists $K > 0$ such that

$$\sup_r \frac{1}{\phi_r}\sum_{i \in J_r} i^{-s}\Big[\|\frac{\Delta^{(\alpha)} x_i}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i} < K.$$

So we get $x = (x_i) \in w_\infty^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$.
(ii) ⇒ (iii). Let (ii) holds. Suppose (iii) does not hold. Then for some $t > 0$

$$\sup_r \frac{1}{\phi_r}\sum_{i \in J_r} i^{-s} M_i(t)^{q_i} = \infty$$

and therefore we can find a subinterval $J_{r(j)}$, of the set of interval J_r such that

$$\frac{1}{\phi_{r(j)}}\sum_{i \in J_{r(j)}} i^{-s} M_i(\frac{1}{j})^{q_i} > j, \ j = 1, 2, 3, \cdots \tag{3.6}$$

Let us define $x = (x_i)$ as follows :

$$\Delta^{(\alpha)} x_i = \begin{cases} \frac{\rho}{j}, & i \in I_{r(j)} \\ 0, & i \notin I_{r(j)} \end{cases}.$$

Then $x = (x_i) \in w_0^\theta(s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$. But by equation (2.6), $x \notin w_\infty^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$, which contradicts (ii). Hence (iii) must holds.
(iii) $\Rightarrow$ (i). Let (iii) holds and suppose $x = (x_i) \in w_\infty^\theta(s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$. Suppose that $x = (x_i) \notin w_\infty^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$, then

$$\sup_r \frac{1}{\phi_r} \sum_{i \in J_r} i^{-s} \mathfrak{F}_i \Big[\| \frac{\Delta^{(\alpha)} x_i}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i} = \infty. \tag{3.7}$$

Let $t = \|\frac{\Delta^{(\alpha)} x_i}{\rho}, z_1, z_2, \cdots, z_{n-1}\|$ for each i, then by equation (2.7)

$$\sup_r \frac{1}{\phi_r} \sum_{i \in J_r} i^{-s} \mathfrak{F}_i(t)^{q_i} = \infty$$

which contradicts (iii). Hence (i) must holds. □

THEOREM 3.6. *Let $\mathcal{M} = (\mathfrak{F}_i)$ be a Musielak-Orlicz function. Then the following statements are equivalent :*

(i) $w_0^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|) \subset w_0^\theta(s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$;

(ii) $w_0^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|) \subset w_\infty^\theta(s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$;

(iii) $\inf_r \frac{1}{h_r} \sum_{i \in I_r} i^{-s} \mathfrak{F}_i(t)^{q_i} > 0$ *for all* $t > 0$.

Proof. (i) $\Rightarrow$ (ii). It is obvious.
(ii) $\Rightarrow$ (iii). Let (ii) holds. Suppose that (iii) does not hold. Then

$$\inf_r \frac{1}{\phi_r} \sum_{i \in J_r} i^{-s} \mathfrak{F}_i(t)^{q_i} = 0 \text{ for some } t > 0,$$

and we can find a subinterval $J_{r(j)}$, of the set of interval J_r such that

$$\frac{1}{\phi_{r(j)}} \sum_{i \in J_{r(j)}} i^{-s} \mathfrak{F}_i(j)^{q_i} < \frac{1}{j}, \; j = 1, 2, 3, \cdots \tag{3.8}$$

Let us define $x = (x_i)$ as follows :

$$\Delta^{(\alpha)} x_i = \begin{cases} \rho j, & i \in J_{r(j)} \\ 0, & i \notin J_{r(j)} \end{cases}.$$

Thus by equation (3.8), $x = (x_i) \in w_0^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$ but $x = (x_i) \notin w_\infty^\theta(s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$, which contradicts (ii). Hence (iii) must holds.
(iii) $\Rightarrow$ (i). Let (iii) holds. Suppose that $x = (x_i) \in w_0^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$. Then

$$\frac{1}{\phi_r} \sum_{i \in J_r} i^{-s} M_k \Big[\| \frac{\Delta^{(\alpha)} x_i}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i} \to 0 \text{ as } r \to \infty. \tag{3.9}$$

Again suppose that $x = (x_i) \notin w_0^\theta(s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$. For some number $\epsilon > 0$ and a subinterval $J_{r(j)}$, of the set of interval J_r. we have

$$\|\frac{\Delta^{(\alpha)} x_i}{\rho}, z_1, z_2, \cdots, z_{n-1}\| \geq \epsilon \text{ for all } i.$$

Then from properties of the Orlicz function, we can write

$$\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)} x_i}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i} \geq M_i(\epsilon)^{q_i}.$$

consequently, by equation (2.9), we have

$$\lim_{r\to\infty} \frac{1}{\phi_r} \sum_{i\in J_r} i^{-s} \mathfrak{F}_i(\epsilon)^{q_i} = 0$$

which contradicts (iii). Hence (i) must holds. □

Theorem 3.7. *Let $0 \leq q_i \leq p_i$ for all i and let $(\frac{p_i}{q_i})$ be bounded. Then*

$$w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, p, \|\cdot, \cdots, \cdot\|) \subseteq w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|).$$

Proof. Let $x = (x_i) \in w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, p, \|\cdot, \cdots, \cdot\|)$, write

$$t_i = \mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)} x_i}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{p_i}$$

and $\mu_i = \frac{q_i}{p_i}$ for all $i \in \mathbb{N}$. Then $0 < \mu_i \leq 1$ for all $i \in \mathbb{N}$. Take $0 < \mu \leq \mu_i$ for $i \in \mathbb{N}$. Define sequences (u_i) and (v_i) as follows :
For $t_i \geq 1$, let $u_i = t_i$ and $v_i = 0$ and for $t_i < 1$, let $u_i = 0$ and $v_i = t_i$. Then clearly for all $i \in \mathbb{N}$, we have

$$t_i = u_i + v_i, t_i^{\mu_i} = u_i^{\mu_i} + v_i^{\mu_i}$$

Now it follows that $u_i^{\mu_i} \leq u_i \leq t_i$ and $v_i^{\mu_i} \leq v_i^{\mu}$. Therefore,

$$\begin{aligned} \frac{1}{\phi_r}\sum_{i\in J_r} t_i^{\mu_i} &= \frac{1}{\phi_r}\sum_{i\in J_r}(u_i^{\mu_i} + v_i^{\mu_i}) \\ &\leq \frac{1}{\phi_r}\sum_{i\in J_r} t_i + \frac{1}{\phi_r}\sum_{i\in J_r} v_i^{\mu}. \end{aligned}$$

Now for each i,

$$\begin{aligned} \frac{1}{\phi_r}\sum_{i\in J_r} v_i^{\mu} &= \sum_{i\in J_r}\Big(\frac{1}{\phi_r} v_i\Big)^{\mu}\Big(\frac{1}{\phi_r}\Big)^{1-\mu} \\ &\leq \Big(\sum_{i\in I_r}\Big[\Big(\frac{1}{\phi_r} v_i\Big)^{\mu}\Big]^{\frac{1}{\mu}}\Big)^{\mu}\Big(\sum_{i\in I_r}\Big[\Big(\frac{1}{\phi_r}\Big)^{1-\mu}\Big]^{\frac{1}{1-\mu}}\Big)^{1-\mu} \\ &= \Big(\frac{1}{\phi_r}\sum_{i\in J_r} v_i\Big)^{\mu} \end{aligned}$$

and so

$$\frac{1}{\phi_r}\sum_{i\in J_r} v_i^{\mu} \leq \frac{1}{\phi_r}\sum_{i\in J_r} t_i + \Big(\frac{1}{\phi_r}\sum_{i\in J_r} v_i\Big)^{\mu}.$$

Hence $x = (x_i) \in w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$. This completes the proof of the theorem. □

THEOREM 3.8. *(i) If* $0 < \inf q_i \leq q_i \leq 1$ *for all* $i \in \mathbb{N}$, *then*

$$w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|) \subseteq w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, \|\cdot, \cdots, \cdot\|).$$

(ii) If $1 \leq q_i \leq \sup q_i = H < \infty$, *for all* $i \in \mathbb{N}$, *then*

$$w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, \|\cdot, \cdots, \cdot\|) \subseteq w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|).$$

Proof. (i) Let $x = (x_i) \in w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$, then

$$\lim_{r\to\infty} \frac{1}{\phi_r} \sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}x_i - L}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i} = 0.$$

Since $0 < \inf q_i \leq q_i \leq 1$. This implies that

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{\phi_r} \sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}x_i - L}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big] \\
&\qquad \leq \lim_{r\to\infty} \frac{1}{\phi_r} \sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}x_i - L}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i},
\end{aligned}$$

therefore,

$$\lim_{r\to\infty} \frac{1}{\phi_r} \sum_{i\in J_r} i^{-s}\mathfrak{F}_i\Big[\|\frac{\Delta^{(\alpha)}x_i - L}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big] = 0.$$

Therefore

$$w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|) \subseteq w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, \|\cdot, \cdots, \cdot\|).$$

(ii) Let $q_i \geq 1$ for each i and $\sup q_i < \infty$. Let $x = (x_i) \in w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, \|\cdot, \cdots, \cdot\|)$, then for each $\rho > 0$, we have

$$\lim_{r\to\infty} \frac{1}{\phi_r} \sum_{i\in J_r} i^{-s}M_i\Big[\|\frac{\Delta^{(\alpha)}x_i - L}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i} = 0 < 1.$$

Since $1 \leq q_i \leq \sup q_i < \infty$, we have

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{\phi_r} \sum_{i\in J_r} i^{-s}M_i\Big[\|\frac{\Delta^{(\alpha)}x_i - L}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{q_i} \\
&\qquad \leq \lim_{r\to\infty} \frac{1}{\phi_r} \sum_{i\in J_r} i^{-s}M_i\Big[\|\frac{\Delta^{(\alpha)}x_i - L}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big] \\
&\qquad = 0 \\
&\qquad < 1.
\end{aligned}$$

Therefore $x = (x_i) \in w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|)$, for each $\rho > 0$. Hence

$$w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, \|\cdot, \cdots, \cdot\|) \subseteq w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|).$$

This completes the proof of the theorem. □

THEOREM 3.9. *If* $0 < \inf q_i \leq q_i \leq \sup q_i = H < \infty$, *for all* $i \in \mathbb{N}$, *then*

$$w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, q, \|\cdot, \cdots, \cdot\|) = w^\theta(\mathcal{M}, s, \Delta^{(\alpha)}, \|\cdot, \cdots, \cdot\|).$$

Proof. It is easy to prove so we omit the details. □

Conclusion

We have introduced here some new sequence spaces defined by fractional difference operator and Musielak-Orlicz function. We have studied their topological properties and proved some inclusion relations between these newly defined spaces.

Competing interests

The authors declare that they have no competing interests.

Authors contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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