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ON ALMOST PSEUDO-VALUATION DOMAINS, II

Gyu Whan Chang

Abstract. Let D be an integral domain, D^w be the w-integral closure of D, X be an indeterminate over D, and $N_v = \{f \in D[X] | c(f)_v = D\}$. In this paper, we introduce the concept of t-locally APVD. We show that D is a t-locally APVD and a UMT-domain if and only if D is a t-locally APVD and D^w is a PvMD, if and only if $D[X]$ is a t-locally APVD, if and only if $D[X]_{N_v}$ is a locally APVD.

1. Introduction

Let D be an integral domain, K be the quotient field of D, and $\bar{D}$ be the integral closure of D in K. An *overring* of D is a ring between D and K.

A prime ideal P of D is called *strongly prime* if $xy \in P$ and $x, y \in K$ imply $x \in P$ or $y \in P$. As in [13], we say that D is a *pseudo-valuation domain* (PVD) if every prime ideal of D is strongly prime; equivalently, if D is quasi-local whose maximal ideal is strongly prime. It is known that if D is a PVD, then Spec(D) is linearly ordered under inclusion [13, Corollary 1.3] and if (D, M) is a PVD which is not a valuation domain, then $M^{-1} = \{x \in K | xM \subseteq D\}$ is a valuation domain such that Spec(M^{-1}) = Spec(D) (in particular, M is the maximal ideal of M^{-1}) [13, Theorem 2.10]. For a survey article on PVDs, we recommend [1]. In [3], the authors introduced the notions of strongly primary ideals and almost PVDs as follows: an ideal I of D is *strongly primary* if, whenever $xy \in I$ with $x, y \in K$ implies $x \in I$ or $y^n \in I$ for some integer $n \geq 1$,

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while D is an *almost PVD* (APVD) if each prime ideal of D is strongly primary. They showed that if D is quasi-local with maximal ideal M, then D is an APVD if and only if there exists a valuation overring V of D such that $M = MV$ and $\sqrt{MV}$ is the maximal ideal of V [3, Theorem 3.4]. They also proved that if D is an APVD, then Spec(D) is linearly ordered under inclusion (and hence D is quasi-local) and $\bar{D}$ is a PVD [3, Propositions 3.2 and 3.7].

As in [9], we say that D is a *locally pseudo valuation domain* (LPVD) if D_M is a PVD for all maximal ideals M of D. In [5], we studied when $D[X]_{N_v}$ is an LPVD. To do this, we introduced the notion of t-locally PVD; D is a *t-locally PVD* (t-LPVD) if D_P is a PVD for all maximal t-ideals P of D. (Definitions related to the t-operation will be reviewed in the sequel.) Then we proved that $D[X]_{N_v}$ is an LPVD if and only if $D[X]$ is a t-LPVD, if and only if D is an LPVD and a UMT-domain [5, Theorem 3.8]. In [6], we defined a locally APVD as follows: D is a *locally APVD* (LAPVD) if D_M is an APVD for all maximal ideals M of D. We proved that $D(X)$, the Nagata ring of D, is an LAPVD if and only if D is an LAPVD and $\bar{D}$ is a Prüfer domain [6, Corollary 8].

In this paper, we study when $D[X]_{N_v}$ is an LAPVD. Precisely, we introduce the concept of t-locally APVDs. We prove that if D is a t-locally APVD, then D^w is a t-locally PVD; and D is a UMT-domain if and only if D^w is a Prüfer v-multiplication domain. We also prove that D is a t-locally APVD and a UMT-domain if and only if $D[X]$ is a t-locally APVD, if and only if $D[X]_{N_v}$ is a locally APVD.

We would like to point out that other classes of integral domains that are closely related to the classes of PVDs and APVDs are introduced in [2].

1.1. Definitions related to the t-operation. Throughout this paper, D denotes an integral domain, $qf(D)$ is the quotient field of D, $\bar{D}$ is the integral closure of D in $qf(D)$, X is an indeterminate over D, and $D[X]$ is the polynomial ring over D.

Let $K = qf(D)$. For any nonzero fractional ideal A of D, let $A^{-1} = \{x \in K | xA \subseteq D\}$, $A_v = (A^{-1})^{-1}$, and $A_t = \cup\{I_v | I \subseteq A$ is a nonzero finitely generated fractional ideal$\}$, and $A_w = \{x \in K | xJ \subseteq A$ for J a nonzero finitely generated ideal of D with $J^{-1} = D\}$. A fractional ideal A is called a *divisorial ideal* (resp., *t-ideal*) if $A_v = A$ (resp., $A_t = A$), while A is called a *maximal t-ideal* if A is maximal among proper integral t-ideals of D. It is well known that each maximal t-ideal is a prime ideal;

each proper integral t-ideal is contained in a maximal t-ideal; and D has at least one maximal t-ideal if D is not a field.

We denote by $c(f)$ the ideal of D generated by the coefficients of a polynomial $f \in D[X]$. Let $N_v = \{f \in D[X] | c(f)_v = D\}$ and $S = \{f \in D[X] | c(f) = D\}$; then N_v and S are saturated multiplicative subsets of $D[X]$ with $S \subseteq N_v$. The quotient ring $D[X]_{N_v}$ (resp., $D(X) := D[X]_S$) is called the (v-)Nagata (resp., Nagata) ring of D. An $x \in K$ is said to be w-integral over D if there is a nonzero finitely generated ideal I of D such that $xI_w \subseteq I_w$. Let $D^w = \{x \in K | x$ is w-integral over $D\}$. We know that D^w is an integrally closed domain; $D \subseteq \bar{D} \subseteq D^w \subseteq K$; and $D^w = \bar{D}[X]_{N_v} \cap K$ [8, Theorem 1.3]. The ring D^w is called the *w-integral closure* of D. An upper to zero in $D[X]$ is a (height-one) prime ideal of $D[X]$ of the form $fK[X] \cap D[X]$, where $f \in D[X]$ is irreducible in $K[X]$. Recall that D is a *UMT-domain* if each upper to zero in $D[X]$ is a maximal t-ideal of $D[X]$ and that D is a *Prüfer v-multiplication domain* (PvMD) if each nonzero finitely generated ideal I of D is t-invertible, i.e., $(II^{-1})_t = D$. The concept of UMT-domains was introduced by Houston and Zafrullah [14]. It is well known that D is a PvMD if and only if D_P is a valuation domain for each maximal t-ideal P of D [12, Theorem 5], if and only if D is an integrally closed UMT-domain [14, Proposition 3.2], if and only if $D[X]_{N_v}$ is a Prüfer domain [15, Theorem 3.7].

2. t-locally almost pseudo-valuation domains

Let D be an integral domain with $qf(D) = K$, $\bar{D}$ be the integral closure of D, D^w be the w-integral closure of D, and $N_v = \{f \in D[X] | c(f)_v = D\}$.

We first introduce the concept of t-locally APVDs: D is a *t-locally APVD* (t-LAPVD) if D_P is an APVD for all maximal t-ideals P of D.

LEMMA 1. (1) *Each nonzero prime ideal of an LAPVD D is a t-ideal.*

(2) *D is an LAPVD if and only if D is a t-LAPVD and each maximal ideal of D is a t-ideal.*

Proof. (1) Let P be a nonzero prime ideal of D, and let M be a maximal ideal of D with $P \subseteq M$. Then D_M is an APVD, and hence

Spec(D_M) is linearly ordered under inclusion. Hence PD_M is a t-ideal of D_M, and thus $P = PD_M \cap D$ is a t-ideal [15, Lemma 3.17].

(2) If D is an LAPVD, then each maximal ideal of D is a t-ideal by (1), and, in particular, D is a t-LAPVD. The converse is clear. □

An overring R of D is said to be *t-linked* over D if for any nonzero finitely generated ideal I of D, $I^{-1} = D$ implies $(IR)^{-1} = R$. It is known that R is t-linked over D if and only if $(Q \cap D)_t \subsetneq D$ for all prime t-ideals Q of R [10, Proposition 2.1], if and only if $R[X]_{N_v} \cap K = R$ [4, Lemma 3.2].

Lemma 2. *Let D be a t-LAPVD, and let P be a nonzero prime ideal of D with $P_t \subsetneq D$.*

(1) *P is a prime t-ideal of D.*
(2) *If P is not a maximal t-ideal, then D_P is a valuation domain.*
(3) *$\bar{D}_{D\setminus P} = (D^w)_{D\setminus P}$ and $\bar{D}_{D\setminus P}$ is a PVD.*

Proof. (1) and (2) Let Q be a maximal t-ideal of D such that $P_t \subseteq Q$; then D_Q is an APVD and PD_Q is a proper prime ideal of D_Q. Hence PD_Q, and thus $P = PD_Q \cap D$, is a t-ideal [15, Lemma 3.17]. Moreover, if P is not a maximal t-ideal, then PD_Q is not a maximal ideal, and hence $D_P = (D_Q)_{PD_Q}$ is a valuation domain [7, Lemma 3.1].

(3) Note that $\bar{D}_{D\setminus P}$ is an integrally closed t-linked overring of D [10, Proposition 2.9]; so $D^w \subseteq \bar{D}_{D\setminus P}$ (cf. [8, Theorem 1.3]), and thus $\bar{D}_{D\setminus P} = (D^w)_{D\setminus P}$. Moreover, since $\bar{D}_{D\setminus P}$ is the integral closure of D_P and D_P is an APVD, we have that $\bar{D}_{D\setminus P}$ is a PVD [3, Proposition 3.7]. □

Lemma 3. *The following statements are equivalent.*

(1) *D is a UMT-domain.*
(2) *D_P is a UMT-domain and PD_P is a t-ideal for each prime t-ideal P of D.*
(3) *D_P has Prüfer integral closure for each maximal t-ideal P of D.*

Proof. This appears in [11, Propositions 1.2 and 1.4, Theorem 1.5]. □

Proposition 4. *Let D be a t-LAPVD.*

(1) *D^w is a t-LPVD.*
(2) *D is a UMT-domain if and only if D^w is a PvMD.*

Proof. (1) Let Q be a maximal t-ideal of D^w, and set $P = Q \cap D$. Since D^w is t-linked over D [8, Lemma 1.2], we have $P_t \subsetneq D$. Hence $(D^w)_{D\setminus P}$ is a PVD by Lemma 2(3). Thus $(D^w)_Q = ((D^w)_{D\setminus P})_{Q_{D\setminus P}}$ is a PVD since $Q_{D\setminus P}$ is a maximal ideal of $(D^w)_{D\setminus P}$ (cf. [8, Corollary 1.4(3)]).

(2) Assume that D^w is a PvMD. Let P be a maximal t-ideal of D, and let Q be a prime ideal of D^w such that $Q \cap D = P$ (cf. [8, Corollary 1.4(3)]). Then $(D^w)_{D\setminus P}$ is a PVD by Lemma 2(3). So $(D^w)_{D\setminus P} = (D^w)_Q$ and $Q_{D\setminus P}$ is a t-ideal of $(D^w)_{D\setminus P}$, and hence $Q = Q_{D\setminus P} \cap D^w$ is a t-ideal of D^w [15, Lemma 3.17]. Hence $(D^w)_{D\setminus P}$ is a valuation domain [12, Theorem 5]. Thus D is a UMT-domain by Lemma 3. The converse holds without assumption that D is a t-LAPVD (see [8, Theorem 2.6]). □

We next give the main result of this paper.

Theorem 5. *The following statements are equivalent for an integral domain D.*

(1) *D is a t-LAPVD and a UMT-domain.*
(2) *D is a t-LAPVD and D^w is a PvMD.*
(3) *$D[X]$ is a t-LAPVD.*
(4) *$D[X]_{N_v}$ is an LAPVD, where $N_v = \{f \in D[X] | c(f)_v = D\}$.*
(5) *$D[X]_{N_v}$ is a t-LAPVD.*

Proof. (1) ⇔ (2) Proposition 4.

(1) ⇒ (3) Assume that D is a t-LAPVD and a UMT-domain. Let Q be a maximal t-ideal of $D[X]$; then either $Q \cap D = (0)$ or $Q = (Q \cap D)[X]$ with $Q \cap D$ maximal t-ideal of D [11, Proposition 2.2]. If $Q \cap D = (0)$, then $D[X]_Q$ is a valuation domain. Assume that $Q = (Q \cap D)[X]$, and note that $D_{Q\cap D}$ is an APVD and the integral closure of $D_{Q\cap D}$ is a Prüfer domain by Lemma 3. Thus $D[X]_Q = (D_{Q\cap D}[X])_{Q_{Q\cap D}} = D_{Q\cap D}(X)$, the Nagata ring of $D_{Q\cap D}$, is an APVD [6, Theorem 7].

(3) ⇒ (4) Let $D[X]$ be a t-LAPVD. Let Q be a maximal ideal of $D[X]_{N_v}$; then $Q = P[X]_{N_v}$ for a maximal t-ideal P of D [15, Proposition 2.1]. Note that $P[X]$ is a maximal t-ideal of $D[X]$ [11, Lemma 2.1(4)]. Thus $(D[X]_{N_v})_Q = (D[X]_{N_v})_{P[X]_{N_v}} = D[X]_{P[X]}$ is an APVD.

(4) ⇒ (1) Let P be a maximal t-ideal of D. Then $P[X]_{N_v}$ is a maximal ideal of $D[X]_{N_v}$ [15, Proposition 2.1], and so $(D[X]_{N_v})_{P[X]_{N_v}} = D[X]_{P[X]} = D_P(X)$, the Nagata ring of D_P, is an APVD. Thus D_P is an APVD and the integral closure of D_P is a valuation domain [6, Theorem 7]. Thus D is a UMT-domain by Lemma 3.

(4) $\Leftrightarrow$ (5) This follows because each maximal ideal of $D[X]_{N_v}$ is a t-ideal (cf. [15, Propositions 2.1 and 2.2]). $\square$

Lemma 1(2) shows that APVD $\Rightarrow$ LAPVD $\Rightarrow$ t-LAPVD. Clearly, PVD $\Rightarrow$ APVD, and thus

$$\begin{array}{ccc} \text{LPVD} & \longrightarrow & t\text{-LPVD} \\ \downarrow & & \downarrow \\ \text{LAPVD} & \longrightarrow & t\text{-LAPVD} \end{array}$$

We end this paper with an example of t-LAPVDs that are neither LAPVDs nor t-LPVDs.

Example 6. Let $\mathbb{Q}[\![t]\!]$ be the power series ring over the field $\mathbb{Q}$ of rational numbers, and let $D = \mathbb{Q}[\![t^2, t^3]\!]$. Then D is a one-dimensional Noetherian APVD such that $\bar{D} = \mathbb{Q}[\![t]\!]$ and D is not a PVD [7, Example 2.1]. Thus $D[X]$ is a t-LAPVD by Theorem 5 but not a t-LPVD [5, Theorem 3.8]. Note also that if M is the maximal ideal of D, then $Q := (X, M)$ is a maximal ideal of $D[X]$ such that $XD[X]_Q$ and $MD[X]_Q$ are not comparable; so $D[X]_Q$ is not an APVD, and thus $D[X]$ is not an LAPVD.

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Department of Mathematics
University of Incheon
Incheon 406-772, Korea
E-mail: whan@incheon.ac.kr