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A NEW SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION

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Abstract. In the present paper we introduce a new subclass of analytic functions in the unit disc defined by convolution $(f_\mu)^{(-1)} * f(z)$, where

$$f_\mu = (1-\mu)z\ {}_2F_1(a,b;c;z) + \mu z(z\ {}_2F_1(a,b;c;z))'.$$

Several interesting properties of the class and integral preserving properties of the subclasses are also considered.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open disc $U = \{z : |z| < 1\}$. If f and g are analytic in U, we say that f is subordinate to g, written as $f(z) \prec g(z)$ if there exists an analytic function w in U with $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$ such that $f(z) = g(w(z))$. Let S^*, K and C be subclasses

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of A consisting of analytic functions which are starlike, convex and close-to-convex in U, respectively.

Consider M as class of functions ϕ which are analytic and univalent in U such that $\phi(U)$ is convex with $\phi(0) = 1$ and $Re\{\phi(z)\} > 0$ for $z \in U$.

Using the subordination principle researchers (cf. [6],[13]) have investigated the subclasses $S^*(\phi), K(\phi)$, and $C(\phi, \psi)$ of the class A for $\phi, \psi \in M$ defined by

$$S^*(\phi) := \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \phi(z),\ z \in U \right\}, \tag{1.2}$$

$$K(\phi) := \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z),\ z \in U \right\}, \tag{1.3}$$

(1.4)
$$C(\phi, \psi) := \left\{ f \in A : \exists\ g \in S^*(\phi) \text{ such that } \frac{zf'(z)}{g(z)} \prec \psi(z), z \in U \right\}.$$

For $\phi(z) = \psi(z) = \frac{1+z}{1-z}$ in the above definitions, we have the popular classes S^*, K and C respectively. Furthermore for $\phi(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, we obtain the classes

$$S^*\left(\frac{1 + Az}{1 + Bz}\right) = S^*(A, B) \text{ and } K\left(\frac{1 + Az}{1 + Bz}\right) = K(A, B). \tag{1.5}$$

Let P denote the class of functions of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

analytic in U and $Re(p(z)) > 0$. Denote by $D^\lambda : A \to A$, the operator defined by

$$D^\lambda f(z) = \frac{z}{(1 - z)^{\lambda+1}} * f(z) \quad (\lambda > -1). \tag{1.6}$$

The operator $D^\lambda f$ is called the Ruscheweyh derivative of f of order λ. It is obvious that $D^0 f = f, D^1 f = zf'$ and

$$D^\alpha f(z) = \frac{z(z^{\alpha-1} f(z))^{(\alpha)}}{\alpha!} \quad (\alpha \in N_0 = N \cup \{0\}). \tag{1.7}$$

Recently K. I. Noor [16], K. I. Noor and M. A. Noor [17] have defined as integral operator $I_n : A \to A$, analogous to $D^\lambda f$ as follows.

Let $f_n(z) = \frac{z}{(1-z)^{n+1}}$, $n \in N_0$ and $f_n^{(-1)}(z)$ be defined such that

$$f_n(z) * f_n^{(-1)}(z) = \frac{z}{(1-z)^2}. \tag{1.8}$$

Then

$$I_n f(z) = f_n^{(-1)}(z) * f(z) = \left[\frac{z}{(1-z)^{n+1}}\right]^{(-1)} * f(z) \quad (f \in A). \tag{1.9}$$

We notice that $I_0 f(z) = zf'(z)$ and $I_1 f(z) = f(z)$. The operator I_n is called the Noor integral of n-th order of f (see [3], [12]), which is very important operator used in defining several subclasses of analytic functions.

For real or complex numbers a, b, c different from $0, -1, -2, \cdots$, the hypergeometric series is defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k \tag{1.10}$$

where $(v)_k$ is the Pochhammer symbol defined in terms of Gamma function by

$$(v)_k = \frac{\Gamma(v+k)}{\Gamma(v)} = v(v+1)\cdots(v+k-1) \tag{1.11}$$

for $k = 1, 2, 3, \cdots$ and $(v)_0 = 1$.

We notice that the series (1.10) converges absolutely for all $z \in U$, so that it represents an analytic function in U. In particular $z\,{}_2F_1(1, a; c; z) = \phi(a, c; z)$ which is the incomplete beta function. Also $\phi(a, 1; z) = \frac{z}{(1-z)^a}$, where $\phi(2, 1; z)$ is the Koebe function.

N. Shukla and P. Shukla [22] studied the mapping properties of f_μ function defined by

$$\begin{aligned} &f_\mu(a, b, c)(z) \\ &= (1-\mu)z\,{}_2F_1(a, b; c; z) + \mu z(z\,{}_2F_1(a, b; c; z))' \quad (\mu \geq 0). \end{aligned} \tag{1.12}$$

Kim and Shon [11] defined a linear operator $L_\mu : A \to A$ defined by

$$L_\mu(a,b,c)(f(z)) = f_\mu(a,b,c)(z) * f(z).$$

We now define a function $(f_\mu(a,b,c)(z))^{(-1)}$ by

$$\begin{aligned} &f_\mu(a,b,c)(z) * (f_\mu(a,b,c)(z))^{(-1)} \\ &\quad = \frac{z}{(1-z)^{\lambda+1}} \quad (\mu \geq 0,\ \lambda > -1) \end{aligned} \tag{1.13}$$

and introduce the linear operator

$$I_\mu^\lambda(a,b,c)f(z) = (f_\mu(a,b,c)(z))^{(-1)} * f(z). \tag{1.14}$$

For $\mu = 0$ in (1.13) we obtain the operator introduced by K. I. Noor [15]. For $\lambda > -1$ we have

$$\frac{z}{(1-z)^{\lambda+1}} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k}{k!} z^{k+1} \quad (z \in U). \tag{1.15}$$

Using (1.10) and (1.15) in (1.13), we get

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(\mu k+1)(a)_k(b)_k}{(c)_k(1)_k} z^{k+1} * (f_\mu(a,b,c)(z))^{(-1)} \\ &\quad = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k}{k!} z^{k+1}. \end{aligned} \tag{1.16}$$

Thus $(f_\mu(a,b,c)(z))^{(-1)}$ has the form

$$(f_\mu(a,b,c)(z))^{(-1)} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k(c)_k}{(\mu k+1)(a)_k(b)_k} z^{k+1} \quad (z \in U). \tag{1.17}$$

Equation (1.14) now implies that

$$I_\mu^\lambda(a,b,c)f(z) = z + \sum_{k=1}^{\infty} \frac{(\lambda+1)_k(c)_k}{(\mu k+1)(a)_k(b)_k} a_{k+1} z^{k+1}. \tag{1.18}$$

In particular

$$I_0^\lambda(a,\lambda+1,a)f(z)=f(z),\quad I_0^1(a,1,a)f(z)=zf'(z). \tag{1.19}$$

It can be easily shown that

$$z(I_\mu^\lambda(a,b,c)f(z))'=(\lambda+1)I_\mu^{\lambda+1}(a,b,c)f(z)-\lambda I_\mu^\lambda(a,b,c)f(z), \tag{1.20}$$

$$\begin{aligned} &z(I_\mu^\lambda(a+1,b,c)f(z))' \\ &\quad = aI_\mu^\lambda(a,b,c)f(z)-(a-1)I_\mu^\lambda(a+1,b,c)f(z). \end{aligned} \tag{1.21}$$

By using the operator $I_\mu^\lambda(a,b,c)$, we introduce the following classes of analytic functions for $\phi,\ \psi\in M,\ \lambda>-1,\ \mu\geq 0$:

$$S_\mu^\lambda(a,b,c)(\phi):=\{f\in A: I_\mu^\lambda(a,b,c)f(z)\in S^*(\phi)\},$$

$$K_\mu^\lambda(a,b,c)(\phi):=\{f\in A: I_\mu^\lambda(a,b,c)f(z)\in K(\phi)\}, \tag{1.22}$$

$$\begin{aligned} &C_\mu^\lambda(a,b,c)(\phi,\psi) \\ &:=\left\{f\in A: \exists g(z)\in S_\mu^\lambda(a,b,c)(\phi)\text{ s.t.}\frac{z(I_\mu^\lambda(a,b,c)f(z))'}{I_\mu^\lambda(a,b,c)g(z)}\prec\psi(z), z\in U\right\}. \end{aligned}$$

We note that

$$f(z)\in K_\mu^\lambda(a,b,c)(\phi)\text{ if and only if } zf'(z)\in S_\mu^\lambda(a,b,c)(\phi). \tag{1.23}$$

In particular

$$S_\mu^\lambda(a,b,c)\left(\frac{1+Az}{1+Bz}\right)=S_\mu^\lambda(a,b,c,A,B)\quad(-1\leq B<A\leq 1),$$

$$K_\mu^\lambda(a,b,c)\left(\frac{1+Az}{1+Bz}\right)=K_\mu^\lambda(a,b,c,A,B)\quad(-1\leq B<A\leq 1).$$

In this paper we investigate the inclusion properties of the class $S_\mu^\lambda(a,b,c)(\phi)$, $K_\mu^\lambda(a,b,c)(\phi)$ and $C_\mu^\lambda(a,b,c)(\phi,\psi)$. Notice that

$$S_0^\lambda(a,\lambda+1,a)\left(\frac{1+z}{1-z}\right)=S^*,\quad K_0^\lambda(a,\lambda+1,a)\left(\frac{1+z}{1-z}\right)=K$$

$$C_0^\lambda(a,\lambda+1,a)\left(\frac{1+z}{1-z}\right)=C.$$

2. Inclusion properties involving the operator $I_{\mu}^{\lambda}(a,b,c)$

The following lemmas will be required in our investigation.

LEMMA 2.1([14]). *Let $\phi(z)$ be convex univalent in U and $E \geq 0$. Suppose $B(z)$ is analytic in U with $Re\ B(z) \geq E$. If $g \in P$ is analytic in U, then*

$$Ez^2g''(z) + B(z)zg'(z) + g(z) \prec \phi(z) \quad (z \in U) \tag{2.1}$$

implies

$$g(z) \prec \phi(z) \quad (z \in U).$$

LEMMA 2.2([20]). *Let $f \in K$ and $g \in S^*$. Then for every analytic function Q in U,*

$$\frac{(f * Qg)}{f * g}(U) \subset \overline{CO}Q(U), \tag{2.2}$$

where $\overline{CO}Q(U)$ denotes the closed convex hull of $Q(U)$.

LEMMA 2.3([19]). *Let β, γ be complex numbers. Let $\phi(z)$ be convex univalent in U with $\phi(0) = 1$ and $Re[\beta\phi(z) + \gamma] > 0, z \in U$ and $q(z) \in A$ with $q(z) \prec \phi(z), z \in U$. If $p(z) \in P$ is analytic in U, then*

$$p(z) + \frac{zp'(z)}{\beta q(z) + \gamma} \prec \phi(z) \quad (z \in U) \tag{2.3}$$

implies

$$p(z) \prec \phi(z) \quad (z \in U).$$

LEMMA 2.4([7]). *Let δ, η be complex numbers. For $\phi(z)$ convex univalent in U with $\phi(0) = 1$ and $Re[\delta\phi(z) + \eta] > 0, z \in U$. If $p(z) \in P$ is analytic in U, then*

$$p(z) + \frac{zp'(z)}{\delta p(z) + \eta} \prec \phi(z) \quad (z \in U) \tag{2.4}$$

implies

$$p(z) \prec \phi(z) \quad (z \in U).$$

THEOREM 2.5. *Let $\phi(z)$ be convex and univalent in U with $\phi(0) = 1$ and Re $\phi(z) \geq 0$. Then*

$$S_{\mu}^{\lambda+1}(a,b,c)(\phi) \subset S_{\mu}^{\lambda}(a,b,c)(\phi)$$

for $\lambda > -1, \mu \geq 0$.

Proof. Let $f(z) \in S_{\mu}^{\lambda+1}(a,b,c)(\phi)$ and

$$p(z) = \frac{z(I_{\mu}^{\lambda}(a,b,c)f(z))'}{I_{\mu}^{\lambda}(a,b,c)f(z)} \tag{2.5}$$

where $p(z) \in P$. Using (1.20) in (2.5) and differentiating we get

$$\frac{z(I_{\mu}^{\lambda+1}(a,b,c)f(z))'}{I_{\mu}^{\lambda+1}(a,b,c)f(z)} = p(z) + \frac{zp'(z)}{(\lambda+1)q(z)}$$

where

$$q(z) = \frac{I_{\mu}^{\lambda+1}(a,b,c)f(z)}{I_{\mu}^{\lambda}(a,b,c)f(z)}$$

and $q(z) \prec \phi(z)$. Hence by applying Lemma 2.3, we obtain

$$\frac{z(I_{\mu}^{\lambda}(a,b,c)f(z))'}{I_{\mu}^{\lambda}(a,b,c)f(z)} \prec \phi(z).$$

In view of (1.22) we get $f(z) \in S_{\mu}^{\lambda}(a,b,c)(\phi)$. □

THEOREM 2.6. *Let $\phi(z)$ be convex and univalent in U with $\phi(0) = 1$ and Re $\phi(z) \geq 0$. Then*

$$S_{\mu}^{\lambda}(a,b,c)(\phi) \subset S_{\mu}^{\lambda}(a+1,b,c)(\phi)$$

for $\lambda > -1, \mu \geq 0$.

Proof. Applying the same technique as in proof of Theorem 2.5 and using (1.21) with Lemma (2.4) we obtain the required result. □

Taking $\phi(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$ in Theorem 2.5 and Theorem 2.6 we obtain the following result.

Corollary 2.7. *For $\lambda > -1,\ \mu \geq 0$ and Re $a > 1$*

$$S_\mu^{\lambda+1}(a,b,c,A,B) \subset S_\mu^\lambda(a,b,c,A,B),$$

$$S_\mu^\lambda(a,b,c,A,B) \subset S_\mu^\lambda(a+1,b,c,A,B).$$

Further if $\phi(z) = \frac{1+z}{1-z}$ in Theorem 2.5 and Theorem 2.6 we obtain the following result.

Corollary 2.8. *For $\lambda > -1,\ \mu \geq 0$ and Re $a > 0$*

$$I_\mu^{\lambda+1}(a,b,c)f(z) \in S^* \text{ implies } I_\mu^\lambda(a,b,c)f(z) \in S^*.$$

Similarly

$$I_\mu^\lambda(a,b,c)f(z) \in S^* \text{ implies } I_\mu^\lambda(a+1,b,c)f(z) \in S^*.$$

Corollary 2.9. *For $\lambda > -1, \mu \geq 0$ and Re $a > 0$ we have*

$$K_\mu^{\lambda+1}(a,b,c)(\phi) \subset K_\mu^\lambda(a,b,c)(\phi),$$

$$K_\mu^\lambda(a,b,c)(\phi) \subset K_\mu^\lambda(a+1,b,c)(\phi).$$

Proof.

$$\begin{aligned} f(z) \in K_\mu^{\lambda+1}(a,b,c)(\phi) &\Leftrightarrow zf'(z) \in S_\mu^{\lambda+1}(a,b,c)(\phi) \\ &\Rightarrow zf'(z) \in S_\mu^\lambda(a,b,c)(\phi) \\ &\Leftrightarrow I_\mu^\lambda(a,b,c)(zf'(z)) \in S^*(\phi) \\ &\Leftrightarrow z(I_\mu^\lambda(a,b,c)f(z))' \in S^*(\phi) \\ &\Leftrightarrow I_\mu^\lambda(a,b,c)f(z) \in K(\phi) \\ &\Leftrightarrow f(z) \in K_\mu^\lambda(a,b,c)(\phi). \end{aligned}$$

The second relation can be proved similarly. □

THEOREM 2.10. *Let $\phi(z)$ be convex univalent in U with $\phi(0) = 1$ and Re $\phi(z) \geq 0$. If $f(z) \in A$ satisfies the condition*

$$f(z) \in S_{\mu}^{\lambda}(a,b,c)(\phi)$$

then

$$F(z) \in S_{\mu}^{\lambda}(a,b,c)(\phi)$$

where $F(z)$ is the integral operator defined by

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt \quad (c \geq 0). \tag{2.6}$$

Proof. From (2.6) we have

$$z(I_{\mu}^{\lambda}(a,b,c)F(z))' = (c+1)I_{\mu}^{\lambda}(a,b,c)f(z) - cI_{\mu}^{\lambda}(a,b,c)F(z). \tag{2.7}$$

Let

$$p(z) = \frac{z(I_{\mu}^{\lambda}(a,b,c)F(z))'}{I_{\mu}^{\lambda}(a,b,c)F(z)}$$

where $p(z) \in P$. Using (2.7), we get

$$p(z) + c = \frac{(c+1)I_{\mu}^{\lambda}(a,b,c)f(z)}{I_{\mu}^{\lambda}(a,b,c)F(z)}. \tag{2.8}$$

Differentiating both sides of (2.8) logarithmically, we get

$$p(z) + \frac{zp'(z)}{c+p(z)} = \frac{z(I_{\mu}^{\lambda}(a,b,c)f(z))'}{I_{\mu}^{\lambda}(a,b,c)f(z)} \prec \phi(z)$$

by hypothesis. Now applying Lemma 2.4 we obtain

$$\frac{z(I_{\mu}^{\lambda}(a,b,c)F(z))'}{I_{\mu}^{\lambda}(a,b,c)F(z)} \prec \phi(z).$$

That is $F(z) \in S_{\mu}^{\lambda}(a,b,c)(\phi)$. □

For $\phi(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$ in Theorem 2.10 we obtain the following result.

Corollary 2.11. *For* $\lambda > -1$, $\mu \geq 0$ *and* $c > 0$, *if* $f(z) \in S_\mu^\lambda(a,b,c,A,B)$, *then* $F(z) \in S_\mu^\lambda(a,b,c,A,B)$ *where* $F(z)$ *is given by (2.6).*

Corollary 2.12. *For* $\lambda > -1$, $\mu \geq 0$ *and* $c \geq 0$, *if* $f(z) \in K_\mu^\lambda(a,b,c)(\phi)$, *then* $F(z) \in K_\mu^\lambda(a,b,c)(\phi)$.

Proof. We have

$$\begin{aligned} f(z) \in K_\mu^\lambda(a,b,c)(\phi) &\Leftrightarrow zf'(z) \in S_\mu^\lambda(a,b,c)(\phi) \\ &\Rightarrow z(F(z))' \in S_\mu^\lambda(a,b,c)(\phi) \\ &\Leftrightarrow F(z) \in K_\mu^\lambda(a,b,c)(\phi). \end{aligned}$$

□

Theorem 2.13. *Let* $f(z) \in A$. *Then*

$$C_\mu^{\lambda+1}(a,b,c,\phi,\psi) \subset C_\mu^\lambda(a,b,c,\phi,\psi)$$

for $\lambda \geq 0$, $\mu \geq 0$.

Proof. Let $f(z) \in C_\mu^{\lambda+1}(a,b,c,\phi,\psi)$. Then by definition

$$\frac{z(I_\mu^{\lambda+1}(a,b,c,\phi,\psi)f(z))'}{I_\mu^{\lambda+1}(a,b,c,\phi,\psi)g(z)} \prec \psi(z)$$

for some $g(z) \in S_\mu^{\lambda+1}(a,b,c)(\phi)$. Let

$$h(z) = \frac{z(I_\mu^\lambda(a,b,c)f(z))'}{I_\mu^\lambda(a,b,c)g(z)} \text{ and} \tag{2.9}$$

$$H(z) = \frac{z(I_\mu^\lambda(a,b,c)g(z))'}{I_\mu^\lambda(a,b,c)g(z)}. \tag{2.10}$$

Notice that $h(z), H(z) \in P$. By Theorem 2.5 $g(z) \in S_\mu^\lambda(a,b,c)(\phi)$ and so $Re H(z) > 0$. We also note that (2.9) implies

$$z(I_\mu^\lambda(a,b,c)f(z))' = (I_\mu^\lambda(a,b,c)g(z))h(z). \tag{2.11}$$

Differentiating both sides of (2.11) gives

$$\frac{z(z(I_\mu^\lambda(a,b,c)f(z))')'}{I_\mu^\lambda(a,b,c)g(z)} = H(z)h(z) + zh'(z). \tag{2.12}$$

Using identity (1.20), we get

$$\begin{aligned}\frac{z(I_\mu^{\lambda+1}(a,b,c)f(z))'}{I_\mu^{\lambda+1}(a,b,c)g(z)} &= \frac{I_\mu^{\lambda+1}(a,b,c)(zf'(z))}{I_\mu^{\lambda+1}(a,b,c)g(z)} \\ &= \frac{z(I_\mu^\lambda(a,b,c)(zf'(z)))' + \lambda I_\mu^\lambda(a,b,c)(zf'(z))}{z(I_\mu^\lambda(a,b,c)g(z))' + \lambda I_\mu^\lambda(a,b,c)g(z)} \\ &= \frac{\frac{z(I_\mu^\lambda(a,b,c)(zf'(z)))'}{I_\mu^\lambda(a,b,c)g(z)} + \frac{\lambda I_\mu^\lambda(a,b,c)(zf'(z))}{I_\mu^\lambda(a,b,c)g(z)}}{\frac{z(I_\mu^\lambda(a,b,c)g(z))'}{I_\mu^\lambda(a,b,c)g(z)} + \lambda} \\ &= \frac{H(z)h(z) + zh'(z) + \lambda h(z)}{H(z)+\lambda} \\ &= h(z) + \frac{zh'(z)}{H(z)+\lambda} \prec \psi(z).\end{aligned}$$

Now from Lemma 2.1, for $E = 0$ and $B(z) = \frac{1}{H(z)+\lambda}$ with $Re(B(z)) = \frac{1}{|H(z)+\lambda|^2} Re\ (H(z)+\lambda) > 0$. We get $h(z) \prec \psi(z)$. In view of (2.9) we get $f(z) \in C_\mu^\lambda(a,b,c,\phi,\psi)$. □

Theorem 2.14. *Let $f \in A$. Then*

$$C_\mu^\lambda(a,b,c,\phi,\psi) \subset C_\mu^\lambda(a+1,b,c,\phi,\psi)$$

$\lambda \geq 0, \mu \geq 0$.

Proof. By using arguments similar to the proof of Theorem 2.13, we get

$$h(z) + \frac{zh'(z)}{H(z)+a-1} \prec \psi(z)$$

for $h(z) = \frac{z(I_\mu^\lambda(a+1,b,c)f(z))'}{I_\mu^\lambda(a+1,b,c)g(z)}$ and $H(z) = \frac{z(I_\mu^\lambda(a+1,b,c)g(z))'}{I_\mu^\lambda(a+1,b,c)g(z)}$ belonging to P. Taking $E = 0$ and $B(z) = \frac{1}{H(z)+a-1}$ with

$$Re(B(z)) = \frac{1}{|H(z)+a-1|^2} Re(H(z)+a-1) > 0.$$

Now applying Lemma 2.1 we obtain the required result. □

THEOREM 2.15. *If* $f(z) \in C_\mu^\lambda(a,b,c,\phi,\psi)$ *then* $F(z) \in C_\mu^\lambda(a,b,c,\phi,\psi)$ *for* $c \geq 0$, *where* $F(z)$ *is given by (2.6).*

Proof. Employing same technique as in proof of Theorem 2.13, we get

$$\frac{zh'(z)}{H(z)+c} + h(z) \prec \psi(z)$$

for $h(z) = \frac{z(I_\mu^\lambda(a,b,c)F(z))'}{I_\mu^\lambda(a,b,c)g(z)}$ and $H(z) = \frac{z(I_\mu^\lambda(a,b,c)g(z))'}{I_\mu^\lambda(a,b,c)g(z)}$ belonging to P. Taking $E = 0$ and $B = \frac{1}{H(z)+c}$, we obtain

$$Re(B(z)) = \frac{1}{|H(z)+c|^2} Re(H(z)+c) > 0.$$

Now by Lemma 2.1 we derive the required result. □

3. Inclusion properties by convolution

In this Section we show that the classes $S_\mu^\lambda(a,b,c)(\phi), K_\mu^\lambda(a,b,c)(\phi)$ and $C_\mu^\lambda(a,b,c,\phi,\psi)$ are invariant under convolution with convex functions.

THEOREM 3.1. *Let* $a,b > 0, c \in I\!R \setminus Z_0^-, \phi, \psi \in M$ *and let* $g \in K$. *Then*

(i) $f \in S_\mu^\lambda(a,b,c)(\phi) \Rightarrow g * f \in S_\mu^\lambda(a,b,c)(\phi)$,
(ii) $f \in K_\mu^\lambda(a,b,c)(\phi) \Rightarrow g * f \in K_\mu^\lambda(a,b,c)(\phi)$,
(iii) $f \in C_\mu^\lambda(a,b,c,\phi,\psi) \Rightarrow g * f \in C_\mu^\lambda(a,b,c,\phi,\psi)$.

Proof. (i) Let $f \in S_\mu^\lambda(a,b,c)(\phi)$, then $\frac{z(I_\mu^\lambda(a,b,c)f(z))'}{I_\mu^\lambda(a,b,c)f(z)} = \phi(w(z))$. Consider the following

(3.1)
$$\begin{aligned}\frac{z(I_\mu^\lambda(a,b,c)(g*f)(z))'}{I_\mu^\lambda(a,b,c)(g*f)(z)} &= \frac{g(z) * z(I_\mu^\lambda(a,b,c)f(z))'}{g(z) * I_\mu^\lambda(a,b,c)f(z)} \\ &= \frac{g(z) * \phi(w(z))I_\mu^\lambda(a,b,c)f(z)}{g(z) * I_\mu^\lambda(a,b,c)f(z)}.\end{aligned}$$

Using Lemma 2.2, we conclude that

$$\frac{\{g * \phi(w) I_\mu^\lambda(a,b,c)f\}}{\{g * I_\mu^\lambda(a,b,c)f\}}(U) \subset \overline{CO}(\phi(U)) \subset \phi(U)$$

since ϕ is convex univalent and $I_\mu^\lambda(a,b,c)f \in S^*(\phi)$. By definition of subordination we conclude that (3.1) is subordinated to ϕ in U and so $g * f \in S_\mu^\lambda(a,b,c)(\phi)$.

(ii) Let $f \in K_\mu^\lambda(a,b,c)(\phi)$. Then by (1.23), $zf'(z) \in S_\mu^\lambda(a,b,c)(\phi)$ and hence by (i) $g * zf'(z) \in S_\mu^\lambda(a,b,c)(\phi)$. Notice that

$$g(z) * zf'(z) = z(g*f)'(z).$$

Now applying (1.23) again, we get $g * f \in K_\mu^\lambda(a,b,c)(\phi)$.

(iii) Let $f \in C_\mu^\lambda(a,b,c,\phi,\psi)$. Then there exists $q \in S_\mu^\lambda(a,b,c)(\phi)$ such that

$$\frac{z(I_\mu^\lambda(a,b,c)f(z))'}{I_\mu^\lambda(a,b,c)q(z)} \prec \psi(z).$$

Therefore

$$z(I_\mu^\lambda(a,b,c)f(z))' = \psi(w(z)) I_\mu^\lambda(a,b,c)q(z) \ (z \in U) \tag{3.2}$$

where w is an analytic function in U with $|w(z)| < 1$ $(z \in U)$ and $w(0) = 0$.

In view of $I_\mu^\lambda(a,b,c)q \in S^*(\phi)$, we have

$$\begin{aligned}\frac{z(I_\mu^\lambda(a,b,c)(g*f)(z))'}{g * I_\mu^\lambda(a,b,c)q} &= \frac{g(z) * z(I_\mu^\lambda(a,b,c)f(z))'}{g(z) * I_\mu^\lambda(a,b,c)q(z)} \\ &= \frac{g(z) * \psi(w(z)) I_\mu^\lambda(a,b,c)q(z)}{g(z) * I_\mu^\lambda(a,b,c)q(z)} \\ &\prec \psi(z) \ (z \in U).\end{aligned}$$

Thus (iii) is proved. □

Next, we investigate the following operators ([18], [21]) defined by

$$\eta_1(z) = \sum_{k=1}^{\infty} \frac{1+c}{k+c} z^k \quad (Re\ c \geq 0; z \in U), \tag{3.3}$$

$$\eta_2(z) = \frac{1}{1-x} \log \left[\frac{1-xz}{1-z} \right] \quad (\log 1 = 0;\ |x| \le 1,\ x \ne 1;\ z \in U). \tag{3.4}$$

It is known that the operators η_1 and η_2 are convex univalent in U([1], [21]). Therefore, we have the following results which immediately follow from Theorem 3.1.

COROLLARY 3.2. *Let* $a, b > 0$; $c \in \mathbb{R} \setminus Z_0^-$; $\phi, \psi \in M$ *and let* η_i $(i = 1, 2)$ *be as defined by (3.3) and (3.4). Then*
(i) $f \in S_\mu^\lambda(a,b,c)(\phi) \Rightarrow \eta_i * f \in S_\mu^\lambda(a,b,c)(\phi)$,
(ii) $f \in K_\mu^\lambda(a,b,c)(\phi) \Rightarrow \eta_i * f \in K_\mu^\lambda(a,b,c)(\phi)$,
(iii) $f \in C_\mu^\lambda(a,b,c,\phi,\psi) \Rightarrow \eta_i * f \in C_\mu^\lambda(a,b,c,\phi,\psi)$.

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