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SECOND CLASSICAL ZARISKI TOPOLOGY ON SECOND SPECTRUM OF LATTICE MODULES

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Abstract. Let M be a lattice module over a C-lattice L. Let $Spec^s(M)$ be the collection of all second elements of M. In this paper, we consider a topology on $Spec^s(M)$, called the second classical Zariski topology as a generalization of concepts in modules and investigate the interplay between the algebraic properties of a lattice module M and the topological properties of $Spec^s(M)$. We investigate this topological space from the point of view of spectral spaces. We show that $Spec^s(M)$ is always T_0-space and each finite irreducible closed subset of $Spec^s(M)$ has a generic point.

1. Introduction

The dual notion of prime submodules (i.e. second submodules) was introduced and studied by S. Yassemi in [19]. H. Ansari-Toroghy and F. Farshadifar studied the Zariski topology on second spectrum of a module over a commutative ring in [3]. The second classical Zariski topology on second spectrum of a module over a commutative ring was introduced and studied by H. Ansari-Toroghy et al. in [4].
The concept of a second element of a lattice module M over a $C-$ lattice L was introduced and studied the Zariski topology on the second spectrum $Spec^s(M)$, i.e., the collection of all second elements of a lattice

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module M over a $C-$lattice L by N. Phadatare et al. in [15]. In [11], P. Girase et al. studied the topology on classical prime spectrum of a lattice module over a $C-$lattice and in [6], V. Borkar et al. studied the classical Zariski topology on prime spectrum of a lattice module over a $C-$lattice.
As a generalization of second classical Zariski topology on second spectrum of a module over a commutative ring in [4], we introduce and study the dual notion of classical Zariski topology on prime spectrum of a lattice module over a $C-$lattice as a second classical Zariski topology on second spectrum of a lattice module M over a $C-$lattice L.
A lattice L is said to be *complete*, if for any subset S of L, we have $\vee S, \wedge S \in L$. A complete lattice L with least element 0_L and greatest element 1_L is said to be a *multiplicative lattice*, if there is defined a binary operation "." called multiplication on L satisfying the following conditions:
(1) $a.b = b.a$, for all $a, b \in L$;
(2) $a.(b.c) = (a.b).c$, for all $a, b, c \in L$;
(3) $a.(\vee_\alpha b_\alpha) = \vee_\alpha(a.b_\alpha)$, for all $a, b_\alpha \in L$;
(4) $a.1_L = a$, for all $a \in L$.
Henceforth, $a.b$ will be simply denoted by ab.

An element a in L is called *compact*, if $a \leq \vee_{\alpha \in I} b_\alpha$ (I is an indexed set) implies $a \leq b_{\alpha_1} \vee b_{\alpha_2} \vee \cdots \vee b_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of I. By a *C-lattice*, we mean a multiplicative lattice L, with least element 0_L and greatest element 1_L which is compact as well as multiplicative identity, that is generated under joins by a multiplicatively closed subset C of compact elements of L.

An element $a \in L$ is said to be *proper*, if $a < 1_L$. A proper element p of a multiplicative lattice L is said to be *prime*, if $ab \leq p$ implies $a \leq p$ or $b \leq p$ for $a, b \in L$. The collection of all prime elements of L is denoted by $Spec(L)$.

The Zariski topology on the set $Spec(L)$ of all prime elements in multiplicative lattices is being studied in [18] by Thakare, Manjarekar and Maeda and in [17], by Thakare and Manjarekar as a generalization of the Zariski topology of a commutative ring with unity.
A proper element m of a multiplicative lattice L is said to be *maximal*, if for every $x \in L$ with $m < x \leq 1_L$ implies $x = 1_L$.

A complete lattice M with smallest element 0_M and greatest element 1_M is said to be a *lattice module* over a multiplicative lattice L, or

L-module, if there is a multiplication between elements of M and L, denoted by $aN \in M$, for $a \in L$ and $N \in M$, which satisfies the following properties:

1. $(ab)N = a(bN)$;
2. $(\vee_\alpha a_\alpha)(\vee_\beta N_\beta) = (\vee_{\alpha\beta} a_\alpha N_\beta)$;
3. $1_L N = N$;
4. $0_L N = 0_M$; for all $a, b, a_\alpha \in L$ and for all $N, N_\beta \in M$.

Let M be a lattice module over a C-lattice L. For $N \in M, b \in L$, denote $(N : b) = \vee\{K \in M | bK \leq N\}$. If $a, b \in L$, we write $(a : b) = \vee\{x \in L | bx \leq a\}$. If $A, B \in M$, then $(A : B) = \vee\{x \in L | xB \leq A\}$. An element $N \in M$ is said to be *compact*, if $N \leq \vee_{\alpha \in I} A_\alpha$ (I is an indexed set) implies $N \leq A_{\alpha_1} \vee A_{\alpha_2} \vee \cdots \vee A_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of I.

An element $N \in M$ is said to be *proper*, if $N < 1_M$. A proper element N of a lattice module M is said to be *prime*, if $aX \leq N$ implies $X \leq N$ or $a1_M \leq N$, i.e., $a \leq (N : 1_M)$ for every $a \in L$ and $X \in M$. The prime spectrum of a lattice module M is the set of all prime elements of M and it is denoted by $Spec(M)$. In [5], S. Ballal and V. Kharat studied the Zariski topology over $Spec(M)$. Also, in [10], F. Callialp et al. studied the Zariski topology over $Spec(M)$ over a multiplicative lattice L.

A non-zero element $N \in M$ is said to be *second*, if for $a \in L$, either $aN = N$ or $aN = 0_M$. An element $N < 1_M$ of M is said to be *maximal*, if $N \leq B$ implies either $N = B$ or $B = 1_M, B \in M$. A non-zero element $K \neq 1_M$ of M is said to be *minimal*, if $0_M \leq N < K$ implies $N = 0_M, N \in M$.

Let M be a lattice module over a C-lattice L. Set $Spec^s(M) = \{S \in M | S \text{ is a second element of } M\}$. We call this set the second spectrum of M. For any element N of M, we define, $F(N) = \{S \in Spec^s(M) | S \leq N\}$. Note that, $F(0_M)$ is an empty set and $F(1_M) = Spec^s(M)$. It is easy to see that for any family of elements $N_i (i \in I)$ of M, $\cap_{i \in I} F(N_i) = F(\wedge_{i \in I} N_i)$. Thus if $\Upsilon(M)$ denotes the collection of all subsets $F(N)$ of $Spec^s(M)$, then $\Upsilon(M)$ is closed under arbitrary intersections. In general $\Upsilon(M)$ is not closed under finite unions. A lattice module M is called *cotop*, if $\Upsilon(M)$ is closed under finite unions. In this case, $\Upsilon(M)$ is called the quasi-Zariski topology (see [15]).

Let N be an element of M. We define $G(N) = Spec^s(M) - F(N)$ and put $\mathcal{G}(M) = \{G(N) | N \in M\}$. Then we define a topology $\xi(M)$ on $Spec^s(M)$ by the subbasis $\mathcal{G}(M)$ and call it the second classical Zariski

topology of M. In fact, $\xi(M)$ to be the collection U of all unions of finite intersections of elements of $\mathcal{G}(M)$.
Further all these concepts and for more information on multiplicative lattices, lattice modules and topology the reader may refer ([1, 2, 7, 12, 14, 16]).

2. Second Classical Zariski Topology

Let M be a lattice module over a $C-$lattice L and let $Spec^s(M)$ be equipped with the second classical Zariski topology. Let $Y \subseteq Spec^s(M)$, then $Cl(Y)$ denotes the closure of Y in $Spec^s(M)$ and join of all elements of Y denoted by $Z(Y)$. Note that, if $Y = \emptyset$, then $Z(Y) = 0_M$.
A topological space X is called irreducible if $X \neq \emptyset$ and every finite intersection of nonempty open sets of X is nonempty. A nonempty subset Y of a topological space X is called an irreducible set if the subspace Y of X is irreducible, i.e., if $Y \subseteq Y_1 \cup Y_2$, then $Y \subseteq Y_1$ or $Y \subseteq Y_2$, where Y_1 and Y_2 are closed subsets of X (see [8]).
Let Y be a closed subset of a topological space. An element $y \in Y$ is called a generic point of Y if $Y = Cl(\{y\})$. Note that, a generic point of the irreducible closed subset of a topological space is unique if the topological space is a T_0-space.

Lemma 2.1. *Let M be a lattice module over a $C-$lattice L. If Y is a nonempty subset of $Spec^s(M)$, then $Cl(Y) = \cup_{S\in Y} F(S)$.*

Proof. Suppose that Y is a nonempty subset of $Spec^s(M)$. Clearly, $Y \subseteq \cup_{S\in Y} F(S)$. Suppose that A is any closed subset of $Spec^s(M)$ such that $Y \subseteq A$. Thus $A = \cap_{k\in J}(\cup_{l=1}^{n_k} F(N_{kl}))$, for some $N_{kl} \in M$, $k \in J$(Indexed set), $n_k \in \mathbb{N}$. Let $S_1 \in \cup_{S\in Y} F(S)$. Then $S_1 \in F(S')$ for some $S' \in Y$ and therefore $S_1 \leq S'$. Now, $S' \in Y \subseteq A$ and $A = \cap_{k\in J}(\cup_{l=1}^{n_k} F(N_{kl}))$, therefore for each $k \in J$, there exists $l \in \{1, 2, \cdots, n_k\}$ such that $S' \in F(N_{kl})$ and therefore $S' \leq N_{kl}$, hence $S_1 \leq S' \leq N_{kl}$. It follows that $S_1 \in F(N_{kl})$ and hence $S_1 \in \cap_{k\in J}(\cup_{l=1}^{n_k} F(N_{kl})) = A$. Hence, $\cup_{S\in Y} F(S) \subseteq A$. Thus $\cup_{S\in Y} F(S)$ is the smallest closed subset of $Spec^s(M)$ containing Y. Consequently, $Cl(Y) = \cup_{S\in Y} F(S)$. □

Corollary 2.2. *Let M be a lattice module over a $C-$lattice L. Then we have the following:*

1. $Cl(\{S\}) = F(S)$, for all $S \in Spec^s(M)$.

2. *$S_1 \in Cl(\{S\})$ if and only if $S_1 \leq S$ if and only if $F(S_1) \subseteq F(S)$, for $S_1 \in Spec^s(M)$.*
3. *The set $\{S\}$ is closed in $Spec^s(M)$ if and only if S is a minimal second element of M.*

Proof. (1) By Lemma 2.1, for $Y \subseteq Spec^s(M)$, we have $Cl(Y) = \cup_{S\in Y} F(S)$. Let $Y = \{S\}$, then $\cup_{S\in Y} F(S) = F(S)$. Hence, $Cl(\{S\}) = F(S)$.
(2) Suppose that $S_1 \in Cl(\{S\})$. Then by part (1), $S_1 \in Cl(\{S\}) = F(S)$, therefore $S_1 \leq S$ and $S_1 \leq S$ implies that $F(S_1) \subseteq F(S)$. Conversely, suppose that $F(S_1) \subseteq F(S)$. Since $S_1 \in F(S_1) \subseteq F(S)$, we have $S_1 \leq S$ and $S_1 \in F(S) = Cl(\{S\})$, by part (1).
(3) Suppose that S is a minimal second element of M and $S_1 \in Cl(\{S\})$. Then $S_1 \in Cl(\{S\}) = F(S)$ implies that $S_1 \leq S$. But S is minimal, therefore $S_1 = S$ and hence $Cl(\{S\}) = \{S\}$. Consequently, $\{S\}$ is closed in $Spec^s(M)$. Conversely, suppose that $\{S\}$ is closed in $Spec^s(M)$ and S is not minimal. Then there exists S_1 such that $S_1 \leq S$, which implies that $S_1 \in Cl(\{S\})$, by part (2). Since, $\{S\}$ is closed in $Spec^s(M)$, $S_1 \in Cl(\{S\}) = \{S\}$. Hence, $S_1 = S$. Consequently, S is a minimal second element of M. □

Lemma 2.3. *Let M be a lattice module over a $C-$lattice L. If Y is a closed subset of $Spec^s(M)$, then $Y = \cup_{S\in Y} F(S)$.*

Proof. Suppose that Y is a closed subset of $Spec^s(M)$. Clearly, $Y \subseteq \cup_{S\in Y} F(S)$. It is enough to show $\cup_{S\in Y} F(S) \subseteq Y$. To show this, we note that for every element S of Y, $F(S) = Cl(\{S\}) \subseteq Cl(Y) = Y$, by Corollary 2.2(1). Hence, $\cup_{S\in Y} F(S) \subseteq Y$. Therefore, $Y = \cup_{S\in Y} F(S)$. □

Lemma 2.4. *Let M be a lattice module over a $C-$lattice L. If M is a cotop lattice module and Y is a subset of $Spec^s(M)$, then $Cl(Y) = F(Z(Y))$.*

Proof. Suppose that M is a cotop lattice module and $Y \subseteq Spec^s(M)$. Then each closed subset is of the form of $F(N)$ for some $N \in M$. Since for each $S \in Y$, $S \leq Z(Y)$, we have, $Y \subseteq F(Z(Y))$. Now, let $F(N)$ be any closed subset of $Spec^s(M)$ containing Y. Then for each $S \in Y$, we have $S \in F(N)$, so that $S \leq N$. Hence, $Z(Y) \leq N$. Thus, if $S \in F(Z(Y))$, then $S \leq Z(Y) \leq N$. Hence $S \in F(N)$ and $F(Z(Y)) \subseteq F(N)$. Thus $F(Z(Y))$ is the smallest closed subset of $Spec^s(M)$ which contains Y. This shows that $Cl(Y) = F(Z(Y))$. □

Lemma 2.5. *Let M be a lattice module over a $C-$lattice L. Then for each $S \in Spec^s(M)$, $F(S)$ is irreducible.*

Proof. Let $F(S) \subseteq X_1 \cup X_2$, where X_1 and X_2 are closed subsets of $Spec^s(M)$. Since $S \in F(S)$ and $F(S) \subseteq X_1 \cup X_2$, we have, $S \in X_1 \cup X_2$, which implies that either $S \in X_1$ or $S \in X_2$. Suppose that $S \in X_1$. Since X_1 is closed in $Spec^s(M)$, we have, $X_1 = \cap_{k\in J}(\cup_{l=1}^{n_k} F(N_{kl}))$, for some $N_{kl} \in M$, $k \in J$, $n_k \in \mathbb{N}$. Thus $S \in \cup_{l=1}^{n_k} F(N_{kl})$ for each $k \in J$. It follows that $F(S) \subseteq \cup_{l=1}^{n_k} F(N_{kl})$ for each $k \in J$. Hence, $F(S) \subseteq \cap_{k\in J}(\cup_{l=1}^{n_k} F(N_{kl})) = X_1$. Consequently, $F(S)$ is irreducible. □

Theorem 2.6. *Let M be a lattice module over a $C-$lattice L and Y be a subset of $Spec^s(M)$. If $Z(Y)$ is a second element and $Z(Y) \in Cl(Y)$, then Y is irreducible.*

Proof. Suppose that $Z(Y)$ is a second element of M and $Z(Y) \in Cl(Y)$. Since for each $S \in Y$, $S \leq Z(Y)$, we have, $F(S) \subseteq F(Z(Y))$ for each $S \in Y$ by Corollary 2.2(2). Therefore, $\cup_{S\in Y} F(S) \subseteq F(Z(Y))$, that is, $Cl(Y) \subseteq F(Z(Y))$, by Lemma 2.1. Now, since $Z(Y)$ is a second element and $Z(Y) \in Cl(Y), F(Z(Y)) \subseteq Cl(Y)$. Consequently, $Cl(Y) = F(Z(Y))$. Now, let $Y \subseteq X_1 \cup X_2$, where X_1 and X_2 are closed subsets of $Spec^s(M)$. Then we have, $F(Z(Y)) = Cl(Y) \subseteq X_1 \cup X_2$. Since $F(S)$ is irreducible for each $S \in Spec^s(M)$, by Lemma 2.5, $F(Z(Y))$ is irreducible. Therefore, $F(Z(Y)) \subseteq X_1$ or $F(Z(Y)) \subseteq X_2$. Hence, $Y \subseteq X_1$ or $Y \subseteq X_2$, consequently, Y is irreducible. □

Definition 2.7. [16] Let M be a lattice module over a $C-$lattice L and N be an element of M. Then the *second radical* of N is defined to be the join of all second elements contained in N, that is, $\sqrt[s]{N} = \vee\{S \in Spec^s(M) | S \leq N\}$.
Note that, $\sqrt[s]{N} = 0_M$, if there is no second element contained in N. If $N = \sqrt[s]{N}$, then N is called a *second radical element*.

Corollary 2.8. *Let M be a lattice module over a $C-$lattice L and N be an element of M. If $\sqrt[s]{N}$ is a second element of M, then the subset $F(N)$ of $Spec^s(M)$ is irreducible.*

Proof. Suppose that $\sqrt[s]{N}$ is a second element of M. By Lemma 2.5, $F(S)$ is irreducible for each $S \in Spec^s(M)$, therefore $F(\sqrt[s]{N})$ is irreducible subset of $Spec^s(M)$. Clearly, for each N in M, $F(N) = F(\sqrt[s]{N})$. Hence, the subset $F(N)$ of $Spec^s(M)$ is irreducible. □

The following Lemma shows that for any lattice module M over a $C-$lattice L, $Spec^s(M)$ is always a T_0-space.

Lemma 2.9. *Let M be a lattice module over a $C-$lattice L. Then the following hold:*

1. *$Spec^s(M)$ is a T_0-space.*
2. *Every $S \in Spec^s(M)$ is a generic point of the irreducible closed subset $F(S)$.*
3. *Every finite irreducible closed subset of $Spec^s(M)$ has a generic point.*

Proof. (1) Suppose that $S, S_1 \in Spec^s(M)$. Then by Corollary 2.2(1), $Cl(\{S\}) = F(S)$, $Cl(\{S_1\}) = F(S_1)$ and therefore $Cl(\{S\}) = Cl(\{S_1\})$ if and only if $F(S) = F(S_1)$ if and only if $S = S_1$ by Corollary 2.2(2). Now, by the fact that a topological space is a T_0-space if the closures of distinct points are distinct, we conclude that $Spec^s(M)$ is a T_0-space.
(2) By Corollary 2.2(1), for each $S \in Spec^s(M)$, $F(S) = Cl(\{S\})$. Hence, S is a generic point of the irreducible closed subset $F(S)$.
(3) Suppose that Y is an irreducible closed subset of $Spec^s(M)$ and $Y = \{S_1, S_2, \cdots, S_n\}$, where $S_i \in Spec^s(M)$, $n \in \mathbb{N}$. By Lemma 2.1, $Y = Cl(Y) = F(S_1) \cup F(S_2) \cup \cdots \cup F(S_n)$. Since Y is irreducible, $Y = F(S_i)$, for some i $(1 \leq i \leq n)$. Hence, by part (2), S_i is a generic point of $F(S_i) = Y$. □

A topological space X is a spectral space if X is homeomorphic to $Spec(S)$, with Zariski topology, for some commutative ring S. Spectral spaces have been characterized by Hochster (see [13]) as the topological spaces X which satisfy the following conditions.

1. X is a T_0-space.
2. X is a quasi-compact.
3. The quasi-compact open subsets of X are closed under finite intersection and form an open basis.
4. Each irreducible closed subset of X has a generic point.

Theorem 2.10. *Let M be a lattice module over a $C-$lattice L with finite second spectrum. Then $Spec^s(M)$ is a spectral space(with second classical Zariski topology).*

Proof. Since $Spec^s(M)$ is finite, by Lemma 2.9, $Spec^s(M)$ is a T_0-space and every irreducible closed subset of $Spec^s(M)$ has a generic point. Also, since $Spec^s(M)$ is finite, every subset of $Spec^s(M)$ is quasi-compact and the quasi-compact open subsets of $Spec^s(M)$ are closed under finite intersections(see [9]). Further $\mathbb{B} = \{G(N_1) \cap G(N_2) \cap \cdots \cap G(N_n) | N_i \in M, 1 \leq i \leq n$ for some $n \in \mathbb{N}\}$ is basis for $Spec^s(M)$ with the property that each basis element, in particular $G(0_M) = Spec^s(M)$, is quasi-compact. Now, by Hochster's characterization of a spectral space, we conclude that $Spec^s(M)$ is a spectral space. □

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