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THE PROPERTIES OF RESIDUATED CONNECTIONS AND ALEXANDROV TOPOLOGIES

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Abstract. In this paper, we investigate the properties of residuated connections and Alexandrov topologies based on $[0,\infty]$. Under various relations, we investigate the residuated and dual residuated connections on Alexandrov toplogies. Moreover, we study their properties and give their examples.

1. Introduction

Blyth and Janovitz [2] introduced the residuated connection as a pair of maps on partially ordered sets. Recently, Orłowska and Rewitzky [7,8] investigated various residuated connections from the viewpoint of many valued logics and rough sets.

Pawlak [9,10] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. Ward et al.[13] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool as algebraic structures for many valued logics [1,3-6,11,12].

For an extension of Pawlak's rough sets, many researchers developed L-lower and L-upper approximation operators in complete residuated

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lattices [1,3-6,11,12]. Using this concepts, information systems and decision rules were investigated in complete residuated lattices [1,11,12].

An interesting and natural research topic in rough set theory is the study of rough set theory and topological structures. Lai [5] and Ma [6] investigated the Alexandrov L-topology and lattice structures of L-fuzzy rough sets determined by lower and upper sets. Kim [3,4] introduce the notion of Alexandrov topologies as a topological viewpoint of fuzzy rough sets and studied the relations among fuzzy preorders, L-lower and L-upper approximation operators and Alexandrov topologies in complete residuated lattices.

In this paper, we introduced the residuated and dual-residuated connection as maps from a non-symmetric pseudo-metric to another non-symmetric pseudo-metric. We investigate the notion of residuated and dual residuated connection on Alexandrov topologies. Under various relations, we study their properties and give their examples.

2. Preliminaries

Let $([0,\infty], \leq, \vee, +, \wedge, \to, \infty, 0)$ be a structure where

$$x \to y = \bigwedge\{z \in [0,\infty] \mid z + x \geq y\} = (y - x) \vee 0,$$

$$\infty + a = a + \infty = \infty, \forall a \in [0,\infty], \infty \to \infty = 0.$$

Definition 2.1. Let X be a set. A function $d_X : X \times X \to [0,\infty]$ is called a *non-symmetric pseudo-metric* if it satisfies the following conditions:

(M1) $d_X(x,x) = 0$ for all $x \in X$,

(M2) $d_X(x,y) + d_X(y,z) \geq d_X(x,z)$, for all $x,y,z \in X$.

The pair (X, d_X) is called a *non-symmetric pseudo-metric space.*

Remark 2.2. (1) We define a function $d_{[0,\infty]^X} : [0,\infty]^X \times [0,\infty]^X \to [0,\infty]$ as $d_{[0,\infty]^X}(A,B) = \bigvee_{x\in X}(A(x) \to B(x)) = \bigvee_{x\in X}((B(x) - A(x)) \vee 0)$. Then $([0,\infty]^X, d_{[0,\infty]^X})$ is a non-symmetric pseudo-metric space.

(2) If (X, d_X) is a non-symmetric pseudo-metric space and we define a function $d_X^{-1}(x,y) = d_X(y,x)$, then (X, d_X^{-1}) is a non-symmetric pseudo-metric space.

(3) Let (X, d_X) be a non-symmetric pseudo-metric space and define $(d_X \oplus d_X)(x,z) = \bigwedge_{y\in X}(d_X(x,y) + d_X(y,z))$ for each $x,z \in X$. By (M2),

$(d_X \oplus d_X)(x,z) \geq d_X(x,z)$ and $(d_X \oplus d_X)(x,z) \leq d_X(x,x) + d_X(x,z) = d(x,z)$. Hence $(d_X \oplus d_X) = d_X$.

(4) If d_X is a non-symmetric pseudo-metric and $d_X(x,y) = d_X(y,x)$ for each $x, y \in X$, then d_X is a pseudo-metric

Example 2.3. (1) Let $X = \{a,b,c\}$ be a set and define maps $d_X^i : X \times X \to [0,\infty]$ for $i = 1,2,3$ as follows:

$$d_X^1 = \begin{pmatrix} 0 & 6 & 5 \\ 6 & 0 & 1 \\ 15 & 7 & 0 \end{pmatrix} d_X^2 = \begin{pmatrix} 0 & 6 & 3 \\ 7 & 0 & 4 \\ 0 & 5 & 0 \end{pmatrix} d_X^3 = \begin{pmatrix} 0 & 3 & 7 \\ 6 & 0 & 9 \\ 5 & 4 & 0 \end{pmatrix}.$$

Since $d_X^1(c,b) + d_X^1(b,a) = 13 < d_X^1(c,a) = 15$ and $d_X^2(b,c) + d_X^2(c,a) = 4 < d_X^2(b,a) = 15$, d_X^1 and d_X^2 are not non-symmetric pseudo-metrics. Since d_X^3 is a non-symmetric pseudo-metric, $d_X^3 \oplus d_X^3 = d_X^3$.

3. The properties of residuated connections and Alexandrov topologies

Definition 3.1. Let (X, d_X) and (Y, d_Y) be non-symmetric pseudo-metric spaces and $f : X \to Y$ and $g : Y \to X$ maps.

(1) (d_X, f, g, d_Y) is called a *residuated connection* if for all $x \in X, y \in Y$, $d_Y(f(x), y) = d_X(x, g(y))$.

(2) (d_X, f, g, d_Y) is called a *dual residuated connection* if for all $x \in X, y \in Y$, $d_Y(y, f(x)) = d_X(g(y), x)$.

Remark 3.2. Let (X, d_X) be a non-symmetric pseudo-metric space. For $A, B \in [0,\infty]^X$,

$$F(A)(y) = \bigwedge_{x \in X} (d_X(x,y) + A(x)),\ G(B)(x) = \bigvee_{y \in X} \Big(d_X(x,y) \to B(y)).$$

Then $(d_{[0,\infty]^X}, F, G, d_{[0,\infty]^X})$ is a residuated connection because for all $A, B \subset X$,

$$\begin{aligned}
& d_{[0,\infty]^Y}(F(A), B) = \bigvee_{y\in X}(F(A)(y) \to B(y)) \\
& = \bigvee_{y\in X}\Big(\bigwedge_{x\in X}(d_X(x,y) + A(x)) \to B(y)\Big) \\
& = \bigvee_{y\in X}\Big((B(y) - \bigwedge_{x\in X}(d_X(x,y) + A(x))) \vee 0\Big) \\
& = \bigvee_{x\in X}\Big((\bigvee_{y\in X}(B(y) - d_X(x,y)) \vee 0) - A(x))) \vee 0\Big) \\
& = \bigwedge_{x\in X}(A(x) \to G(B)(x)) = d_{[0,\infty]^X}(A, G(B)).
\end{aligned}$$

THEOREM 3.3. *Let (X, d_X) and (Y, d_Y) be non-symmetric pseudo-metric spaces and $f : X \to Y$ and $g : Y \to X$ maps.*

(1) (d_X, f, g, d_Y) is a residuated connection iff $d_Y(f(x), f(z)) \le d_X(x,z)$ for all $x, z \in X$, $d_X(g(y), g(w)) \le d_Y(y,w)$ for all $y, w \in Y$, and $d_Y(f(g(y)), y) = d_X(x, g(f(x))) = 0$.

(2) (d_X, f, g, d_Y) is a dual residuated connection iff $d_Y(f(x), f(z)) \le d_X(x,z)$ for all $x, z \in X$, $d_X(g(y), g(w)) \le d_Y(y,w)$ for all $y, w \in Y$, and $d_Y(y, f(g(y))) = d_X(g(f(x)), x) = 0$.

Proof. (1) Let (d_X, f, g, d_Y) be a residuated connection. Since $d_Y(f(x), y) = d_X(x, g(y))$, we have $0 = d_Y(f(x), f(x)) = d_X(x, g(f(x)))$ and $d_Y(f(g(y)), y) = d_X(g(y), g(y)) = 0$. Furthermore,

$$\begin{aligned}
& d_Y(f(x_1), f(x_2)) = d_X(x_1, g(f(x_2))) \\
& \le d_X(x_1, x_2) + d_X(x_2, g(f(x_2))) = d_X(x_1, x_2), \\
& d_X(g(y_1), g(y_2)) = d_Y(f(g(y_1)), y_2) \\
& \le d_Y(f(g(y_1)), y_1) + d_Y(y_1, y_2) = d_Y(y_1, y_2).
\end{aligned}$$

Conversely, $d_Y(f(x), y) \le d_Y(f(g(y)), y) + d_Y(f(x), f(g(y))) = d_Y(f(x), f(g(y))) \le d_X(x, g(y))$. Similarly, $d_Y(f(x), y) \ge d_X(x, g(y))$.

(2) It is similarly proved as (1). □

EXAMPLE 3.4.(1) Let $(X = \{a, b, c\}, d_i)$, $i = 1, 2, 3$, be a non-symmetric pseudo-metric space as follows:

$$d_1 = \begin{pmatrix} 0 & 6 & 5 \\ 6 & 0 & 5 \\ 7 & 7 & 0 \end{pmatrix} d_2 = \begin{pmatrix} 0 & 6 & 5 \\ 6 & 0 & 7 \\ 7 & 5 & 0 \end{pmatrix} d_3 = \begin{pmatrix} 0 & 10 & 6 \\ 7 & 0 & 6 \\ 6 & 6 & 0 \end{pmatrix}$$

(1) Let $f : X \to X$ be a function as $f(a) = b, f(b) = a, f(c) = c$. Since $d_1(x,y) = d_1(f(x), f(y))$, $d_1(x, f(f(x))) = d_1(f(f(x)), x) = 0$,

by Theorem 3.3, (d_1, f, f, d_1) are both residuated and dual residuated connections.

(2) Since $7 = d_2(c, a) \geq d_2(f(c), f(a)) = d_2(c, b) = 5$ and $5 = d_2(c, b) \not\geq d_2(f(c), f(b)) = d_2(c, a) = 7$, (d_2, f, f, d_2) are neither residuated nor dual residuated connections.

(3) Let $g, h : X \to X$ a function as $g(a) = g(b) = a, g(c) = c$ and $h(a) = h(b) = b, h(c) = c$. Since $d_3(x, y) \geq d_3(g(x), g(y))$, $d_3(x, y) \geq d_3(h(x), h(y))$, $g(h(a)) = g(h(b)) = a, g(h(c)) = c$, $h(g(a)) = h(g(b)) = b, g(h(c)) = c$, then $d_X(g(h(b)), b) = d_X(a, b) = 10 = d_X(a, h(g(a)))$, $d_X(h(g(a)), a) = d_X(b, g(h(b))) = d_X(b, a) = 7$. Hence (d_3, g, h, d_3) are neither a residuated connection nor a dual residuated connection.

We redefine the following definition as a sense in [3-6].

Definition 3.5.A subset $\tau_X \subset [0, \infty]^X$ is called an *Alexandrov topology* on X iff it satisfies the following conditions:

(AT1) $\alpha_X \in \tau_X$ where $\alpha_X(x) = \alpha$ for each $x \in X$ and $\alpha \in [0, \infty]$.

(AT2) If $A_i \in \tau_X$ for all $i \in I$, then $\bigvee_{i \in I} A_i, \bigwedge_{i \in I} A_i \in \tau_X$.

(AT3) If $A \in \tau_X$ and $\alpha \in [0, \infty]$, then $\alpha + A, \alpha \to A \in \tau_X$ where $(\alpha \to A)(x) = (A(x) - \alpha) \vee 0$.

The pair (X, τ_X) is called an *Alexandrov topological space.*

Theorem 3.6. *Let $\tau_X \subset [0, \infty]^X$ be an Alexandrov topology. Define $d_{\tau_X} : \tau_X \times \tau_X \to L$ as $d_{\tau_X}(A, B) = \bigvee_{x \in X}(A(x) \to B(x)) = \bigvee_{x \in X}((B(x) - A(x)) \vee 0)$. Then the followings hold.*

(1) (τ_X, d_{τ_X}) is a non-symmetric pseudo-metric space.

(2) If $d_{\tau_X}(A, C) = d_{\tau_X}(B, C)$ for all $C \in \tau_X$, then $A = B$.

Proof. (1) (M1) $d_{\tau_X}(A, A) = \bigvee_{x \in X}(A(x) \to A(x)) = 0$ for all $A \in \tau_X$,

(M2) Since $d_{\tau_X}(A, B) + d_{\tau_X}(B, C) = \bigvee_{x \in X}(A(x) \to B(x)) + \bigvee_{x \in X}(B(x) \to C(x)) \geq \bigvee_{x \in X}((B(x) - A(x)) \vee 0) + (C(x) - B(x)) \vee 0 \geq \bigvee_{x \in X}((C(x) - A(x)) \vee 0) = d_{\tau_X}(A, C)$, for all $A, B, C \in \tau_X$,

(2) Since $d_{\tau_X}(A, B) = d_{\tau_X}(B, B) = 0 = d_{\tau_X}(A, A) = d_{\tau_X}(B, A)$, $A = B$. □

Theorem 3.7. *Let (X, d_X) be a non-symmetric pseudo-metric. Define*

$$\tau_{d_X} = \{A \in [0, \infty]^X \mid A(x) + d_X(x, z) \geq A(z)\}.$$

Then the followings hold.

(1) τ_{d_X} is an Alexandrov topology on X.

(2)*If* $(d_X)_x = d_X(x,-) \in [0,\infty]^X$ *and* $((d_X)_x^{-1} \to \alpha)(z) = (d_X)_x^{-1}(z) \to \alpha = d_X(z,x) \to \alpha$ *, then* $(d_X)_x \in \tau_{d_X}$ *and* $(d_X)_x^{-1} \to \alpha \in \tau_{d_X}$*. Moreover,* $\bigvee_{y\in X}(d_X(-,y) \to B(y)) \in \tau_{d_X}$ *and* $\bigwedge_{y\in X}(B(x) + d_X(x,-)) \in \tau_{d_X}$*.*

Proof. (1) Since $\alpha_X(x) + d_X(x,y) \geq \alpha_X(y)$, we have $\alpha_X \in \tau_{d_X}$.

If $A_i \in \tau_{d_X}$ for all $i \in I$, then

$$(\bigwedge_{i\in I} A_i) + d_X(x,y) = \bigwedge_{i\in I}(A_i + d_X(x,y)) \geq \bigwedge_{i\in I} A_i,$$
$$(\bigvee_{i\in I} A_i) + d_X(x,y) = \bigvee_{i\in I}(A_i + d_X(x,y)) \geq \bigvee_{i\in I} A_i,$$

then $\bigwedge_{i\in I} A_i, \bigvee_{i\in I} A_i \in \tau_{d_X}$.

If $A \in \tau_{d_X}$ and $\alpha \in L$, then $\alpha + (\alpha \to A(x)) + d_X(x,y) \geq A(x) + d_X(x,y) \geq A(y)$ implies $(\alpha \to A(x)) + d_X(x,y) \geq (\alpha \to A(y))$. So, $\alpha \to A \in \tau_{d_X}$. Easily, $\alpha + A \in \tau_{d_X}$. Hence τ_{d_X} is an Alexandrov topology on X.

(2) Since $(d_X)_x(y) + d_X(y,z) \leq (d_X)_x(z)$, $(d_X)_x \in \tau_{d_X}$. Moreover, $(d_X)_x^{-1} \to \alpha) \in \tau_{d_X}$ from

$$\begin{aligned}
&(d_X(z,x) \to \alpha) + d_X(z,w) + d_X(w,x)\\
&\geq (\alpha - d_X(z,x)) \vee 0 + d_X(z,x) \geq \alpha,\\
&(\Rightarrow)(d_X(z,x) \to \alpha) + d_X(z,w) \geq (\alpha - d_X(w,x)) \vee 0\\
&(\Rightarrow)(d_x^{-1}(z) \to \alpha) + d_X(z,w) \geq d_x^{-1}(w) \to \alpha
\end{aligned}$$

By (1), $\bigwedge_{x\in X}(d_X(x,-) + A(x)) \in \tau_{d_X}$ and $\bigvee_{x\in X}(d_X(-,x) \to A(x)) \in \tau_{d_X}$. □

Theorem 3.8. *Let* (X, d_X) *and* (Y, d_Y) *be non-symmetric pseudo-metric spaces and* $f : X \to Y$ *be a map such that* $d_Y(f(x), f(y)) \leq d_X(x,y)$ *for all* $x, y \in X$*. Then the followings hold.*

(1) *A map* $f : (X, \tau_{d_X}) \to (Y, \tau_{d_Y})$ *is continuous, that is,* $f^{\leftarrow}(B) \in \tau_{d_X}$ *for each* $B \in \tau_{d_Y}$*.*

(2) *For each* $B \in [0,\infty]^Y$, $f^{\leftarrow}(F_2(B)) \leq F_1(f^{\leftarrow}(B))$ *where*

$$F_1(A)(z) = \bigwedge_{x\in X}(A(x) + d_X(x,z)),\ F_2(B)(y) = \bigwedge_{w\in Y}(B(w) + d_Y(w,y)).$$

(3) *For each* $B \in [0,\infty]^Y$, $G_1(f^{\leftarrow}(B)) \leq f^{\leftarrow}(G_2(B))$ *where*

$$G_1(A)(z) = \bigvee_{x\in X}(d_X(z,x) \to A(x)), G_2(B)(y) = \bigvee_{w\in Y}(d_Y(y,w) \to B(w)).$$

Proof. (1) For each $B \in \tau_{d_Y}$, $f^{\leftarrow}(B) \in \tau_{d_X}$ from

$$f^{\leftarrow}(B)(x) + d_X(x,z) = B(f(x)) + d_X(x,z)$$
$$\geq B(f(x)) + d_Y(f(x), f(z)) \geq B(f(z)) = f^{\leftarrow}(B)(z).$$

(2) For each $B \in [0,\infty]^Y$,

$$f^{\leftarrow}(F_2(B))(x) = F_2(B)(f(x)) = \bigwedge_{y\in X}(B(y) + d_Y(y, f(x)))$$
$$\leq \bigwedge_{z\in X}(B(f(z)) + d_Y(f(z), f(x)) \leq \bigwedge_{z\in X}(f^{\leftarrow}(B)(z) + d_X(z,x))$$
$$= F_1(f^{\leftarrow}(B))(x).$$

(3) For each $B \in [0,\infty]^Y$,

$$f^{\leftarrow}(G_2(B))(x) = G_2(B)(f(x)) = \bigvee_{y\in X}(d_Y(f(x), y) \to B(y))$$
$$\geq \bigvee_{y\in X}(d_Y(f(x), y) \to B(y)) \geq \bigvee_{z\in X}(d_Y(f(x), f(z)) \to B(f(z)))$$
$$\geq \bigvee_{z\in X}(d_X(x,z) \to B(f(z))) = G_1(f^{\leftarrow}(B))(x).$$

□

THEOREM 3.9. *Let (X, d_X) and (Y, d_Y) be non-symmetric pseudo-metrics and $f : X \to Y$ and $g : Y \to X$ maps. Then the following statements hold:*

(1) *(d_X, f, g, d_Y) is a residuated connection iff $d_X(x_1, x_2) \geq d_Y(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$ and $(d_{\tau_{d_X}}, F_1, G_1, d_{\tau_{d_Y}})$ is a residuated connection where*

$$F_1(A)(y) = \bigwedge_{y\in Y}(d_Y(f(x), y) + A(x)), \;\; G_1(B)(x) = \bigvee_{x\in X}(d_X(x, g(y)) \to B(y)).$$

(2) *(d_X, f, g, d_Y) is a dual residuated connection iff $d_Y(y_1, y_2) \geq d_X(g(y_1), g(y_2))$ for all $y_1, y_2 \in Y$ and $d_{\tau_{d_Y}}(B, F_2(A)) = d_{\tau_{d_X}}(G_2(B), A)$ where*

$$F_2(A)(y) = \bigvee_{x\in X}(d_Y(y, f(x)) \to A(x)), \;\; G_2(B)(x) = \bigwedge_{y\in Y}(d_X(g(y), x) + B(y)).$$

Proof. (1) Let $d_X(x, g(y)) = d_Y(f(x), y)$. Since $d_Y(f(x), y) + d_Y(y, w) \geq d_Y(f(x), w)$, $(d_Y)_{f(x)} \in \tau_{d_Y}$. Thus $F_1(A) = \bigwedge_{y\in Y}(d_Y(f(x), -) + A(x)) \in \tau_{d_Y}$. Since $(d_X(x, g(y)) \to B(y)) + d_X(x,z) + d_X(z, g(y)) \geq B(y)$,

$G(B) \in \tau_{d_X}$. Moreover,

$$
\begin{aligned}
& d_{\tau_{d_Y}}(F_1(A), B) = \bigvee_{y\in X}(F_1(A)(y) \to B(y)) \\
& = \bigvee_{y\in Y} \Big(\bigwedge_{x\in X}(d_Y(f(x), y) + A(x)) \to B(y)\Big) \\
& = \bigvee_{y\in Y} \bigvee_{x\in X} \Big((B(y) - d_Y(f(x), y) - A(x)) \vee 0\Big) \\
& = \bigvee_{y\in Y} \bigvee_{x\in X} \Big(((B(y) - d_Y(f(x), y)) \vee 0) - A(x)) \vee 0\Big) \\
& = \bigvee_{x\in X} \bigvee_{y\in Y} \Big(A(x) \to (d_X(x, g(y)) \to B(y))\Big) \\
& = \bigvee_{x\in X} \Big(A(x) \to \bigvee_{y\in Y}(d_X(x, g(y)) \to B(y))\Big) \\
& = \bigvee_{x\in X} \Big(A(x) \to G_1(B)(x)\Big) = d_{\tau_{d_X}}(A, G_1(B)).
\end{aligned}
$$

Conversely, since $F_1((d_X)_x)(y) = \bigwedge_{z\in X}(d_Y(f(z), y) + (d_X)_x(z)) \leq d_Y(f(x), y)$ and $d_Y(f(z), y) + d_X(x, z) \geq d_Y(f(z), y) + d_Y(f(x), f(z)) \geq d_Y(f(x), y)$, $F_1((d_X)_x)(y) = d_Y(f(x), y)$, that is, $F_1((d_X)_x) = (d_Y)_{f(x)}$.

$$
\begin{aligned}
& d_{\tau_{d_Y}}((d_Y)_{f(x)}, B) = d_{\tau_{d_Y}}(F_1((d_X)_x), B) = d_{\tau_{d_X}}((d_X)_x, G_1(B)) \\
& = \bigvee_{z\in X} \bigvee_{y\in Y} \Big((d_X)_x(z) \to (d_X(z, g(y)) \to B(y))\Big) \\
& = \bigvee_{z\in X} \bigvee_{y\in Y} \Big(((B(y) - d(f(x), y)) \vee 0 - d_X(x, z)) \vee 0\Big) \\
& = \bigvee_{y\in Y}(\bigwedge_{z\in X} \Big(d_X(x, z) + d_X(z, g(y))) \to B(y))\Big) \\
& = \bigvee_{y\in Y}(d_X(x, g(y))) \to B(y)) = d_{\tau_{d_Y}}(g^{\leftarrow}((d_X)_x, B).
\end{aligned}
$$

Since $d_{\tau_{d_Y}}((d_Y)_{f(x)}, B) = d_{\tau_{d_Y}}(g^{\leftarrow}((d_X)_x, B)$ for all $B \in \tau_{d_Y}$, by Theorem 3.6(2), $(d_Y)_{f(x)}(y) = d_Y(f(x), y) = g^{\leftarrow}((d_X)_x)(y) = d_X(x, g(y))$ for all $x \in X, y \in Y$.

(2) Let $d_Y(y, f(x)) = d_X(g(y), x)$. Since $d_X(g(y), x) + d_X(x, z) \geq d_X(g(y), z)$, $(d_X)_{g(y)} \in \tau_{d_X}$. Thus $G(B) \in \tau_{d_X}$. Since $(d_Y(y, f(x)) \to A(x)) + d_Y(y, w) + d_Y(w, f(x)) \geq A(x)$, $F(A) \in \tau_{d_Y}$. Thus,

$$
\begin{aligned}
& d_{\tau_{d_X}}(G_1(B), A) = \bigvee_{x\in X}(G_2(B)(x) \to A(x)) \\
& = \bigvee_{x\in X} \Big(\bigvee_{y\in Y}(d_X(g(y), x) + B(y)) \to A(x)\Big) \\
& = \bigvee_{Y\in Y} \bigvee_{x\in X} \Big(B(y) \to (d_Y(y, f(x)) \to A(x))\Big) \\
& = \bigvee_{y\in Y} \Big(B(y) \to \bigvee_{x\in X}(d_Y(y, f(x)) \to A(x))\Big) \\
& = \bigvee_{y\in Y} \Big(B(y) \to F_2(A)(y)\Big) = d_{\tau_{d_Y}}(B, F_2(A))
\end{aligned}
$$

Conversely, since $G_2((d_Y)_y)(x) = \bigwedge_{w\in X}(d_X(g(w), x) + (d_Y)_y(w)) \leq d_X(g(y), x)$ and $d_X(g(w), x) + d_Y(y, w) \geq d_X(g(w), x) + d_X(g(y), g(w)) \leq$

$d_X(g(y),x)$, $G_2((d_Y)_y)(x) = d_X(g(y),x)$.

$$\begin{aligned}
& d_{\tau_{d_X}}((d_X)_{g(y)}, A) = d_{\tau_{d_X}}(G_2((d_Y)_y), A) = d_{\tau_{d_Y}}((d_Y)_y, F_2(A)) \\
& = \bigvee_{w\in Y}\bigvee_{x\in X}\Big((d_Y)_y(w) \to (d_Y(w, f(x)) \to A(x))\Big) \\
& = \bigvee_{x\in X}(\bigwedge_{w\in Y}\Big(d_Y(y,w) + d_X(w, f(x))) \to A(x))\Big) \\
& = \bigvee_{x\in X}(d_Y(y, f(x))) \to A(x)) = d_{\tau_{d_X}}(f^{\leftarrow}((d_Y)_y, A).
\end{aligned}$$

Since $d_{\tau_{d_X}}((d_X)_{g(y)}, A) = d_{\tau_{d_X}}(f^{\leftarrow}((d_Y)_y, A)$ for all $A \in \tau_{d_X}$, by Theorem 3.6(2), $(d_X)_{g(y)}(x) = d_X(g(y),x) = f^{\leftarrow}((d_Y)_y)(x) = d_Y(y, f(x))$. for all $x \in X, y \in Y$. □

THEOREM 3.10. *Let (X, d_X) and (Y, d_Y) be non-symmetric pseudo-metric spaces and $f : X \to Y$ with $d_Y(f(x), f(y)) \le d_X(x,y)$ for all $x, y \in X$. If $F : \tau_{d_X} \to \tau_{d_Y}$ is a function with $F((d_X)_x)(y) = (d_Y)_{f(x)}(y)$ such that $F(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} F(A_i)$ and $F(\alpha+A) = \alpha+F(A)$, then there exists $G : \tau_{d_Y} \to \tau_{d_X}$ with $G(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} G(A_i)$ and $G(\alpha \to A) = \alpha \to G(A)$. Moreover, $(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$ is a residuated connection.*

Proof. Let $F : \tau_{d_X} \to \tau_{d_Y}$ be a function with $F((d_X)_x)(y) = (d_Y)_{f(x)}(y)$ such that $F(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} F(A_i)$ and $F(\alpha + A) = \alpha + F(A)$. Since $(d_Y)_{f(x)}(y) + d_Y(y,w) \ge (d_Y)_{f(x)}(w)$, $F((d_X)_x) \in \tau_{d_Y}$. Moreover, $F(A)(y)$ $= F(\bigwedge_{x\in X}(A(x) + (d_X)_x))(y) = \bigwedge_{x\in X}(A(x) + F((d_X)_x)(y)) = \bigwedge(A(x) + d_Y(f(x),y))$ and $F(A) \in \tau_{d_Y}$. Hence F is well defined. Define $G : \tau_{d_Y} \to \tau_{d_X}$ as

$$G(B)(x) = \bigwedge\{A(x) \mid F(A) \ge B\} = \bigvee(d_Y(f(x),y) \to B(y)).$$

Since $G(B)(x) + d_X(x,z) + d_Y(f(z),y) \ge G(B)(x) + d_Y(f(x), f(z)) + d_Y(f(z),y) \ge G(B)(x) + d_Y(f(x),y) \ge B(y)$, $G(B) \in \tau_{d_X}$. Moreover, $(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$ is a residuated connection. □

EXAMPLE 3.11.(1) Let $(X = \{a,b,c\}, d_1)$ be a non-symmetric pseudo-metric and $f : X \to X$ a function in Example 3.4(1). Then $d_1(x,y) = d_1(f(x), f(y))$ and $d_1(f(x),y) = d_1(x, f(y))$ for all $x, y \in X$. Let $F_1, G_1 : \tau_{d_1} \to \tau_{d_1}$ be functions with $F_1(A)(y) = \bigwedge(A(x) + d_1(f(x),y))$ and $G_1(B)(x) = \bigvee_{y\in X}(d_1(f(x),y) \to B(y)) = \bigvee_{y\in X}(d_1(x, f(y)) \to B(y))$. By Theorems 3.9(1) and 3.10, $(d_{\tau_{d_1}}, F_1, G_1, d_{\tau_{d_1}})$ is a residuated connection.

Let $F_2, G_2 : \tau_{d_1} \to \tau_{d_1}$ be functions with $F_2(A)(y) = \bigvee_{x\in X}(d_1(y, f(x)) \to A(x))$ and $G_2(B)(x) = \bigwedge_{y\in X}(B(y)+d_1(f(y), x))$. By Theorem 3.9(2), $(d_{\tau_{d_1}}, F_2, G_2, d_{\tau_{d_1}})$ is a dual residuated connection.

(2) Let $(X = \{a, b, c\}, d_3)$ be a non-symmetric pseudo-metric and $g, h : X \to X$ functions in Example 3.4(3). Then $d_3(x, y) \geq d_3(g(x), g(y))$, $d_3(x, y) \geq d_3(h(x), h(y))$. Let $F_3, G_3 : \tau_{d_3} \to \tau_{d_3}$ be functions with $F_3(A)(y) = \bigwedge(A(x) + d_3(g(x), y))$ and $G_3(B)(x) = \bigvee_{y\in X}(d_3(g(x), y) \to B(y))$. By Theorem 3.10, $(d_{\tau_{d_3}}, F_3, G_3, d_{\tau_{d_3}})$ is a residuated connection.

Let $F_4, G_4 : \tau_{d_3} \to \tau_{d_3}$ be a function with $F_4(A)(y) = \bigwedge(A(x) + d_3(h(x), y))$ and $G_4(B)(x) = \bigvee_{y\in X}(d_3(h(x), y) \to B(y))$. By Theorem 3.10, $(d_{\tau_{d_3}}, F_4, G_4, d_{\tau_{d_3}})$ is a residuated connection.

References

[1] R. Bělohlávek, *Fuzzy Relational Systems*, Kluwer Academic Publishers, New York, 2002.

[2] T.S. Blyth, M.F. Janovitz, *Residuation Theory*, Pergamon Press, New York, 1972.

[3] Y.C. Kim, *Join-meet preserving maps and fuzzy preorders*, Journal of Intelligent & Fuzzy Systems **28**(2015), 1089–1097.

[4] Y.C. Kim, *Categories of fuzzy preorders, approximation operators and Alexandrov topologies*, Journal of Intelligent & Fuzzy Systems **31** (2016), 1787–1793.

[5] H. Lai, D. Zhang, *Fuzzy preorder and fuzzy topology*, Fuzzy Sets and Systems **157** (2006), 1865–1885.

[6] Z.M. Ma, B.Q. Hu, *Topological and lattice structures of L-fuzzy rough set determined by lower and upper sets*, Inf. Sci. **218** (2013), 194–204.

[7] E. Orłowska,I. Rewitzky, *Context algebras, context frames and their discrete duality*, Transactions on Rough Sets IX, Springer, Berlin, 2008, 212–229.

[8] E. Orłowska, I. Rewitzky *Algebras for Galois-style connections and their discrete duality,* Fuzzy Sets and Systems, **161** (2010), 1325–1342.

[9] Z. Pawlak, *Rough sets*, Internat. J. Comput. Inform. Sci. **11** (1982), 341–356.

[10] Z. Pawlak, *Rough sets: Theoretical Aspects of Reasoning about Data, System Theory, Knowledge Engineering and Problem Solving*, Kluwer Academic Publishers, Dordrecht, The Netherlands (1991).

[11] A. M. Radzikowska, E.E. Kerre, *A comparative study of fuzy rough sets*, Fuzzy Sets and Systems, **126** (2002), 137–155.

[12] Y.H. She, G.J. Wang, *An axiomatic approach of fuzzy rough sets based on residuated lattices*, Computers and Mathematics with Applications, **58** (2009), 189–201.

[13] M. Ward, R.P. Dilworth, *Residuated lattices*, Trans. Amer. Math. Soc. **45** (1939), 335–354,

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