Korean J. Math. **19** (2011), No. 4, pp. 367–379

MAXIMAL EXPONENTS OF PRIMITIVE GRAPHS WITH MINIMUM DEGREE 3

Byung Chul Song and Byeong Moon Kim*

Abstract. In this paper, we find the maximum exponent of primitive simple graphs G under the restriction $\deg(v) \geq 3$ for all vertex v of G. Our result is also an answer of a Klee and Quaife type problem on exponent to find minimum number of vertices of graphs which have fixed even exponent and the degree of whose vertices are always at least 3.

1. Introduction

A digraph $D = (V, A)$ is primitive if there is a positive integer k such that for any pair of vertices u, v, there is a $u \to v$ walk, a walk from u to v, of length k. We say that the smallest such k is the exponent of D, which is denoted by $\exp(D)$.

The exponent of D is the same with the minimum k such that for an adjacency matrix A of D, $A^k > 0$, which means that every entry of A^k is positive. Note that the diameter, $\mathrm{diam}(D)$, of a connected digraph is the minimum k such that $I + A + A^2 + \cdots + A^k > 0$.

Wielandt [14] found that the maximum exponent of a primitive digraph on n vertices is $n^2 - 2n + 2$. Dulmage and Mendelsohn [3] found the upper bound $n + s(n-2)$ of exponents of primitive digraphs on n vertices with girth s. Zhang [15] proved for all k with $2 \leq k \leq \frac{n^2-2n+4}{2}$, there is a primitive digraph on n vertices whose exponent is k. Holladay and Varga [4] and Lewin [9] computed the maximum exponent of

Received September 15, 2011. Revised December 3, 2011. Accepted December 5, 2011.

2000 Mathematics Subject Classification: 15A48, 05C20, 05C50.

Key words and phrases: maximal exponent, minimum degree, primitive graph.

This work was supported by the Research Institute of Natural Science of Gangneung-Wonju National University.

*Corresponding author.

primitive graphs. Moon and Pullman [10] proved that the maximum exponent of a primitive tournament on n vertices is $n+2$. Brualdi and Ross [1] computed the lower and upper bounds of primitive nearly reducible $n \times n$ matrices and classified the maximal cases. Ross [11] computed the upper bound of the exponent of nearly reducible primitive $n \times n$ matrix A such that the girth of the associated digraph of A is s. Shao [12] proved for all k with $6 \leq k \leq \frac{n^2-2n+10}{9}$, there is a nearly reducible primitive $n \times n$ matrix whose exponent is k. Shen [13] computed the maximum exponent of 2 regular digraphs. The authors [5] of this paper found the maximum exponent of primitive Cartesian product graphs.

Klee and Quaife [7, 8], and Klee [6] obtained some interesting results on diameter. They computed the minimum order of a simple graph with specified diameter, connectivity and degree. They also classified all 3-regular graphs which have the minimum order with given diameter and connectivity.

In this paper, we find the maximum exponent of a primitive simple graph $G = (V, E)$ with $|V| = n$ such that $\deg(v) \geq 3$ for all $v \in V$. As a consequence, we obtain a Klee and Quaife type result for exponent instead of diameter, which finds the minimum number of vertices of a graph of minimum degree 3 with fixed even exponent.

2. Main theorem

Theorem 1. *Let $G = (V, E)$ be a primitive graph on n vertices and let $\deg(v) \geq 3$ for all $v \in V$. Then, for $t \geq 2$,*

$$\exp(G) \leq \begin{cases} 8t-4 \text{ for } n = 6t, 6t+1, \\ 8t-2 \text{ for } n = 6t+2, 6t+3, \\ 8t \text{ for } n = 6t+4, \\ 8t+2 \text{ for } n = 6t+5. \end{cases}$$

Moreover this upper bound is extremal for $n \geq 8$, i.e., for each $n \geq 8$, there is a primitive graph on n vertices with minimum degree 3 and exponent the smallest value of above inequality.

Proof. This Theorem follows from Propositions 1-4 in section 4. □

As a consequence of Theorem 1, we have the following Klee and Quaife type result.

Corollary 1. *Let* $G=(V,E)$ *be a primitive graph,* $\deg(v)\geq 3$ *for all* $v\in V$ *and* $\exp(G)=2k$*. Then, the number of vertices of* G *is less than or equal to*

$$\begin{cases} \frac{3}{2}k+4, \text{ if } k\equiv 0 \pmod 4; \\ \\ \frac{3k+7}{2}, \text{ if } k \text{ is odd;} \\ \\ \frac{3}{2}k+3, \text{ if } k\equiv 2 \pmod 4. \end{cases}$$

3. Some lemmas

Throughout this paper, we assume that $G=(V,E)$ is a primitive graph and $\deg(v)\geq 3$ for all $v\in V$. For a subgraph H of G, let V_H and E_H be the set of vertices and edges of H respectively. Let Γ be the set of all odd cycles in G. For $C\in\Gamma$, let l_C be the length of C. We define

$$\begin{aligned} S_0 &= (\bigcup_{C\in\Gamma} V_C)\bigcup\{v\in V|\mathrm{dist}(C_0,v)+\mathrm{dist}(v,C_1) \\ &= \mathrm{dist}(C_0,C_1) \quad \text{for some} \quad C_0,C_1\in\Gamma\}. \end{aligned}$$

For $T\subset V$, $< T >= (T,E_T)$ is a subgraph of G where $E_T=E\cap\{\{v,w\}|v,w\in T\}$. Usually $< T >$ is called the subgraph of G generated by T. It is not difficult to see that $< S_0 >$ is connected. Let s_0 be the number of the elements of S_0. For a subgraph H of G and $v,w\in V_H$, we say $v\overset{\alpha}{\longrightarrow}w$ along H if there is a $v\to w$ walk in H with length α. Also $\mathrm{dist}_H(v,w)$ is the minimum k such that $v\overset{k}{\longrightarrow}w$ along H and $\exp_H(v,w)$ is the minimum k such that for all $\alpha\geq k$, $v\overset{\alpha}{\longrightarrow}w$ along H. We briefly write $v\overset{\alpha}{\longrightarrow}w$, $\mathrm{dist}(v,w)$ and $\exp(v,w)$ instead of $v\overset{\alpha}{\longrightarrow}w$ along G, $\mathrm{dist}_G(v,w)$ and $\exp_G(v,w)$, respectively. Note that if $v\overset{\alpha}{\longrightarrow}w$, $v\overset{\beta}{\longrightarrow}w$ and $\alpha\not\equiv\beta \pmod 2$, then $\exp_G(v,w)\leq\max\{\alpha,\beta\}-1$.

Lemma 1. *If* $C_0,C_1\in\Gamma, v\in S_0$, $\mathrm{dist}(C_0,C_1)=\mathrm{dist}(C_0,v)+\mathrm{dist}(C_1,v)$, $\mathrm{dist}\ (C_0,v)=t$*, and* $v=v_0\to v_1\to\cdots\to v_t=w$ *for some* $w\in V_{C_0}$*, then* $v_0,v_1,\cdots,v_t$ *are distinct elements of* S_0*.*

Proof. Let $k = \text{dist}(C_0, C_1) = \text{dist}(C_0, v) + \text{dist}(C_1, v)$. For all $i = 0, \cdots, t$, we have $k = \text{dist}(C_0, C_1) \le \text{dist}(C_0, v_i) + \text{dist}(v_i, C_1) \le \text{dist}(w, v_i) + \text{dist}(v_i, C_1) \le t - i + \text{dist}(v_i, v) + \text{dist}(v, C_1) \le t - i + i + k - t = k$. Hence $v_i \in S_0$. Since $\text{dist}(v_0, v_i) = t - i$, we have $v_i \neq v_j$ for $i \neq j$. □

LEMMA 2. *Let $T \subset S_0$ and $w \in S_0$. If $\text{dist}(w, T) = k$, then $|S_0 - T| \ge k$.*

Proof. Since $\text{dist}_{<S_0>}(w, T) = k$, there is a walk $w = w_0 \to w_1 \to \cdots \to w_k$ such that $w_i \in S_0 - T$ and $w_i \neq w_j$ for $i \neq j < k$ and $w_k \in T$. So $S_0 - T \supset \{w_0, \cdots, w_{k-1}\}$. Therefore, we have $|S_0 - T| \ge k$. □

LEMMA 3. *For each pair $v, w \in S_0$,*

$$\exp_{<S_0>}(v, w) \le s_0 - 1.$$

Proof. Case I) $v \in V_C$ for some $C \in \Gamma$.
There is $\tilde{w} \in V_C$ such that $\text{dist}_{<S_0>}(\tilde{w}, w) = \text{dist}_{<S_0>}(C, w) = t$. Let α be an integer such that $v \xrightarrow{\alpha} \tilde{w}$ along C with $0 \le \alpha \le \frac{l_C - 1}{2}$. Since $v \xrightarrow{\alpha} \tilde{w}, \xrightarrow{t} w$, $v \xrightarrow{l_C - \alpha} \tilde{w}, \xrightarrow{t} w$ and $\alpha + t \not\equiv l_C - \alpha + t \pmod 2$, we have $\exp_{<S_0>}(v, w) \le l_C - \alpha + t - 1 \le l_C + t - 1$. By Lemma 2 with $T = V_C$, $|S_0 - V_C| = s_0 - l_C \ge t$. So $\exp_{<S_0>}(v, w) \le s_0 - 1$.
Case II) $v \notin \bigcup_{C \in \Gamma} V_C$.
There are $C_0, C_1 \in \Gamma$ such that $\text{dist}(C_0, v) + \text{dist}(v, C_1) = \text{dist}(C_0, C_1)$. Let $v_0 (\in V_{C_0}), v_1 (\in V_{C_1})$ be vertices with $\text{dist}(C_0, C_1) = \text{dist}(v_0, v_1) = h$. Let $v_0 = u_0 \to u_1 \to \cdots \to u_h = v_1$, $W_1 = \{u_i | 1 \le i \le h - 1\}$, and $W_2 = V_{C_0} \cup V_{C_1} \cup W_1$. We may assume that $v \in W_1$. Let $\tilde{w} \in W_2$ such that $\text{dist}_{<S_0>}(w, \tilde{w}) = \text{dist}_{<S_0>}(w, W_2) = t$. From Lemmas 1 and 2, $s_0 \ge |W_2| + t \ge |V_{C_0}| + |V_{C_1}| + h - 1 + t = l_{C_0} + l_{C_1} + h + t - 1$. Let $\text{dist}(v, C_i) = h_i$ for $i = 0, 1$. Note that $h_0 + h_1 = h$.

Subcase i) $\tilde{w} \in V_{C_1}$.
There is α such that $v_1 \xrightarrow{\alpha} \tilde{w}$ along C_1 with $0 \le \alpha \le \frac{l_{C_1} - 1}{2}$. Since $v \xrightarrow{h_1} v_1 \xrightarrow{\alpha} \tilde{w} \xrightarrow{t} w, v \xrightarrow{h_1} v_1 \xrightarrow{l_{C_1} - \alpha} \tilde{w} \xrightarrow{t} w$ and $h_1 + \alpha + t \not\equiv h_1 + l_{C_1} - \alpha + t \pmod 2$, $\exp_{<S_0>}(v, w) \le h_1 + l_{C_1} - \alpha + t - 1 \le (h - 1) + l_{C_1} + t - 1 = h + l_{C_1} + t - 2 \le s_0 - l_{C_0} - 1 \le s_0 - 4$.

Subcase ii) $\tilde{w} \in W_1$.
Let $k_i = \text{dist}_{<S_0>}(\tilde{w}, C_i)$ for $i = 0, 1$. Since $k_0 + h_0 + k_1 + h_1 = 2h$, $k_0 + h_0 \le h$ or $k_1 + h_1 \le h$. If $k_0 + h_0 \le h$, there are walks $v \xrightarrow{h_0} v_0 \xrightarrow{k_0} \tilde{w} \xrightarrow{t} w$ and

$v \xrightarrow{h_0} v_0 \xrightarrow{l_{C_0}} v_0 \xrightarrow{k_0} \tilde{w} \xrightarrow{t} w$. Since $h_0+k_0+t \not\equiv h_0+l_{C_0}+k_0+t \pmod 2$, we have $\exp_{<S_0>}(v,w) \leq h_0+l_{C_0}+k_0+t-1 \leq l_{C_0}+h+t-1 \leq s_0-l_{C_1} \leq s_0-3$. If $k_1+h_1 \leq h$, similarly we can show $\exp_{<S_0>}(v,w) \leq s_0-3$.

Subcase iii) $\tilde{w} \in V_{C_0}$.
This is similar to subcase i). □

Let $S_i = \{v \in V | \text{dist}(v, S_0) = i\}$, $|S_i| = s_i$ and $k = \max\{i | s_i \geq 1\}$. Then, if $u \in S_i$, $v \in S_j$ and $\{u,v\} \in E$ for some i,j, then $|i-j| \leq 1$.

Lemma 4. *If $s_i = 1$ and $u,v \in S_j$ for some $1 \leq i \leq j \leq k$, then $\{u,v\} \notin E$.*

Proof. If $\{u,v\} \in E$, since $i < j$ and $s_i = 1$, $u \xrightarrow{j-i} w$ and $v \xrightarrow{j-i} w$ for $w \in S_i$. Thus, $u \xrightarrow{j-i} w \xrightarrow{j-i} v \xrightarrow{1} u$ is a closed walk of odd length which must contains an odd cycle not included in $< S_0 >$, which is a contradiction. □

Lemma 5. *If $0 \leq i \leq k-2$, then $s_i + s_{i+2} \geq 3$.*

Proof. If not, $s_i = s_{i+2} = 1$. For any $v \in S_{i+1}$, since $s_i = 1$, by Lemma 4, there is no vertex in S_{i+1} adjacent to v. So $\deg(v) \leq s_i + s_{i+2} = 2$. This is a contradiction. □

Corollary 2. *If $1 \leq i \leq k-3$, then $s_i + s_{i+1} + s_{i+2} + s_{i+3} \geq 6$.*

Lemma 6. *If $k \geq 3$, then*

$$s_{k-2} + s_{k-1} + s_k \geq 6.$$

Proof. Suppose $s_{k-2}+s_{k-1}+s_k \leq 5$. If there are $u,v \in S_k$ such that $u \neq v$ and $\{u,v\} \in E$, there are $x_1,x_2,y_1,y_2 \in V \setminus \{u,v\}$ such that $x_1 \neq x_2, y_1 \neq y_2$ and $\{x_i,u\},\{y_i,v\} \in E$ for $i = 1,2$. Since $u,v,x_1,x_2,y_1,y_2 \in S_k \cup S_{k-1}$ and $s_{k-1}+s_k \leq 4$, $x_i = y_j$ for some $i,j = 1,2$. Thus, $x_i \to u \to v \to y_j = x_i$ is a cycle of length 3, which is impossible. So if $u,v \in S_k$, $\{u,v\} \notin E$. Since $\deg(v) \geq 3$, $s_{k-1} \geq 3$. So $s_{k-2}+s_{k-1}+s_k = s_{k-1}+(s_{k-2}+s_k) \geq 3+3 = 6$. This is a contradiction. Therefore $s_{k-2}+s_{k-1}+s_k \geq 6$.
□

Lemma 7. *If $i \geq 1$ and $s_i+s_{i+1}+\cdots+s_k = m \geq 8$, then $k \leq i+\frac{2}{3}m-\frac{4}{3}$. And if $n \geq s_0+8$, then $k \leq \frac{2}{3}n-\frac{2}{3}s_0-\frac{1}{3}$.*

Proof. Let $k-i=4t+r$ and $0\le r\le 3$. Since $5\le i+\frac{2}{3}m-\frac{4}{3}$, we may assume that $k\ge 5$. If $r=0$, since $t\ge 2$, by Corollary 2 and Lemma 6,

$$m=(s_i+s_{i+1}+s_{i+2}+s_{i+3})+(s_{i+4}+s_{i+5}+s_{i+6}+s_{i+7})+\cdots+(s_{i+4t-8}+\cdots+s_{i+4t-5})$$
$$+s_{i+4t-4}+s_{i+4t-3}\ +\ (s_{k-2}+s_{k-1}+s_k)\ \ge\ 6(t-1)+2+6\ =\ 6t+2.$$

So $k=i+4t=i+\frac{2}{3}(6t+2)-\frac{4}{3}\le i+\frac{2}{3}m-\frac{4}{3}$. For $r=1,2,3$, by using Lemma 5, Corollary 2 and Lemma 6, we can prove it similarly. In particular, if $i=1$, then $k\le 1+\frac{2}{3}(n-s_0)-\frac{4}{3}=\frac{2}{3}n-\frac{2}{3}s_0-\frac{1}{3}$. □

LEMMA 8. *If $s_0=3$, then $k\le\frac{2}{3}n-\frac{19}{3}$.*

Proof. Since $s_0=3$, $< S_0 >$ is a cycle of length 3. For each $v\in S_0$, there is $w\in V\setminus S_0$ such that $\{v,w\}\in E$. Suppose $S_0=\{v_1,v_2,v_3\}$. If $1\le i<j\le 3$ and $\{v_i,w'\},\{v_j,w'\}\in E$ for some $w'\in S_1$, since $v_i\to w'\to v_j\to v_i$ is a cycle of length 3, $w'\in S_0$, which is a contradiction. So $s_1\ge 3$ and for each $w\in S_1$ there is only one $v\in S_0$ such that $w\longrightarrow v$. Suppose $w_1,w_2\in S_1$ and $\{w_1,w_2\}\in E$. If $\{w_1,v_i\},\{w_2,v_i\}\in E$, $w_1\to v_i\to w_2\to w_1$ is a circuit of length 3. This is a contradiction. If $\{w_1,v_i\},\{w_2,v_j\}\in E$ for $1\le i<j\le 3$, $w_1\to v_i\to v_k\to v_j\to w_2\to w_1$ is a circuit of length 5 where v_k is an element of S_0 different from v_i and v_j. This is a contradiction. So any two elements of S_1 are not adjacent. For each $w\in S_1$ such that $\{v_i,w\}\in E$, the number of vertices $u\in S_2$ satisfying $\{u,w\}\in E$ is at least two.

If $u\in S_2$ and $u\xrightarrow{2}v_i$ and $u\xrightarrow{2}v_j$ for $1\le i<j\le 3$, $u\xrightarrow{2}v_i\xrightarrow{1}v_j\xrightarrow{2}u$ is a closed walk of length 5. If v_k is an element of S_0 different from v_i and v_j, this walk does not pass through v_k. So there is an odd cycle different from $< S_0 >$. This is a contradiction. So for all $u\in S_2$, there is only one $v\in S_0$ such that $v\xrightarrow{2}u$. Thus S_2 has at least six elements. Since $s_3+s_4+\cdots+s_k=n-s_0-s_1-s_2\le n-12$, by Lemma 7, $k\le 3+\frac{2}{3}(n-12)-\frac{4}{3}=\frac{2}{3}n-\frac{19}{3}$. □

LEMMA 9.
$$\exp(G)\le\exp(< S_0 >)+2k.$$

Proof. If $v,w\in V$, $v\in S_i,w\in S_j$ for some $i,j\le k$. There are $v_0,w_0\in S_0$ such that $v\xrightarrow{i}v_0$ and $w\xrightarrow{j}w_0$. If $t=\exp(< S_0 >)+2k-i-j\ge\exp(< S_0 >)$, there is a walk $v_0\xrightarrow{t}w_0$. So $v\xrightarrow{i}v_0\xrightarrow{t}w_0\xrightarrow{j}w$. So $\exp(G)\le\exp(< S_0 >)+2k$. □

Corollary 3. *If* $s_0 = 3$, *then*

$$\exp(G) \leq \frac{4}{3}n - \frac{32}{3}.$$

Lemma 10. *If* $\exp(< S_0 >) = s_0 - 1$, *then* s_0 *is odd and* $< S_0 >$ *is Hamiltonian.*

Proof. Since $\exp(< S_0 >) = s_0 - 1$, there are $v, w \in S_0$ such that $\exp_{<S_0>}(v, w) = s_0 - 1$. If $v \in \bigcup_{C \in \Gamma} V_C$, there exists $C_1 \in \Gamma$ such that $v \in V_{C_1}$. There is $w' \in V_{C_1}$ such that $\mathrm{dist}_{<S_0>}(w, w') = \mathrm{dist}_{<S_0>}(w, C_1) = t$. If $w' \xrightarrow{\alpha} v$ along C_1 with $\alpha \leq \frac{l_{C_1} - 1}{2}$, there is $w' \xrightarrow{l_{C_1} - \alpha} v$ along C_1. We have $s_0 - 1 = \exp_{<S_0>}(w, v) \leq l_{C_1} - \alpha + t - 1 \leq s_0 - \alpha - 1$. So $w' = v$ and $l_{C_1} + t = s_0$. Thus $S_0 = V_{C_1} \cup \{w_i | 0 \leq i \leq t - 1\}$. Suppose $t \geq 1$. Choose $w = w_0 \to w_1 \to w_2 \to \cdots \to w_t = v$ where $w_i \in S_0$. Since $\mathrm{dist}_{<S_0>}(w, C_1) = t$, if $j - i \geq 2$, w_i and w_j are not adjacent. In particular, w is not adjacent to w_i for $2 \leq i \leq t - 1$. Since $\deg_{<S_0>}(w) \geq 2$ and $S_0 = V_{C_1} \cup \{w_i | 0 \leq i \leq t - 1\}$, w is adjacent to at least one elements of V_{C_1}. So $t = 1$ and there is $\tilde{v} \in V_{C_1}$ such that $\tilde{v} \neq v$ and $\{w, \tilde{v}\} \in E$. If $\tilde{v} \xrightarrow{\beta} v$ along $C_!$ with $1 \leq \beta \leq \frac{l_{C_1} - 1}{2}$, since $\tilde{v} \xrightarrow{l_{C_1} - \beta} v$ along C_1, $s_0 - 1 = \exp_{<S_0>}(v, w) \leq s_0 - \beta - 1 \leq s_0 - 2$. This is a contradiction. So $t = 0$. Since $l_{C_1} = s - 0$, $< S_0 >$ is Hamiltonian. Since $C_1 \in \Gamma$, $s_0 = l_{C_1}$ is odd. If $v \notin \bigcup_{C \in \Gamma} V_C$, by similar method as used in case II of Lemma 3, we can obtain $\exp_{<S_0>}(v, w) \leq s_0 - 3$, which is a contradiction. □

Note that if there are two vertices whose degree in $< S_0 >$ is 1 or 2 then $s_1 \geq 2$. And so $s_2 \geq 4$.

4. Proof of main theorem

Let $G = (V, E)$ be a primitive graph and $\deg(v) \geq 3$ for all $v \in V$.

Proposition 1. *If* G *has* $6t$ *or* $6t + 1$ *vertices and* $t \geq 2$, *then*

$$\exp(G) \leq 8t - 4.$$

Proof. If $s_0 = 3$, then

$$\exp(G) \leq 8t - \frac{32}{3} < 8t - 4,$$

by Corollary 3. Suppose that $s_0 \geq 8$. If $n \leq s_0 + 7$, then by Lemma 6, $k \leq 4$. By Lemma 9,

$$\exp(G) \leq s_0 + 7 \leq n + 7 \leq 6t + 7 < 8t - 4.$$

If $n \geq s_0 + 8$, then by Lemma 3, 7 and 9,

$$\exp(G) \leq s_0 + 2k - 1 \leq s_0 + 2(\frac{2}{3} \cdot 6t - \frac{2}{3}s_0 - \frac{1}{3}) - 1 \leq 8t - \frac{8}{3} - \frac{5}{3} < 8t - 4.$$

If $s_0 = 7$, by Lemma 3, 7 and 9,

$$\exp(G) \leq 7 + 2k - 1 \leq 6 + 2(\frac{2}{3} \cdot 6t - \frac{2}{3} \cdot 7 - \frac{1}{3}) = 8t - 4.$$

If $s_0 = 6$, by Lemma 7,

$$k \leq \frac{2}{3} \cdot 6t - \frac{2}{3} \cdot 6 - \frac{1}{3} = 4t - \frac{13}{3}.$$

So $k \leq 4t - 5$. By Lemma 9,

$$\exp(G) \leq 5 + 2k \leq 8t - 5 < 8t - 4.$$

If $s_0 = 4$, then $\exp(< S_0 >) = 2$. By Lemma 3, 7 and 9,

$$\exp(G) \leq 2 + 2(\frac{2}{3} \cdot 6t - \frac{2}{3} \cdot 4 - \frac{1}{3}) = 8t - 4.$$

If $s_0 = 5$, by Lemma 7, $k \leq \frac{2}{3} \cdot 6t - \frac{2}{3} \cdot 5 - \frac{1}{3} = 4t - \frac{11}{3}$. So $k \leq 4t - 4$. By Lemma 9,

$$\exp(G) \leq 4 + 2(4t - 4) = 8t - 4.$$

□

Proposition 2. *If G has $6t + 2$ or $6t + 3$ vertices and $t \geq 2$, then*

$$\exp(G) \leq 8t - 2.$$

Proof. If $s_0 = 3$, by Corollary 3,

$$\exp(G) \leq \frac{4}{3}(6t + 2) - \frac{32}{3} = 8t - 8.$$

If $s_0 \geq 10$, by Lemma 3, 7 and 9,

$$\begin{aligned}\exp(G) &\leq s_0 - 1 + 2k \leq s_0 - 1 + 2(\frac{2}{3}(6t+2) - \frac{2}{3}s_0 - \frac{1}{3}) \\ &= 8t - \frac{s_0}{3} + 1 \leq 8t - \frac{7}{3} < 8t - 2.\end{aligned}$$

If $s_0 = 9$, by Lemma 7, $k \leq \frac{2}{3}(6t+2) - \frac{2}{3}9 - \frac{1}{3} = 4t - 5$. By Lemma 9,

$$\exp(G) \leq \exp(< S_0 >) + 2k \leq 8 + 2(4t-5) = 8t - 2.$$

If $s_0 = 8$, by Lemma 7, $k \leq \frac{2}{3}(6t+2) - \frac{2}{3} \cdot 8 - \frac{1}{3} = 4t - \frac{13}{3}$. So $k \leq 4t - 5$. Then by Lemma 9,

$$\exp(G) \leq \exp(< S_0 >) + 2k \leq 7 + 2(4t-5) = 8t - 3.$$

If $s_0 = 6$, by Lemma 3, 7 and 9,

$$k \leq \frac{2}{3}(6t+2) - \frac{12}{3} - \frac{1}{3} = 4t - 3.$$

So

$$\exp(G) \leq \exp(< S_0 >) + 2k \leq 5 + 2(4t-3) = 8t - 1.$$

Since $8t - 1 \geq 6t + 3$ and $8t - 1$ is odd, by [2], $\exp(G) \neq 8t - 1$. So $\exp(G) \leq 8t - 2$.
If $s_0 = 4$, by Lemma 7, $k \leq \frac{2}{3}(6t+2) - \frac{2}{3}4 - \frac{1}{3} \leq 4t - \frac{5}{3}$. So $k \leq 4t - 2$. By Lemma 9,

$$\exp(G) \leq \exp(< S_0 >) + 2k \leq 2 + 2(4t-2) = 8t - 2.$$

If $s_0 = 7$, by Lemma 7, $k \leq 4t - \frac{11}{3}$. So $k \leq 4t - 4$. By Lemma 9,

$$\exp(G) \leq \exp(< S_0 >) + 2k \leq 6 + 2(4t-4) = 8t - 2.$$

If $s_0 = 5$, by Lemma 7, $k \leq 4t - \frac{7}{3}$. So $k \leq 4t - 3$. By Lemma 9,

$$\exp(G) \leq \exp(< S_0 >) + 2k \leq 4 + 2(4t-3) = 8t - 2.$$

□

PROPOSITION 3. *If G has $6t + 4$ vertices and $t \geq 2$, then*

$$\exp(G) \leq 8t.$$

Proof. If $s_0 = 3$, by Lemma 8 and Corollary 3, $\exp(G) \le \frac{4}{3}(6t+4) - \frac{32}{3} = 8t - \frac{16}{3} < 8t$.

If $s_0 \ge 10$, by Lemma 7, $k \le \frac{2}{3}(6t+4) - \frac{20}{3} - \frac{1}{3} = 4t - \frac{13}{3}$. Thus $k \le 4t-5$. By Lemma 9, $\exp(G) \le 9 + 8t - 10 = 8t - 1$.

If $s_0 = 8$, by Lemma 7, $k \le \frac{2}{3}(6t+4) - \frac{16}{3} - \frac{1}{3} = 4t - 3$. By Lemma 9, $\exp(G) \le 7 + 2(4t-3) = 8t+1$. Since $8t > 6t+3 = n-1$, by [2], $\exp(G)$ is even. So $\exp(G) \le 8t$.

If $s_0 = 6$, by Lemma 7, $k \le 4t - \frac{5}{3}$ and thus we have $k \le 4t-2$. So $\exp(G) \le 5 + 2(4t-2) = 8t+1$. Since $8t > 6t+3 = n-1$, by [2], $\exp(G)$ is even. So $\exp(G) = 8t$.

If $s_0 = 9$, by Lemma 7, $k \le 4t-4$ and $\exp(G) \le 8 + 2(4t-4) = 8t$.

If $s_0 = 7$, we have $k \le 4t-3$ and $8t \le \exp(G) \le 6 + 2(4t-3) = 8t$.

If $s_0 = 4$, then $\exp(< S_0 >) = 2$. Since $k \le \frac{2}{3}(6t+4) - \frac{8}{3} - \frac{1}{3} = 4t - \frac{1}{3}$, $k \le 4t-1$, we have $\exp(G) \le 2 + 2(4t-1) = 8t$.

Finally, if $s_0 = 5$, $k \le \frac{2}{3}(6t+4) - \frac{10}{3} - \frac{1}{3} = 4t-1$. If $k = 4t-1$, $6t+4 = s_0 + (s_1 + \cdots + s_{4t-4}) + (s_{4t-3} + s_{4t-2} + s_{4t-1}) \ge 5 + 6t - 6 + 6 = 6t+5$. This is a contradiction. So $k \le 4t-2$. By Lemma 9, $\exp(G) \le 5 + 8t - 4 - 1 = 8t$. □

Proposition 4. *If G has $6t+5$ vertices and $t \ge 2$, then*

$$\exp(G) \le 8t+2.$$

Proof. If $s_0 = 3$, by Corollary 3,

$$\exp(G) \le \frac{4}{3}(6t+5) - \frac{32}{3} = 8t - 4 < 8t+2.$$

If $s_0 \ge 10$, by Lemma 3, 7 and 9,

$$\exp(G) \le 9 + 2(\frac{2}{3}n - \frac{2}{3}s_0 - \frac{1}{3}) = 8t + \frac{5}{3} < 8t+2.$$

If $s_0 = 9$, by Lemma 7, $k \le \frac{2}{3}(6t+5) - \frac{18}{3} - \frac{1}{3} = 4t-3$. By Lemma 9 ,

$$\exp(G) \le \exp(< S_0 >) + 2k \le 9 + 2(4t-3) = 8t+2.$$

If $s_0 = 8$, by Lemma 7, $k \le \frac{2}{3}(6t+5) - \frac{16}{3} - \frac{1}{3} = 4t - \frac{7}{3}$. So $k \le 4t-3$. By Lemma 9,

$$\exp(G) \le \exp(< S_0 >) + 2k \le 8t+1 < 8t+2.$$

If $s_0 = 7$, by Lemma 7, $k \le 4t - \frac{5}{3}$. So $k \le 4t-2$. By Lemma 9, $\exp(G) \le 6 + 2(4t-2) = 8t+2$.

If $s_0 = 6$, by Lemma 7, $k \le \frac{2}{3}(6t+5) - \frac{12}{3} - \frac{1}{3} = 4t-1$. So $\exp(G) \le 5 + 2(4t-1) = 8t+3$.

If $s_0 = 4$, $k \le 4t + \frac{1}{3}$. So $k \le 4t$. By Lemma 9, $\exp(G) \le 2 + 2(4t) = 8t+2$.

If $s_0 = 5$, since $k \le \frac{2}{3}(6t+5) - \frac{10}{3} - \frac{1}{3} = 4t - \frac{1}{3}$, $k \le 4t-1$. By Lemma 9, $\exp(G) \le 4 + 2(4t-1) = 8t+2$. □

The following figure gives examples which assert the upper bound given in Propositions 1 - 4 are extremal.

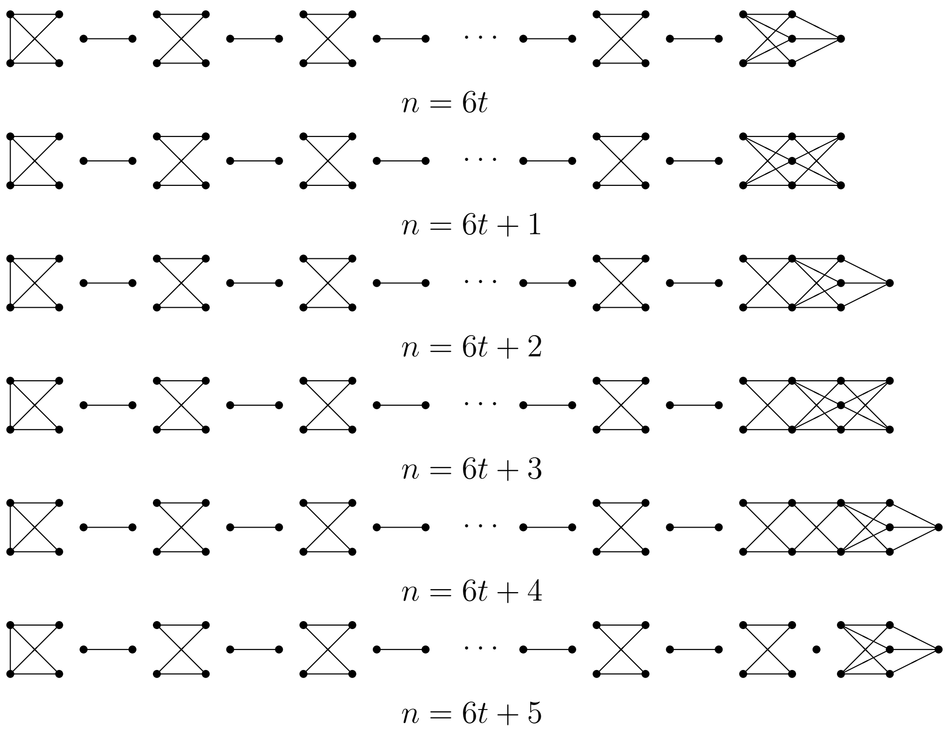

Figure 1.

So we have the following Proposition.

Proposition 5. *If $t \geq 2$ and $0 \leq r \leq 5$, then there is a primitive graph G on $6t + r$ vertices such that the minimum degree of G is 3 and*

$$\exp(G) = \begin{cases} 8t - 4, \text{ for } r = 0, 1 \\ 8t - 2 \text{ for } r = 2, 3, \\ 8t \text{ for } r = 4, \\ 8t + 2 \text{ for } r = 5. \end{cases}$$

Thus Theorem 1 is proved.

Remark 1. For $n \leq 11$, the upper bound of $\exp(G)$ in Theorem 1 is still true except $n = 4$. In that case, $G \simeq K_4$ and $\exp(G) = 2$. The graphs in Figure 2 are extremal cases.

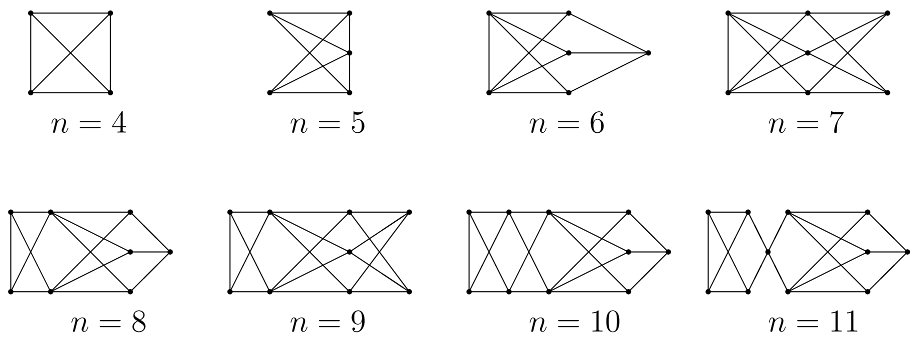

Figure 2.

References

[1] R.A. Brualdi and J.A. Ross, *On the exponent of a primitive, nearly reducible matrix*, Math. Oper. Res. **5**(1980), 229–241.

[2] J. Cai, K.M. Zhang, *The characterization of symmertric primitive matrices with exponent $2n - 2r$*, Linear Multilinear Algebra **39**(1995), 391–396.

[3] A. Dulmage and N. Mendelsohn, *On the exponent of a primitive,nearly reducible matrix*, Math. Oper. Res. **5**(1980), 229–241.

[4] J.C. Holladay and R.S. Varga, *On powers of non-negative matrices*, Proc. Amer. Math. Soc. **9**(1958), 631–634.

[5] B.M. Kim, B.C. Song and W. Hwang , *Wielandt type theorem for cartesian product of digraphs*, Linear Algebra Appl. **429**(2008), 841–848.

[6] V. Klee, *Classification and enumeration of minimum* $(d, 3, 3)$*-graphs for odd* d, J. Combin. Theory Ser. B **28**(1980), 184–207.

[7] V. Klee and H. Quaife, *Minimum graphs of specified diameter, connectivity and valence*, Math. Oper. Res. **1**(1)(1976), 28–31.

[8] V. Klee and H. Quaife, *Classification and enumeration of minimum* $(d, 1, 3)$*-graphs and minimum* $(d, 2, 3)$*-graphs*, J. Combin. Theory Ser. B **23**(1977), 83–93.

[9] M. Lewin, *On exponents of primitive matrices*, Numer. Math. **18**(1971), 154–161.

[10] J.W. Moon and N.J. Pullman, *On the powers of tournament matrices*, J. Combinatorial Theory **3**(1967), 1–9.

[11] J. Ross, *On the exponent of a primitive, nearly reducible matrix. II*, SIAM J. Alg. Discr. Meth. **3**(1982), 395–410.

[12] J. Shao, *The exponent set of primitive, nearly reducible matrices*, SIAM J. Alg. Discr. Meth. **8**(1987), 578–584.

[13] J. Shen, *Exponents of 2-regular digraphs*, Discrete Math. **214**(2000), 211–219.

[14] H. Wielandt, *Unzerleghare, nicht negative Matrizen*, Math. Z. **52**(1950) 642–645.

[15] K. Zhang, *On Lewin and Vitek's conjecture about the exponent set of primitve matrices*, Linear Algebra Appl. **96**(1987), 101–108.

Department of Mathematics
Gangneung-Wonju National University
Gangneung, Korea
E-mail: bcsong@gwnu.ac.kr

Department of Mathematics
Gangneung-Wonju National University
Gangneung, Korea
E-mail: kbm@gwnu.ac.kr