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SOME DESCRIPTION OF ESSENTIAL STRUCTURED APPROXIMATE AND DEFECT PSEUDOSPECTRUM

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Abstract. In this paper, we introduce and study the structured essential approximate and defect pseudospectrum of closed, densely defined linear operators in a Banach space. Beside that, we discuss some results of stability and some properties of these essential pseudospectra. Finally, we will apply the results described above to investigate the essential approximate and defect pseudospectra of the following integro-differential transport operator.

1. Introduction

Throughout the paper, let $(X, \|.\|)$ be an infinite-dimensional Banach space. We denote by $\mathcal{L}(X)$ (resp. $\mathcal{C}(X)$) the set of all bounded (resp. closed, densely defined) linear operators from X into X. The set of all compact operators of $\mathcal{L}(X)$ is denoted by $\mathcal{K}(X)$. We denote by T' (resp. $T = I$) the adjoint operator (resp. the identity operator). Let $T \in \mathcal{C}(X)$, the set

$$\rho(T) := \big\{\lambda \in \mathbb{C} : \lambda - T \ \text{ is injective and } (\lambda - T)^{-1} \in \mathcal{L}(X)\big\}.$$

The spectrum of T is the set $\sigma(T) := \mathbb{C}\backslash\rho(T)$. The set $\rho(T)$ is open, whereas the spectrum $\sigma(T)$ of a closed linear operator T is closed. The

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approximate point spectrum of T is the set defined by

$$\sigma_{ap}(T) = \big\{\lambda \in \mathbb{C} : \lambda - T \text{ is not bounded below}\big\}.$$

For the definition of bounded below we refer [9]. The defect spectrum of T is the set defined by

$$\sigma_{\delta}(T) = \big\{\lambda \in \mathbb{C} : \lambda - T \text{ is not surjective}\big\}.$$

From duality we have

$$\sigma_{ap}(T) = \sigma_{\delta}(T') \text{ and } \sigma_{ap}(T') = \sigma_{\delta}(T).$$

Now, we define the minimum modulus of an operator T

$$\mathrm{m}(T) \;\; := \;\; \inf\Big\{\|Tx\| : \; x \in \mathcal{D}(T) \text{ and } \|x\| = 1\Big\},$$

and the surjectivity modulus

$$\mathrm{q}(T) \;\; := \;\; \sup\Big\{r > 0 : rB_X \subset TB_X\Big\},$$

where, B_X is the closed unit ball of X. It is clear that

$$\sigma_{ap}(T) := \Big\{\lambda \in \mathbb{C} : \; \mathrm{m}(\lambda - T) = 0\Big\}$$

and

$$\sigma_{\delta}(T) := \Big\{\lambda \in \mathbb{C} : \; \mathrm{q}(\lambda - T) = 0\Big\}.$$

We are concerned with the following the essential approximate and defect spectrum of a closed, densely defined linear operator T

$$\begin{aligned} \sigma_{eap}(T) &= \bigcap_{K \in \mathcal{K}(X)} \sigma_{ap}(T + K), \\ \sigma_{e\delta}(T) &= \bigcap_{K \in \mathcal{K}(X)} \sigma_{\delta}(T + K). \end{aligned}$$

Pseudospectra for unstructured perturbations have been studied in the context of numerical analysis and fluid dynamics by L. N. Trefethen we refer the reader for [1, 12, 13]. The definition of structured pseudospectra of a closed densely defined linear operator T, for every $\varepsilon > 0$ and for $B, C \in \mathcal{L}(X)$ is given by:

$$\sigma_{\varepsilon}(T, B, C) = \sigma(T) \bigcup \left\{\lambda \in \mathbb{C} : \; \|B(\lambda - T)^{-1}C\| > \frac{1}{\varepsilon}\right\}.$$

For $\varepsilon > 0$, it can be shown that $\sigma_{\varepsilon}(T, B, C)$ is a larger set and is never empty.

Many problems arising from the most diverse areas of natural science, when modeled under the mathematical point of view, involve the study of unstructured pseudospectra and structured pseudospectra. We refer to E. B. Davies [4] who defined the structured pseudospectra, or spectral value sets of a closed densely defined linear operator T on X by

$$\sigma_\varepsilon(T,B,C)=\bigcup_{\|D\|<\varepsilon}\sigma(T+CDB).$$

The structured pseudospectra of T are a family of strictly nested closed sets, which grow to fill the whole complex plane as $\varepsilon\to\infty$ (see [4]). From these definitions, it follows that the structured pseudospectra associated with various ε are nested sets. Then, for all $0<\varepsilon_1<\varepsilon_2$, we have

$$\sigma(T)\subseteq\sigma_{\varepsilon_1}(T,B,C)\subseteq\sigma_{\varepsilon_2}(T,B,C),$$

and that the intersections of all the structured pseudospectra is the spectra,

$$\bigcap_{\varepsilon>0}\sigma_\varepsilon(T,B,C)=\sigma(T).$$

In this paper, the notion of structured essential approximate and defect pseudospectrum can be extended by devoting our studies to the essential approximate and defect spectrum . For $\varepsilon>0$, $T\in\mathcal{C}(X)$ and $B,C\in\mathcal{L}(X)$ such that $0\in\rho(B)\cap\rho(C)$ we define

$$\begin{aligned}\sigma_{eap,\varepsilon}(T)&=\bigcap_{K\in\mathcal{K}(X)}\sigma_{ap,\varepsilon}(T+K,B,C)\\ \sigma_{e\delta,\varepsilon}(T)&=\bigcap_{K\in\mathcal{K}(X)}\sigma_{\delta,\varepsilon}(T+K,B,C)\end{aligned}$$

where,

$$\sigma_{ap,\varepsilon}(T,B,C)=\sigma_{ap}(T)\bigcup\left\{\lambda\in\mathbb{C}:\ \mathrm{m}(C^{-1}(\lambda-T)B^{-1})\le\varepsilon\right\}$$

and

$$\sigma_{\delta,\varepsilon}(T,B,C)=\sigma_{\delta}(T)\bigcup\left\{\lambda\in\mathbb{C}:\ \mathrm{q}(C^{-1}(\lambda-T)B^{-1})\le\varepsilon\right\}.$$

The subsets $\sigma_{ap,\varepsilon}(\cdot,\cdot,\cdot)$ (resp. $\sigma_{\delta,\varepsilon}(\cdot,\cdot,\cdot)$) are called the structured approximate pseudospectrum (resp. The structured defect pseudospectrum). If we take $B=I=C$, then the subsets $\sigma_{ap,\varepsilon}(\cdot,\cdot,\cdot)$ (resp. $\sigma_{\delta,\varepsilon}(\cdot,\cdot,\cdot)$)

coincide with usual definitions of the approximate pseudospectrum $\sigma_{ap,\varepsilon}(\cdot)$ (resp. the defect pseudospectrum $\sigma_{\delta,\varepsilon}(\cdot)$) (see [2, 14]).

The seminal motivation behind this research comes from the papers of A. Ammar, A. Jeribi and K. Mahfoudhi [1, 2] concerned with characterizing the unstructured approximate pseudospectra and the unstructured essential approximate pseudospectra. The principal aim of this work is to use the new definitions of the structured approximate and defect pseudospectra to measure the sensitivity of the essential approximate and defect spectrum with respect to additive perturbations of T by an bounded operator D of a norm less than ε (Theorem 2.2) and we characterize the structured essential approximate and defect pseudospectrum by means of semi-Fredholm operators (Theorems 3.1 and 3.2). Furthermore, we establish some results for perturbation and properties of the structured approximate and defect pseudospectra (Theorems 4.1, 4.2 and 4.3). In the end, we will apply the results described above to investigate the essential approximate and defect pseudospectra of the following integro-differential transport operator.

The contents of the paper are as follows. After this introduction where several basic definitions and facts will be recalled, in Section 3, we devoted to characterize the structured essential approximate and defect pseudospectrum of closed, densely defined linear operators on a Banach space. In Section 4, we will prove the invariance of the structured essential approximate and defect pseudospectrum and establish some results of perturbation on the context of closed, densely defined linear operators on a Banach space. In Section 5, we apply the results obtained to investigate the essential approximate and defect pseudospectra of the following integro-differential transport operator with boundary conditions in L_1-spaces.

2. Preliminaries and auxiliary results

The goal of this section consists in establishing some preliminary results which will be needed in the sequel. For $T \in \mathcal{C}(X)$, we denote by $\mathcal{N}(T)$ and $\mathcal{R}(T)$ (resp. the null space and the range of T). The nullity of T, $\alpha(T)$, is defined as the dimension of $\mathcal{N}(T)$ and the deficiency of T, $\beta(T)$ is defined as the codimension of $\mathcal{R}(T)$ in X.

In what follows, we need to introduce some important classes of operators. The set of upper semi-Fredholm operators from X into X is defined by

$$\Phi_+(X) := \big\{T \in \mathcal{C}(X) : \ \alpha(T) < \infty, \ \mathcal{R}(T) \text{ is closed in } X\big\},$$

the set of all lower semi-Fredholm operators is defined by

$$\Phi_-(X) := \big\{T \in \mathcal{C}(X) : \ \beta(T) < \infty, \ \mathcal{R}(T) \text{ is closed in } X\big\}.$$

The set of all semi-Fredholm operators is defined by

$$\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X),$$

and the class $\Phi(X)$ of all Fredholm operators is defined by

$$\Phi(X) := \Phi_+(X) \cap \Phi_-(X).$$

The index of a semi-Fredholm operator T is defined by $i(T) = \alpha(T) - \beta(T)$, and if it is finite then we say that T is Fredholm. The set of bounded Fredholm operators from X into X is defined by

$$\Phi^b(X) := \Phi(X) \cap \mathcal{L}(X).$$

The set of bounded upper (resp. lower) semi-Fredholm operators from X into X is defined by

$$\Phi^b_+(X) := \Phi_+(X) \cap \mathcal{L}(X) \text{ (resp. } \Phi^b_-(X) := \Phi_-(X) \cap \mathcal{L}(X)).$$

Definition 2.1. Let X be a Banach space and $K \in \mathcal{K}(X)$.

(i) The operator K is called an upper semi-Fredholm perturbation if $T + K \in \Phi_+(X)$

whenever $T \in \Phi_+(X)$. The set of upper semi-Fredholm perturbations denote by $\mathcal{F}_+(X)$.

(ii) The operator K is called an lower semi-Fredholm perturbation if $T + K \in \Phi_-(X)$

whenever $T \in \Phi_-(X)$. The set of lower semi-Fredholm perturbations denote by $\mathcal{F}_-(X)$.

If we replace in Definition 2.1 respectively, $\Phi(X), \Phi_+(X)$ and $\Phi_-(X)$ by $\Phi(X), \Phi_+(X)$ and $\Phi_-(X)$ we obtain respectively the sets $\mathcal{F}^b(X), \mathcal{F}^b_+(X)$ and $\mathcal{F}^b_-(X)$. In general, we have the following inclusions

$$\mathcal{K}(X) \subset \mathcal{F}^b_+(X) \subset \mathcal{F}^b(X) \text{ and } \mathcal{K}(X) \subset \mathcal{F}^b_-(X) \subset \mathcal{F}^b(X).$$

Before going further, let us recall the following theorem:

Theorem 2.1. *Let X be a Banach space.*

(i) [8, Lemma 2.1] *Let $T \in \mathcal{C}(X)$ and $K \in \mathcal{L}(X)$.*

(i_1) *If $T \in \Phi_+(X)$ and $K \in \mathcal{F}_+(X)$, then $T + K \in \Phi_+(X)$ and $i(T + K) = i(T)$.*

(i_2) *If $T \in \Phi_-(X)$ and $K \in \mathcal{F}_-(X)$, then $T + K \in \Phi_-(X)$ and $i(T + K) = i(T)$.*

(ii) [8, Theorem 3.9] *An operator $T \in \Phi_+(X)$ with $i(T) \leq 0$ if, and only if, T can be*

expressed in the form $T = S + K$ where $K \in \mathcal{K}(X)$ and $S \in \mathcal{C}(X)$ is an operator with

closed range and $\alpha(S) = 0$.

Let T be a closed linear operator on a Banach space X. For $x \in \mathcal{D}(T)$, the graph norm of x is defined by

$$\|x\|_T := \|x\| + \|Tx\|.$$

It follows from the closedness of T that $\mathcal{D}(T)$ endowed with the norm $\|.\|_T$ is a Banach space. Let X_T denote $(\mathcal{D}(T), \|.\|_T)$. In this new space the operator T satisfies

$$\|Tx\| \leq \|x\|_T$$

and consequently, T is a bounded operator from X_T into X. If $\hat{T}$ denotes the restriction of T to $\mathcal{D}(T)$, we observe that

$$\begin{cases} \alpha(\hat{T}) = \alpha(T), & \mathcal{N}(\hat{T}) = \mathcal{N}(T), \\ \beta(\hat{T}) = \beta(T) & \text{and } \mathcal{R}(\hat{T}) = \mathcal{R}(T). \end{cases} \tag{2.1}$$

Definition 2.2. Let X be a Banach space. A linear operator B from X to X is called T-compact if $\mathcal{D}(T) \subset \mathcal{D}(B)$ and whenever a sequence (x_n) of elements of $\mathcal{D}(T)$ satisfies

$$\|x_n\| + \|Tx_n\| \leq c, \ n = 1, 2, \ldots,$$

then (Bx_n) has a subsequence convergent in X.

Lemma 2.1. *If $T \in \mathcal{C}(X)$ such that $\|C^{-1}B^{-1}\| \neq 0$ and $\varepsilon > 0$, then $\sigma_{ap,\varepsilon}(\cdot, \cdot, \cdot)$ and $\sigma_{\delta,\varepsilon}(\cdot, \cdot, \cdot)$ are closed.*

Proof. Let $\lambda \notin \sigma_{ap,\varepsilon}(T, B, C)$. Then,

$$\delta := \inf_{\|x\|=1, x \in \mathcal{D}(T)} \|C^{-1}(\lambda - T)B^{-1}x\| > \varepsilon.$$

Now, take $\mu \in \mathbb{C}$ such that $\mid \mu - \lambda \mid < \frac{\delta-\varepsilon}{\|C^{-1}B^{-1}\|}$. We have

$$\begin{aligned}
&\inf_{\|x\|=1, x\in\mathcal{D}(T)} \|C^{-1}(\mu - T)B^{-1}x\| \\
&= \inf_{\|x\|=1, x\in\mathcal{D}(T)} \|C^{-1}(\mu - \lambda + \lambda - T)B^{-1}x\| \\
&\geq \inf_{\|x\|=1, x\in\mathcal{D}(T)} \Big| \|C^{-1}(\lambda - T)B^{-1}x\| - |\mu - \lambda| \|C^{-1}B^{-1}\| \Big| \\
&> \varepsilon.
\end{aligned}$$

So, the complement of $\sigma_{ap,\varepsilon}(T, B, C)$ is open. By a similar reasoning, we show that $\sigma_{\delta,\varepsilon}(T, B, C)$ is closed. □

Remark 2.1. For all $\varepsilon > 0$ it may happen that

$$\sigma_\varepsilon(T, B, C) \neq \sigma_{ap,\varepsilon}(T, B, C)$$

holds as the following example shows. Let α, β, δ and $\gamma \in \mathbb{C}$ with $\alpha \neq \beta$ and $\delta \neq 0, \gamma \neq 0$. and let $T = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ \gamma & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}$. A direct computation shows that

$$\|B(\lambda - T)^{-1}C\| = \max\left\{ \frac{1}{\lambda - \beta}, \frac{\gamma\delta}{\lambda - \alpha} \right\}$$

and

$$\left\| C^{-1}(\lambda - T)B^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right\| = \left\| \begin{pmatrix} \frac{\lambda-\beta}{\delta} & 0 \\ 0 & \frac{\lambda-\alpha}{\gamma} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|.$$

Hence, if we take $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we have

$$\begin{aligned}
\sigma_{ap,\varepsilon}(T, B, C) &= \Big\{ \lambda \in \mathbb{C} : |\lambda| \leq \delta\varepsilon + \beta \Big\}, \\
\sigma_\varepsilon(T, B, C) &= \Big\{ \lambda \in \mathbb{C} : |\lambda| \leq \varepsilon + \beta \Big\} \bigcup \Big\{ \lambda \in \mathbb{C} : |\lambda| \leq \gamma\delta\varepsilon + \alpha \Big\}.
\end{aligned}$$

Moreover, if $\delta = 1$, we can see for all $\varepsilon > 0$ that

$$\sigma_\varepsilon(T, B, C) \backslash \sigma_{ap,\varepsilon}(T, B, C) = \Big\{ \lambda \in \mathbb{C} : |\lambda| \leq \gamma\varepsilon + \alpha \Big\}.$$

Proposition 2.1. *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$.*

(i) $\sigma_{ap,\varepsilon}(T, C, B) = \sigma_{\delta,\varepsilon}(T', B', C')$, *for all* $\varepsilon > 0$.

(ii) *If* $\alpha \in \mathbb{C}$ *and* $\varepsilon > 0$, *then* $\sigma_{ap,\varepsilon}(T + \alpha, B, C) = \alpha + \sigma_{ap,\varepsilon}(T, B, C)$.

(iii) *If* $\alpha \in \mathbb{C}\backslash\{0\}$ *and* $\varepsilon > 0$, *then* $\sigma_{ap,|\alpha|\varepsilon}(\alpha T, B, C) = \alpha\sigma_{ap,\varepsilon}(T, B, C)$.

Proof. (i) Let $\lambda \in \sigma_{ap,\varepsilon}(T, C, B)$, then

$$\begin{aligned} \lambda &\in \Big\{\lambda \in \mathbb{C} : \ \mathrm{m}(B^{-1}(\lambda - T)C^{-1}) \le \varepsilon\Big\} \\ &= \Big\{\lambda \in \mathbb{C} : \ \mathrm{q}\left((B^{-1}(\lambda - T)C^{-1})'\right) \le \varepsilon\Big\} \\ &= \Big\{\lambda \in \mathbb{C} : \ \mathrm{q}\left((C'^{-1}(\lambda - T')B'^{-1})\right) \le \varepsilon\Big\} \end{aligned}$$

So, $\lambda \in \sigma_{\delta,\varepsilon}(T', B', C')$. By the same way we deduce the second inclusion.

(ii) Let $\lambda \in \sigma_{ap,\varepsilon}(T+\alpha, B, C)$, then $\inf\limits_{x\in\mathcal{D}(T),\ \|x\|=1} \|C^{-1}((\lambda-\alpha)-T)B^{-1}x\| < \varepsilon$. Hence $\lambda - \alpha \in \sigma_{ap,\varepsilon}(T, B, C)$. This yields to

$$\lambda \in \alpha + \sigma_{ap,\varepsilon}(T, B, C).$$

For the second inclusion it is the same reasoning.

(iii) Let $\lambda \in \sigma_{ap,|\alpha|\varepsilon}(\alpha T, B, C)$, then

$$\begin{aligned} &\inf_{x\in\mathcal{D}(T),\ \|x\|=1} \|C^{-1}(\lambda - \alpha T)B^{-1}x\| \\ &\quad= \inf_{x\in\mathcal{D}(T),\ \|x\|=1} \|C^{-1}\alpha(\frac{\lambda}{\alpha} - T)B^{-1}x\| \ \alpha \neq 0, \\ &\quad= |\alpha| \inf_{x\in\mathcal{D}(T),\ \|x\|=1} \|C^{-1}(\frac{\lambda}{\alpha} - T)B^{-1}x\| \\ &\quad< |\alpha|\varepsilon. \end{aligned}$$

Hence $\dfrac{\lambda}{\alpha} \in \sigma_{ap,\varepsilon}(T, B, C)$. So $\sigma_{ap,|\alpha|\varepsilon}(\alpha T) \subseteq \alpha\sigma_{ap,\varepsilon}(T, B, C)$. However, the reverse inclusion is similar. □

Now, we give a characterization of the structured approximate and defect pseudospectrum of linear operators on a Banach space.

Theorem 2.2. *Let $T \in \mathcal{C}(X)$, $B, C \in \mathcal{L}(X)$ and $\varepsilon > 0$.*

(i) $\sigma_{ap,\varepsilon}(T,B,C) = \bigcup_{\|D\|<\varepsilon} \sigma_{ap}(T+CDB)$.

(ii) $\sigma_{\delta,\varepsilon}(T,B,C) = \bigcup_{\|D\|<\varepsilon} \sigma_{\delta}(T+CDB)$.

Proof. (i) Let $\lambda \in \sigma_{ap,\varepsilon}(T,B,C)$. There are two possible cases:
$\underline{1^{st}case}$: If $\lambda \in \sigma_{ap}(T)$, then it is sufficient to take $D = 0$.

$\underline{2^{nd}case}$: If $\lambda \notin \sigma_{ap}(T)$, then there exists $x_0 \in X$ such that

$$\|x_0\| = 1 \text{ and } \|C^{-1}(\lambda - T)B^{-1}x_0\| < \varepsilon.$$

By the Hahn Banach Theorem, (see [10]) there exists $x^{'} \in X^{'}$ such that $\|x^{'}\| = 1$ and $x^{'}(x_0) = \|x_0\|$. Consider the operator D defined by the formula

$$\begin{cases} D : X \longrightarrow X, \\ \ x \longrightarrow Dx := x^{'}(x)C^{-1}(\lambda - T)B^{-1}x_0. \end{cases}$$

Then, D is a linear operator everywhere defined on X and bounded, since for $x \neq 0$ we have

$$\|Dx\| = \|x^{'}(x)C^{-1}(\lambda - T)B^{-1}x_0\| \leq \|x^{'}\|\|x\|\|C^{-1}(\lambda - T)B^{-1}x_0\|.$$

Therefore,

$$\frac{\|Dx\|}{\|x\|} \leq \|C^{-1}(\lambda - T)B^{-1}x_0\|.$$

Hence, $\|D\| < \varepsilon$. We claim that $\inf_{\|x\|=1, x\in\mathcal{D}(T)} \|(\lambda - T - CDB)x\| = 0$. Now take $y_0 = B^{-1}x_0$, then

$$\begin{aligned} \inf_{\|x\|=1, x\in\mathcal{D}(T)} \|(\lambda - T - CDB)x\| &\leq \|(\lambda - T - CDB)y_0\| \\ &\leq \|(\lambda - T)y_0 - CDx_0\| \\ &= \|(\lambda - T)y_0 - Cx^{'}(x_0)C^{-1}(\lambda - T)y_0\| \\ &= 0. \end{aligned}$$

Conversely, we assume that there exists a bounded operator $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and $\lambda \in \sigma_{ap}(T+CDB)$, which means that $\inf_{\|x\|=1, x\in\mathcal{D}(T)} \|(\lambda - T - CDB)x\| = 0$. In order to prove that $\inf_{\|x\|=1, x\in\mathcal{D}(T)} \|C^{-1}(\lambda - T)B^{-1}x\| < \varepsilon$. We have

$$\begin{aligned}\|C^{-1}(\lambda - T)B^{-1}x_0\| &= \|C^{-1}(\lambda - T - CDB + CDB)B^{-1}x_0\| \\ &\leq \|C^{-1}(\lambda - T - CDB)B^{-1}x_0\| + \|C^{-1}CDBB^{-1}x_0\|.\end{aligned}$$

Then,

$$\inf_{\|x\|=1, x\in\mathcal{D}(T)} \|C^{-1}(\lambda - T)B^{-1}x\| < \varepsilon.$$

(ii) Let $\lambda \in \sigma_{\delta,\varepsilon}(T, B, C)$. Using proposition 2.1 we have for all $\varepsilon > 0$

$$\sigma_{\delta,\varepsilon}(T, B, C) = \sigma_{ap,\varepsilon}(T', C', B').$$

Then,

$$\begin{aligned}\sigma_{ap,\varepsilon}(T', C', B') &= \bigcup_{\|D\|<\varepsilon} \sigma_{ap}(T' + B'DC') \\ &= \bigcup_{\|D'\|<\varepsilon} \sigma_{ap}\big((T + CD'B)'\big) \\ &= \bigcup_{\|D\|<\varepsilon} \sigma_{\delta}(T + BDC).\end{aligned}$$

Hence

$$\sigma_{\delta,\varepsilon}(T, B, C) \subseteq \bigcup_{\|D\|<\varepsilon} \sigma_{\delta}(T + BDC).$$

The converse is similar. □

3. Essential approximate and defect pseudospectrum

In this section, we will bring a new characterization of the structured essential approximate and defect pseudospectrum. We are now in the position to state the main result of this section.

THEOREM 3.1. *Let $T \in \mathcal{C}(X)$, $B, C \in \mathcal{L}(X)$ and $\varepsilon > 0$.*

(i) $\sigma_{eap,\varepsilon}(T, B, B) = \bigcup_{\|D\|<\varepsilon} \sigma_{eap}(T + CDB).$

(ii) $\sigma_{e\delta,\varepsilon}(T, B, C) = \bigcup_{\|D\|<\varepsilon} \sigma_{e\delta}(T + CDB).$

Proof. (i) Let $\lambda \notin \sigma_{eap,\varepsilon}(T,B,C)$, then there exists a compact operator K on X such that $\lambda \notin \sigma_{ap,\varepsilon}(T+K,B,C)$. Therefore, $\lambda \notin \sigma_{ap}(T+CDB+K)$ for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Hence,

$$\lambda - T - CDB - K \in \Phi_+(X) \text{ and } i(\lambda - T - CDB - K) \leq 0,$$

for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Using Theorem 2.1, we get for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ that

$$\lambda - T - CDB \in \Phi_+(X) \text{ and } i(\lambda - T - CDB) \leq 0.$$

Conversely, we assume that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have

$$\lambda - T - CDB \in \Phi_+(X) \text{ and } i(\lambda - T - CDB) \leq 0.$$

According of Theorem 2.1-(ii), $\lambda - T - CDB$ can be expressed in the form

$$\lambda - T - CDB = S + K$$

where, $K \in \mathcal{K}(X)$ and $S \in \mathcal{C}(X)$ is an operator with closed range and $\alpha(S) = 0$. Then,

$$\lambda - T - CDB - K = S \text{ and } \alpha(\lambda - T - CDB - K) = 0.$$

By using [10, Theorem 3.12], there exists a constant $c > 0$ such that

$$\|(\lambda - T - CDB - K)x\| \geq c\|x\|, \text{ for all } x \in \mathcal{D}(T).$$

Then, $\inf\limits_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T - CDB - K)x\| \geq c > 0$. Hence, $\lambda \notin \sigma_{ap}(T+CDB+K)$. Therefore, $\lambda \notin \sigma_{eap,\varepsilon}(T,B,C)$.

(ii) Statement (ii) can be checked similarly as the assertion (i). $\square$

Remark 3.1. Let $T \in \mathcal{C}(X)$, $B, C \in \mathcal{L}(X)$, $\varepsilon > 0$.

(i) If $0 < \varepsilon_1 < \varepsilon_2$, then $\sigma_j(T,B,C) \subset \sigma_{j,\varepsilon_1}(T,B,C) \subset \sigma_{j,\varepsilon_2}(T,B,C)$ with, $j = eap, e\delta$.

(ii) For all $\varepsilon > 0$, then $\sigma_{j,\varepsilon}(T,B,C) \subset \sigma_{j,\varepsilon}(T,B,C)$ with, $j = eap, e\delta$.

(iii) $\bigcap\limits_{\varepsilon>0} \sigma_{j,\varepsilon}(T,B,C) = \sigma_j(T,B,C)$ with, $j = eap, e\delta$.

(iv) $\sigma_{j,\varepsilon}(T+K,B,C) = \sigma_{j,\varepsilon}(T,B,C)$ for all $K \in \mathcal{K}(X)$ with, $j = eap, e\delta$.

THEOREM 3.2. *Let $T \in \mathcal{C}(X)$, $B, C \in \mathcal{L}(X)$ and $\varepsilon > 0$.*

$$\sigma_{eap,\varepsilon}(T,B,C) = \bigcap_{F \in \mathcal{F}_+^b(X)} \sigma_{ap,\varepsilon}(T+F,B,C)$$

and

$$\sigma_{e\delta,\varepsilon}(T,B,C) = \bigcap_{F \in \mathcal{F}_-^b(X)} \sigma_{\delta,\varepsilon}(T+F,B,C).$$

Proof. Let $\lambda \notin \bigcap_{F \in \mathcal{F}_+^b(X)} \sigma_{ap,\varepsilon}(T+F,B,C)$, then there exists $F \in \mathcal{F}_+^b(X)$ such that
$\lambda \notin \sigma_{ap,\varepsilon}(T+F,B,C)$. Therefore, $\lambda \notin \sigma_{ap}(T+F+CDB)$ for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Therefore,

$$\lambda - T - F - CDB \in \Phi_+(X) \text{ and } i(\lambda - T - F - CDB) \leq 0.$$

Using Theorem 2.1, we conclude that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$,

$$\lambda - T - CDB \in \Phi_+(X) \text{ and } i(\lambda - T - CDB) \leq 0.$$

Finally, Theorem 3.1-(i) shows that $\lambda \notin \sigma_{eap,\varepsilon}(T,B,C)$. For the second inclusion, it is clear that

$$\bigcap_{F \in \mathcal{F}_+^b(X)} \sigma_{ap,\varepsilon}(T+F,B,C) \subset \bigcap_{F \in \mathcal{K}(X)} \sigma_{ap,\varepsilon}(T+F,B,C) = \sigma_{eap,\varepsilon}(T,B,C),$$

because, $\mathcal{K}(X) \subset \mathcal{F}_+^b(X)$. The proof of the second part of this theorem is of the same way that the first part. □

REMARK 3.2. Let $T \in \mathcal{C}(X)$, $B, C \in \mathcal{L}(X)$ and $\varepsilon > 0$.

(i) From Theorem 3.2, we obtain

$$\sigma_{eap,\varepsilon}(T+F,B,C) = \sigma_{eap,\varepsilon}(T,B,C) \text{ for all } F \in \mathcal{F}_+^b(X)$$
$$\text{and } \sigma_{e\delta,\varepsilon}(T+F,B,C) = \sigma_{e\delta,\varepsilon}(T,B,C) \text{ for all } F \in \mathcal{F}_-^b(X).$$

(ii) Let $\mathcal{I}(X)$ and $\mathcal{V}(X)$ be subsets of $\mathcal{L}(X)$.

(ii_1) If $\mathcal{K}(X) \subset \mathcal{I}(X) \subset \mathcal{F}_+^b(X)$, then

$$\sigma_{eap,\varepsilon}(T,B,C) = \bigcap_{M \in \mathcal{I}(X)} \sigma_{ap,\varepsilon}(T+M,B,C).$$

(ii_2) If $\mathcal{K}(X) \subset \mathcal{V}(X) \subset \mathcal{F}_-^b(X)$, then

$$\sigma_{e\delta,\varepsilon}(T,B,C) = \bigcap_{M\in\mathcal{V}(X)} \sigma_{\delta,\varepsilon}(T+M,B,C).$$

4. Some stability result of $\sigma_{eap,\varepsilon}(\cdot,\cdot,\cdot)$ and $\sigma_{e\delta,\varepsilon}(\cdot,\cdot,\cdot)$

In this section, we have also the following useful stability result for the structured essential approximate and defect pseudospectrum.

THEOREM 4.1. *Let T and A be two elements of $\mathcal{C}(X)$ and $\varepsilon > 0$. Assume that for $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and A is $(T + CDB)$-compact,*

(i) $\sigma_{eap,\varepsilon}(T,B,C) = \sigma_{eap,\varepsilon}(T+A,B,C)$.

(ii) $\sigma_{e\delta,\varepsilon}(T,B,C) = \sigma_{e\delta,\varepsilon}(T+A,B,C)$.

Proof. (i) Let $\lambda \notin \sigma_{eap,\varepsilon}(T,B,C)$, then for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, we have

$$\lambda - T - CDB \in \Phi_+(X) \text{ and } i(\lambda - T - CDB) \leq 0.$$

Since, A is $(T+CDB)$-compact and applying [11, Theorem 3.3], we get

$$\lambda - T - A - CDB \in \Phi_+(X) \text{ and } i(\lambda - T - A - CDB) \leq 0.$$

Then, $\lambda \notin \sigma_{eap,\varepsilon}(T+A,B,C)$. We conclude that

$$\sigma_{eap,\varepsilon}(T+A,B,C) \subset \sigma_{eap,\varepsilon}(T,B,C).$$

Let us prove now the converse inclusion, let $\lambda \notin \sigma_{eap,\varepsilon}(T+A,B,C)$. Then for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, we have

$$\lambda - T - A - CDB \in \Phi_+(X) \text{ and } i(\lambda - T - A - CDB) \leq 0.$$

On the other hand, A is $(T+CDB)$-compact. Using [11, Theorem 2.12], we deduce that A is $(T + A + CDB)$-compact, then

$$\lambda - T - CDB \in \Phi_+(X) \text{ and } i(\lambda - T - CDB) \leq 0.$$

Therefore, $\lambda \notin \sigma_{eap,\varepsilon}(T,B,C)$. This proves that $\sigma_{eap,\varepsilon}(T,B,C) \subset \sigma_{eap,\varepsilon}(T+A,B,C)$.

(ii) The proof of (ii) can be checked in a similar way to that in (i). □

Theorem 4.2. *Let $\varepsilon > 0$ and $T, A \in \mathcal{C}(X)$ such that $0 \notin \sigma_{eap}(T) \cup \sigma_{eap}(A)$. Assume that there exist two operators T_0 and $A_0 \in \mathcal{L}(X)$ such that*

$$TT_0 = I - F_1, \tag{4.1}$$

$$AA_0 = I - F_2. \tag{4.2}$$

(i) If $T_0 - A_0 \in \mathcal{F}_+(X)$, $i(T) = i(A)$ and $F_i \in \mathcal{F}_+(X)$ with $i = 1, 2$, then

$$\sigma_{eap,\varepsilon}(T, B, C) = \sigma_{eap,\varepsilon}(A, B, C).$$

(ii) If $T_0 - A_0 \in \mathcal{F}_-(X)$, $i(T) = i(A)$ and $F_i \in \mathcal{F}_-(X)$, with $i = 1, 2$, then

$$\sigma_{e\delta,\varepsilon}(T, B, C) = \sigma_{e\delta,\varepsilon}(A, B, C).$$

Proof. (i) Using Eqs. (4.1) and (4.2), we can show for any scalar λ the relation

$$\begin{aligned}(\lambda - T - CDB)T_0 &- (\lambda - A - CDB)A_0 \\ &= F_2 - F_1 + (\lambda - CDB)(T_0 - A_0).\end{aligned} \tag{4.3}$$

Let $\lambda \notin \sigma_{eap,\varepsilon}(T, B, C)$, then for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have that,

$$\lambda - T - CDB \in \Phi_+(X) \text{ and } i(\lambda - T - CDB) \leq 0.$$

Since $T + CDB$ is closed and $\mathcal{D}(T + CDB) = \mathcal{D}(T)$ endowed with the graph norm is a Banach space denoted by X_{T+BDC}. we can show that

$$\lambda - \hat{T} - \hat{C}\hat{D}\hat{B} \in \Phi_+^b(X_{T+CDB}, X).$$

Moreover, $F_1 \in \mathcal{F}_+(X)$ and using Eq. (4.1) and [6, Theorem 2.1] we infer that,

$$T_0 \in \Phi_+^b(X, X_{T+D}).$$

So,

$$(\lambda - \hat{T} - \hat{C}\hat{D}\hat{B})T_0 \in \Phi_+^b(X). \tag{4.4}$$

Next, if the difference $T_0 - A_0 \in \mathcal{F}_+(X)$, applying Eq. (4.3) we obtain that,

$$(\lambda - T - CDB)T_0 - (\lambda - A - CDB)A_0 \in \mathcal{F}_+(X).$$

Also, it follows from Eq. (4.4) and Lemma 2.1-(i) that

$$(\lambda - A - CDB)A_0 \in \Phi_+^b(X)$$

and

$$i[(\lambda - \hat{A} - \hat{C}\hat{D}\hat{B})A_0] = i[(\lambda - \hat{T} - \hat{C}\hat{D}\hat{B})T_0]. \tag{4.5}$$

Since $A \in \mathcal{C}(X)$, proceeding as above, Eq. (4.2) implies that

$$A_0 \in \Phi_+^b(X, X_{B+D}).$$

Thus, since $(\lambda - A - CDB)A_0 \in \Phi_+(X)$, the use of [9, Theorem 6] shows that

$$\lambda - \hat{A} - \hat{C}\hat{D}\hat{B} \in \Phi_+^b(X_{B+D}, X).$$

This implies that

$$\lambda - A - CDB \in \Phi_+(X).$$

On the other hand, $0 \notin \sigma_{eap}(T) \cup \sigma_{eap}(A)$. Then,

$$i(T) = i(A) \leq 0.$$

Therefore, from Eqs. (4.1), (4.2) and using Atkinson Theorem we have that

$$-i(T_0) = i(T) \text{ and } -i(A_0) = i(A).$$

This together with Eq. (4.3) shows that

$$i(\lambda - T - CDB) = i(\lambda - A - CDB) \leq 0.$$

Hence, $\lambda \notin \sigma_{eap,\varepsilon}(A, B, C)$. This proves that $\sigma_{eap,\varepsilon}(A, B, C) \subset \sigma_{eap,\varepsilon}(T, B, C)$. The proof of the opposite inclusion follows by symmetry.

(ii) Similarly, we can prove the statement

$$\sigma_{e\delta,\varepsilon}(T, B, C) = \sigma_{e\delta,\varepsilon}(A, B, C).$$

□

THEOREM 4.3. *Let $T, A \in \mathcal{C}(X)$, $B, C \in \mathcal{L}(X)$ and $\varepsilon > 0$.*

(i) *If $T(A + CDB) \in \mathcal{F}_+(X)$ for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, then*

$$\sigma_{eap,\varepsilon}(T + A, B, C)\backslash\{0\} \subseteq [\sigma_{eap}(T) \cup \sigma_{eap,\varepsilon}(A, B, C)]\backslash\{0\}.$$

Moreover, if $(A + CDB)T \in \mathcal{F}_+(X)$, then

$$\sigma_{eap,\varepsilon}(T + A, B, C)\backslash\{0\} = [\sigma_{eap}(T) \cup \sigma_{eap,\varepsilon}(A, B, C)]\backslash\{0\}.$$

(ii) *If $T(A + CDB) \in \mathcal{F}_-(X)$ for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, then*

$$\sigma_{e\delta,\varepsilon}(T + A, B, C)\backslash\{0\} \subseteq [\sigma_{e\delta}(T) \cup \sigma_{e\delta,\varepsilon}(A, B, C)]\backslash\{0\}.$$

Moreover, if $(A + CDB)T \in \mathcal{F}_-(X)$, then

$$\sigma_{e\delta,\varepsilon}(T + A, B, C)\backslash\{0\} = [\sigma_{e\delta}(T) \cup \sigma_{e\delta,\varepsilon}(A, B, C)]\backslash\{0\}.$$

Proof. (i) Let $\lambda \in \mathbb{C}$. We can write,

$$(\lambda - T)(\lambda - A - CDB) = T(A + CDB) + \lambda(\lambda - T - A - CDB) \tag{4.6}$$

and

$$(\lambda - A - CDB)(\lambda - T) = (A + CDB)T + \lambda(\lambda - T - A - CDB). \tag{4.7}$$

Let $\lambda \notin \sigma_{eap}(T) \cup \sigma_{eap,\varepsilon}(A, B, C)\backslash\{0\}$, then $\lambda - T \in \Phi_+(X)$ and for all $\|D\| < \varepsilon$, we have

$$\lambda - A - CDB \in \Phi_+(X).$$

From, [10, Theorem 7.32, p.175] we deduce that,

$$(\lambda - T)(\lambda - A - CDB) \in \Phi_+(X) \text{ and } i((\lambda - T)(\lambda - A - CDB)) \leq 0.$$

Since, $T(A + CDB) \in \mathcal{F}_+(X)$, and using Eq. (4.6), we infer that,

$$\lambda - T - A - CDB \in \Phi_+(X)$$

and

$$i(\lambda - T - A - CDB) \leq 0.$$

Then, $\lambda \notin \sigma_{eap,\varepsilon}(T + A, B, C)$ and so this proves the inclusion

$$\sigma_{eap,\varepsilon}(T + A, B, C)\backslash\{0\} \subseteq [\sigma_{eap}(T) \cup \sigma_{eap,\varepsilon}(A, B, C)]\backslash\{0\}. \tag{4.8}$$

Now, it remains to prove the inverse inclusion of Eq. (4.8). Let $\lambda \notin \sigma_{eap,\varepsilon}(T + A, B, C)\backslash\{0\}$. Then, for all $\|D\| < \varepsilon$ we have

$$\lambda - T - A - CDB \in \Phi_+(X) \text{ and } i(\lambda - T - A - CDB) \leq 0.$$

Since, $T(A + CDB) \in \mathcal{F}_+(X)$, $(A + CDB)T \in \mathcal{F}_+(X)$ and using Eqs. (4.7) and (4.6) we obtain that

$$(\lambda - T)(\lambda - B - CDB) \in \Phi_+(X) \text{ and } (\lambda - A - CDB)(\lambda - T) \in \Phi_+(X).$$

It is clear from [9, Theorem 6] that $\lambda - T \in \Phi_+(X)$ and for all $\|D\| < \varepsilon$ we have

$$\lambda - A - CDB \in \Phi_+(X) \text{ and } i(\lambda - A - CDB) \leq 0.$$

Therefore, $\lambda \notin \sigma_{eap}(T) \cup \sigma_{eap,\varepsilon}(A, B, C)$. This proves that,

$$\sigma_{eap,\varepsilon}(T + A, B, C)\backslash\{0\} = [\sigma_{eap}(T) \cup \sigma_{eap,\varepsilon}(A, B, C)]\backslash\{0\}.$$

The proof of (ii) is a straightforward adoption of the proof of (i). $\square$

5. Application to transport equation

In this section, we will apply the results described above to investigate the structured essential approximate pseudospectrum of the following integro-differential operator:

$$\begin{aligned}\mathcal{A}_{\mathcal{K}}\psi(x,\xi) &= -\xi\frac{\partial\psi}{\partial x}(x,\xi)-\sigma(x,\xi)\psi(x,\xi)+\int_a^b \kappa(x,\xi,\xi')\psi(x,\xi')d\xi' \\ &= \mathcal{T}_{\mathcal{K}}\psi+\mathcal{B}\psi\end{aligned}$$

with the following boundary operator :

$$\xi\psi(0,\xi)=p\int_a^b \kappa(\xi,\xi')\psi(1,\xi')\xi'd\xi',$$

where $x\in[0,1]$, $\xi,\ \xi'\in[a,b]$ with $0<a<b<\infty$, $p\geq 0$ denote the medium number of daughter cells which are descended from mother cells and the kernel of correlation $\kappa(.,.)$ satisfies the normalisation's condition

$$\int_a^b \kappa(\xi,\xi')d\xi=1$$

for ensuring the cells flow's continuity when $p=1$. Our general assumptions are

$$\begin{cases}\sigma(x,\xi)\in L_1([0,1]\times[a,b]), \\ \\ \mathcal{B}\in\mathcal{L}(L_1([0,1]\times[a,b])),\end{cases}$$

where $\mathcal{B}$ is the partially integral operator with kernel $\kappa(x,\xi,\xi')$. Let us introduce the functional setting of the problem:

$$\begin{aligned}\Omega &= [0,1]\times[a,b],\ (0<a<b<\infty), \\ \Omega_0 &= \{0\}\times[a,b], \\ \Omega_1 &= \{1\}\times[a,b].\end{aligned}$$

Ω_0 and Ω_1 represent respectively the outgoing and the incoming boundary of the phase space Ω.

$$X:=L_1(\Omega,dxd\xi).$$

We consider the boundary spaces:

$$X^0:=L_1(\Omega_0,|\xi|d\xi),$$

$$\text{and } X^1:=L_1(\Omega_1,|\xi|d\xi),$$

equipped with their norms. Let $\mathcal{W}_1$ the space defined by:

$$\mathcal{W}_1 := \big\{\varphi \in X : \xi \frac{\partial \varphi}{\partial x} \in X\big\}.$$

It is well-known that any function ψ in $\mathcal{W}_1$ possesses traces on the spatial boundary $\{0\}$ and $\{1\}$ which respectively belong to the space X^0 and X^1. Let $\mathcal{K}$ be the following boundary operator :

$$\begin{aligned} \mathcal{K} \ &: \ X^1 \longrightarrow X^0 \\ &\psi \longrightarrow \frac{p}{\xi}\int_a^b \kappa(\xi,\xi')\psi(1,\xi')\xi' d\xi' \end{aligned}$$

Remark 5.1. [5] If $\kappa(.,.)$ is positive, then the operator $\mathcal{K}$ is positive and $\|\mathcal{K}\| = p$.

We define the bounded collision operator $\mathcal{B}$ by

$$\begin{aligned} \mathcal{B} \ &: \ X \longrightarrow X \\ &\varphi \longrightarrow \mathcal{B}\psi(x,\xi) = -\sigma(x,\xi)\psi(x,\xi) + \int_a^b \kappa(x,\xi,\xi^{'})\psi(x,\xi^{'})d\xi^{'}, \end{aligned}$$

where the kernels $k : [0,1] \times [a,b] \times [a,b] \longrightarrow \mathbb{R}$ is assumed to be measurable. In the following we will make the assumption:

$$(\mathcal{H}) \begin{cases} \text{The function } \mathcal{B}(.) \text{ is strongly measurable. i.e., there is a compact subset} \\ \mathcal{O} \subseteq \mathcal{L}(L_1([a,b],dy)) \text{ such that } \mathcal{B}(x) \in \mathcal{O} \text{ , i.e., on } [0,1], \text{ and } \mathcal{B}(x) \in \mathcal{K}(L_1([a,b],dy)) \\ \text{where, } \mathcal{K}(L_1([a,b],dy)) \text{ denotes the set of all compact operators on } L_1([a,b],dy). \end{cases}$$

Definition 5.1. The operator $\mathcal{B}$ is said to be regular if it satisfies the assumption $(\mathcal{H})$.

Each operator $\mathcal{T}_{\mathcal{K}}$ is defined by

$$\begin{cases} \mathcal{T}_{\mathcal{K}} : \mathcal{D}(\mathcal{T}_{\mathcal{K}}) \subseteq X \longrightarrow X \\ \qquad\qquad \psi \longrightarrow \mathcal{T}_{\mathcal{K}}\psi(x,\xi) = -\xi \dfrac{\partial \psi}{\partial x}(x,\xi) \\ \mathcal{D}(\mathcal{T}_{\mathcal{K}}) = \big\{\psi \in \mathcal{W}_1 : \ \psi^0 = \mathcal{K}\psi^1, \big\}. \end{cases}$$

where $\psi^0 = \psi(0,\xi)$, $\psi^1 = \psi(1,\xi)$ and $\xi \in [a,b]$.

Let $\varphi \in X$, $\lambda \in \mathbb{C}$ and consider the resolvent equation for $\mathcal{T}_{\mathcal{K}}$

$$(\lambda - \mathcal{T}_{\mathcal{K}})\psi = \varphi$$

where φ is a given element of X and the unknown ψ must be sought in $\mathcal{D}(\mathcal{T}_{\mathcal{K}})$.
Let λ^*, be the real defined by

$$\lambda^* := \lim \inf_{(x,\xi)\longrightarrow(0,0)} \sigma(x,\xi).$$

For $\mathrm{Re}\lambda + \lambda^* > 0$, solution is formally by

$$\psi(x,\xi) = \psi(0,\xi)\, e^{-\int_0^x \frac{(\lambda+\sigma(x',\xi))dx'}{|\xi|}} + \frac{1}{\xi}\int_0^x e^{\int_{x'}^x \frac{-(\lambda+\sigma(\tau,\xi))d\tau}{\xi}} \varphi(x',\xi)\, dx' \tag{5.1}$$

and therefore

$$\psi(1,\xi) = \psi(0,\xi)\, e^{-\int_0^1 \frac{(\lambda+\sigma(x',\xi))dx'}{|\xi|}} + \frac{1}{\xi}\int_0^1 e^{\int_{x'}^1 \frac{-(\lambda+\sigma(\tau,\xi))d\tau}{\xi}} \varphi(x',\xi)\, dx'.$$

In order to clarify our subsequent analysis, we introduce the following bounded operators:

$$\begin{cases} P_\lambda : X^0 \to X^1, \\ \psi \to (P_\lambda\psi)(0,\xi) := \psi(0,\xi)\, e^{-\int_0^1 \frac{(\lambda+\sigma(x',\xi))dx'}{|\xi|}}, \end{cases}$$

and

$$\begin{cases} Q_\lambda : X^0 \to X, \\ \psi \to (Q_\lambda\psi)(0,\xi) := \psi(0,\xi)\, e^{-\int_0^x \frac{(\lambda+\sigma(x',\xi))dx'}{|\xi|}}. \end{cases}$$

Let Π_λ and R_λ the following operators :

$$\begin{cases} \Pi_\lambda : X \to X^1, \\ \psi \to (\Pi_\lambda\varphi)(x,\xi) := \dfrac{1}{\xi}\displaystyle\int_0^1 e^{\int_{x'}^1 \frac{-(\lambda+\sigma(\tau,\xi))d\tau}{\xi}} \varphi(x',\xi)\, dx', \end{cases}$$

and

$$\begin{cases} R_\lambda : X \to X, \\ \psi \to (R_\lambda\varphi)(0,\xi) := \dfrac{1}{\xi}\displaystyle\int_0^x e^{\int_{x'}^x \frac{-(\lambda+\sigma(\tau,\xi))d\tau}{\xi}} \varphi(x',\xi)\, dx'. \end{cases}$$

The operators P_λ, Q_λ, Π_λ and R_λ are bounded on their respective spaces. In fact, their norms are bounded above, respectively, by $e^{\frac{-1}{b(Re\lambda+\lambda^*)}}$, $[Re\lambda+\lambda^*]^{-1}$, 1 and $[Re\lambda+\lambda^*]^{-1}$.

It follows from estimate of P_λ that, for $Re\lambda > \lambda^*$, $\|P_\lambda \mathcal{K}\| < 1$ and consequently

$$\psi^o = \sum_{n\geq 0} (P_\lambda \mathcal{K})^n \Pi_\lambda \varphi. \tag{5.2}$$

On the other hand, Eq. (5.1) can be rewritten in the form

$$\psi = Q_\lambda \mathcal{K} \psi^o + R_\lambda \varphi.$$

Substituting Eq. (5.2) into the above equation we get

$$\psi = \sum_{n\geq 0} Q_\lambda \mathcal{K} (P_\lambda \mathcal{K})^n \Pi_\lambda \varphi + R_\lambda \varphi.$$

Finally, the resolvent set of the operator $\mathcal{T}_\mathcal{K}$ contains $\{\lambda \in \mathbb{C}$ such that $Re\lambda > \lambda^*\}$ and for $Re\lambda > \lambda^*$ we have

$$(\lambda - \mathcal{T}_\mathcal{K})^{-1} = \sum_{n\geq 0} Q_\lambda \mathcal{K} (P_\lambda \mathcal{K})^n \Pi_\lambda + R_\lambda. \tag{5.3}$$

In view (5.3) we have

$$\mathcal{B}(\lambda - \mathcal{T}_\mathcal{K})^{-1}\mathcal{B} = \sum_{n\geq 0} \mathcal{B} Q_\lambda \mathcal{K} (P_\lambda \mathcal{K})^n \Pi_\lambda \mathcal{B} + \mathcal{B} R_\lambda \mathcal{B}. \tag{5.4}$$

It will know from [8, Lemma 3.2]) that

$$\mathcal{O}_\lambda := \Big\{ \lambda \in \mathbb{C} : Re\lambda \leq -\lambda^* \Big\} = \sigma_{eap}(\mathcal{T}_\mathcal{K}).$$

Theorem 5.1. *Let $\varepsilon > 0$ and $\kappa(.,.)$ be positive. Then there exists $\delta > 0$ such that*

$$\mathcal{O}_\lambda \subset \sigma_{eap,\varepsilon}(\mathcal{T}_\mathcal{K}, \mathcal{B}, \mathcal{B}) \quad \subset \quad \Theta_\lambda$$

and

$$\mathcal{O}_\lambda \subset \sigma_{e\delta,\varepsilon}(\mathcal{T}_\mathcal{K}, \mathcal{B}, \mathcal{B}) \quad \subset \quad \Theta_\lambda$$

where, $\Theta_\lambda := \Big\{ \lambda \in \mathbb{C} : Re\lambda \leq -\lambda^* + \varepsilon \|\mathcal{B}\|^2 \left(\frac{p}{1-p\delta} + 1 \right) \Big\}$.

Proof. Let $\lambda \in \mathbb{C}$ such that $Re\lambda > \lambda^*$. By using Eq. (5.4) we have

$$\begin{aligned} \|\mathcal{B}(\lambda-\mathcal{T}_{\mathcal{K}})^{-1}\mathcal{B}\| &\leq \|\mathcal{B}\|^2\|Q_\lambda\|\|\mathcal{K}\|\frac{\|\Pi_\lambda\|}{1-\|P_\lambda\mathcal{K}\|}+\|\mathcal{B}\|^2\|R_\lambda\| \\ &\leq \frac{\|\mathcal{B}\|^2}{Re\lambda+\lambda^*}\left(\frac{\|\mathcal{K}\|}{1-\|P_\lambda\|\|\mathcal{K}\|}+1\right) \\ &\leq \frac{\|\mathcal{B}\|^2}{Re\lambda+\lambda^*}\left(\frac{\|\mathcal{K}\|}{1-e^{\frac{-1}{b(Re\lambda+\lambda^*)}}\|\mathcal{K}\|}+1\right). \end{aligned}$$

But, there exists $\delta > 0$ such that $e^{\frac{-1}{b(Re\lambda+\lambda^*)}} \leq \delta$, then we get

$$\|\mathcal{B}(\lambda-\mathcal{T}_{\mathcal{K}})^{-1}\mathcal{B}\| \leq \frac{\|\mathcal{B}\|^2}{Re\lambda+\lambda^*}\left(\frac{p}{1-p\delta}+1\right).$$

Now, let $\lambda \in \sigma_{eap,\varepsilon}(\mathcal{T}_{\mathcal{K}},\mathcal{B},\mathcal{B})$. Since $\|\mathcal{B}(\lambda-\mathcal{T}_{\mathcal{K}})^{-1}\mathcal{B}\| > \frac{1}{\varepsilon}$,

$$\begin{aligned} \frac{1}{\varepsilon} &\leq \frac{\|\mathcal{B}\|^2}{Re\lambda+\lambda^*}\left(\frac{p}{1-p\delta}+1\right) \\ Re\lambda &\leq -\lambda^*+\varepsilon\|\mathcal{B}\|^2\left(\frac{p}{1-p\delta}+1\right). \end{aligned}$$

We obtain that

$$\sigma_{eap,\varepsilon}(\mathcal{T}_{\mathcal{K}},\mathcal{B},\mathcal{B}) \subset \left\{\lambda \in \mathbb{C} : Re\lambda \leq -\lambda^*+\varepsilon\|\mathcal{B}\|^2\left(\frac{p}{1-p\delta}+1\right)\right\}.$$

On the other hand, from [8, Lemma 3.2], we have

$$\left\{\lambda \in \mathbb{C} : Re\lambda \leq -\lambda^*\right\} := \sigma_{eap}(\mathcal{T}_{\mathcal{K}}) \subset \sigma_{eap,\varepsilon}(\mathcal{T}_{\mathcal{K}},\mathcal{B},\mathcal{B}).$$

The proof of the second part of this theorem is of the same way that the first part. □

THEOREM 5.2. *Let $\varepsilon > 0$ and $0 \in \rho(\mathcal{B})$.*

(i) $\sigma_{ap,\varepsilon}(\mathcal{T}_{\mathcal{K}},\mathcal{B}^{-1},\mathcal{B}) \subset \sigma_{ap,\varepsilon}(\mathcal{A}_{\mathcal{K}},\mathcal{B}^{-1},\mathcal{B})$.

(ii) $\sigma_{\delta,\varepsilon}(\mathcal{T}_{\mathcal{K}},\mathcal{B}^{-1},\mathcal{B}) \subset \sigma_{\delta,\varepsilon}(\mathcal{A}_{\mathcal{K}},\mathcal{B}^{-1},\mathcal{B})$.

Proof. *(i)* Let $\lambda \in \sigma_{ap,\varepsilon}(\mathcal{T}_{\mathcal{K}},\mathcal{B}^{-1},\mathcal{B})$. Then, exists a sequence $(\psi_n)_{n\in\mathbb{N}} \in \mathcal{D}(\mathcal{T}_{\mathcal{K}})$ such that

$$\|\psi_n\| = 1 \text{ and } \|\mathcal{B}^{-1}(\lambda-\mathcal{T}_{\mathcal{K}})\mathcal{B}\psi_n\| \leq \varepsilon.$$

Put $\varphi_{n,k}(x,\xi) = \psi_n(x,\xi)e^{-ik\xi}$. We claim that $\|\mathcal{B}^{-1}(\lambda - \mathcal{T}_{\mathcal{K}} - \mathcal{B})\mathcal{B}\varphi_{n,k}(x,\xi)\| \leq \varepsilon$. Indeed,

$$\begin{array}{rcl} \|\mathcal{B}^{-1}(\lambda - \mathcal{T}_{\mathcal{K}} - \mathcal{B})\mathcal{B}\varphi_{n,k}\| & \leq & \|\mathcal{B}^{-1}(\lambda - \mathcal{T}_{\mathcal{K}})\mathcal{B}\varphi_{n,k}\| + \|\mathcal{B}^{-1}\mathcal{B}\mathcal{B}\varphi_{n,k}\| \\ & \leq & \varepsilon + \|\mathcal{B}\varphi_{n,k}\|. \end{array}$$

Let us prove that, $\|\mathcal{B}\varphi_{n,k}\| \to 0$ as $k \to \infty$. Since $\mathcal{B}$ is regular, $\mathcal{B}$ can be approximated, in the uniform topology by a sequence $\mathcal{B}_m$ of finite rank operator on $L_1([a,b],dv)$ which converges, in the operator norm, to B. Then it suffices to establish the result for a finite rank operator, that is

$$\kappa_m(x,\xi,\xi') = \sum_{i=0}^{m} \alpha_i(x) f_i(\xi) g_i(\xi')$$

where $\alpha_i(x) \in L_\infty([a,b],dv)$, $f_i(\xi) \in L_1([a,b],dv)$ and $g_i(\xi') \in L_\infty([a,b],dv)$.

$$\begin{array}{rcl} \|\mathcal{B}\varphi_{n,k}\| & = & \|\mathcal{B}\varphi_{n,k} - \mathcal{B}_m\varphi_{n,k}\| + \|\mathcal{B}_m\varphi_{n,k}\| \\ & = & \|B - B_m\| + \|B_m\varphi_{n,k}\| \end{array}$$

It is sufficient to prove that $\|\mathcal{B}_m\varphi_{n,k}\| \to 0$ as $k \to \infty$. According to Riemann-Lebesgue Lemma and the Dominated Convergence Theorem, it is not difficult to see that

$$\|\mathcal{B}_m\varphi_{n,k}\| \to 0 \text{ as } k \to \infty.$$

In particular, there exists $(k_n)_{n\in\mathbb{N}} \subset \mathbb{Z}$ such that $\|\mathcal{B}_m\varphi_{n,k_n}\| \leq \frac{1}{n}$. Therefore

$$\begin{array}{rcl} \|\mathcal{B}^{-1}(\lambda - \mathcal{T}_{\mathcal{K}} - \mathcal{B})\mathcal{B}\varphi_{n,k_n}\| & \leq & \|\mathcal{B}^{-1}(\lambda - \mathcal{T}_{\mathcal{K}})\mathcal{B}\varphi_{n,k_n}\| + \|\mathcal{B}\varphi_{n,k_n}\| \\ & \leq & \varepsilon + \dfrac{1}{n} \to \varepsilon \text{ as } n \to \infty. \end{array}$$

Then, $\lambda \in \sigma_{ap,\varepsilon}(\mathcal{T}_{\mathcal{K}} + \mathcal{B}, \mathcal{B}^{-1}, \mathcal{B})$.

(ii) The proof of (ii) is the same way by noting that the dual of a regular collision operator is still regular. □

Corollary 5.1. *Let $\varepsilon > 0$ and we assume that the collision operator is regular.*

(i) $\sigma_{ap,\varepsilon}(\mathcal{T}_{\mathcal{K}}, I, I) \subset \sigma_{ap,\varepsilon}(\mathcal{A}_{\mathcal{K}}, I, I)$.

(ii) $\sigma_{\delta,\varepsilon}(\mathcal{T}_{\mathcal{K}}, I, I) \subset \sigma_{\delta,\varepsilon}(\mathcal{A}_{\mathcal{K}}, I, I)$.

THEOREM 5.3. *Let $\varepsilon > 0$ and we assume that the collision operator is regular.*

(i) $\sigma_{eap,\varepsilon}(\mathcal{T}_{\mathcal{K}}, \mathcal{B}^{-1}, \mathcal{B}) \subset \sigma_{eap,\varepsilon}(\mathcal{A}_{\mathcal{K}}, \mathcal{B}^{-1}, \mathcal{B})$.

(ii) $\sigma_{e\delta,\varepsilon}(\mathcal{T}_{\mathcal{K}}, \mathcal{B}^{-1}, \mathcal{B}) \subset \sigma_{e\delta,\varepsilon}(\mathcal{A}_{\mathcal{K}}, \mathcal{B}^{-1}, \mathcal{B})$.

Proof. (i) Using Theorem 5.2 we have for all compact operator $\mathcal{S}$

$$\sigma_{ap,\varepsilon}(\mathcal{T}_{\mathcal{K}} + \mathcal{S} - \mathcal{S}, \mathcal{B}^{-1}, \mathcal{B}) \subset \sigma_{ap,\varepsilon}(\mathcal{T}_{\mathcal{K}} + \mathcal{B} + \mathcal{S} - \mathcal{S}, \mathcal{B}^{-1}, \mathcal{B}).$$

Then,

$$\begin{aligned}
&\sigma_{eap,\varepsilon}(\mathcal{T}_{\mathcal{K}} + \mathcal{S}, \mathcal{B}^{-1}, \mathcal{B})\\
= &\bigcap_{-\mathcal{S}\in\mathcal{K}(X)} \sigma_{ap,\varepsilon}(\mathcal{T}_{\mathcal{K}} + \mathcal{S} - \mathcal{S}, \mathcal{B}^{-1}, \mathcal{B})\\
\subset &\bigcap_{-\mathcal{S}\in\mathcal{K}(X)} \sigma_{ap,\varepsilon}(\mathcal{T}_{\mathcal{K}} + \mathcal{B} + \mathcal{S} - \mathcal{S}, \mathcal{B}^{-1}, \mathcal{B}) = \sigma_{eap,\varepsilon}(\mathcal{T}_{\mathcal{K}} + \mathcal{B} + \mathcal{S}, \mathcal{B}^{-1}, \mathcal{B}).
\end{aligned}$$

From Remark 3.1-(iv), we deduce that $\sigma_{eap,\varepsilon}(\mathcal{T}_{\mathcal{K}}, \mathcal{B}^{-1}, \mathcal{B}) \subset \sigma_{eap,\varepsilon}(\mathcal{T}_{\mathcal{K}} + \mathcal{B}, \mathcal{B}^{-1}, \mathcal{B})$.

(ii) The proof of (ii) is similar to (i). □

COROLLARY 5.2. *Let $\varepsilon > 0$ and we assume that the collision operator is regular.*

(i) $\sigma_{eap,\varepsilon}(\mathcal{T}_{\mathcal{K}}, I, I) \subset \sigma_{eap,\varepsilon}(\mathcal{A}_{\mathcal{K}}, I, I)$.

(ii) $\sigma_{e\delta,\varepsilon}(\mathcal{T}_{\mathcal{K}}, I, I) \subset \sigma_{e\delta,\varepsilon}(\mathcal{A}_{\mathcal{K}}, I, I)$.

6. Final remarks

One of the fundamental ideas investigated in this paper is that of providing conditions under which the structured essential approximate and defect pseudospectrum of closed, densely defined linear operators in a Banach space. has a relationship with Fredholm theory and perturbation theory. This paper shows the relation between structured approximate (defect) pseudospectrum and structured approximate (defect) spectrum also between structured essential approximate (defect) and structured approximate (defect) pseudospectrum. As future research we are trying to extend this concept of structured approximate (defect) pseudospectrum and structured essential approximate (defect) pseudospectrum to the case of multivalued linear relations.

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