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A STUDY ON THE CONTRACTED ES CURVATURE TENSOR IN $g - ESX_n$

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Abstract. This paper is a direct continuation of [1]. In this paper we derive tensorial representations of contracted ES curvature tensors of $g - ESX_n$ and prove several generalized identities involving them. In particular, a variation of the generalized Bianchi's identity in $g - ESX_n$, which has a great deal of useful physical applications, is proved in Theorem (2.9).

1. Preliminaries

This paper is a direct continuation of our previous paper [1], which will be denoted by I in the present paper. All considerations in this paper are based on our results and symbolism of I([1],[2],[3],[4],[5],[6],[7],[8],[9]). Whenever necessary, these results will be quoted in the text. In this section, we introduce a brief collection of basic concepts, notations, and results of I, which are frequently used in the present paper.

(a) Let X_n be a generalized n-dimensional Riemannian manifold referred to a real coordinate system x^ν, which obeys the coordinate transformations $x^\nu \to x^{\nu'}$ for which

$$\det\left(\frac{\partial x'}{\partial x}\right) \neq 0. \tag{1.1}$$

In $n - g - UFT$ the manifold X_n is endowed with a real nonsymmetric tensor $g_{\lambda\mu}$, which may be decomposed into its symmetric

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part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}. \tag{1.2}$$

where

$$\mathfrak{g} = \det(g_{\lambda\mu}) \neq 0, \quad \mathfrak{h} = \det(h_{\lambda\mu}) \neq 0, \quad \mathfrak{k} = \det(k_{\lambda\mu}). \tag{1.3}$$

In virtue of (1.3) we may define a unique tensor $h^{\lambda\nu}$ by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta^\nu_\mu. \tag{1.4}$$

which together with $h_{\lambda\mu}$ will serve for raising and/or lowering indices of tensors in X_n in the usual manner. There exists a unique tensor $*g^{\lambda\nu}$ satisfying

$$g_{\lambda\mu}{}^*g^{\lambda\nu} = g_{\mu\lambda}{}^*g^{\nu\lambda} = \delta^\nu_\mu. \tag{1.5}$$

It may be also decomposed into its symmetric part ${}^*h^{\lambda\nu}$ and skew-symmetric part ${}^*k^{\lambda\nu}$:

$${}^*g^{\lambda\nu} = {}^*h^{\lambda\nu} + {}^*k^{\lambda\nu}. \tag{1.6}$$

The manifold X_n is connected by a general real connection $\Gamma_\lambda{}^\nu{}_\mu$ with the following transformation rule:

$$\Gamma_{\lambda'}{}^{\nu'}{}_{\mu'} = \frac{\partial x^{\nu'}}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x^{\lambda'}}\frac{\partial x^\gamma}{\partial x^{\mu'}}\Gamma_\beta{}^\alpha{}_\gamma + \frac{\partial^2 x^\alpha}{\partial x^{\lambda'}\partial x^{\mu'}}\right). \tag{1.7}$$

It may also be decomposed into its symmetric part $\Lambda_\lambda{}^\nu{}_\mu$ and its skew-symmetric part $S_{\lambda\nu}{}^\nu$, called the torsion tensor of $\Gamma_\lambda{}^\nu{}_\mu$:

$$\Gamma_\lambda{}^\nu{}_\mu = \Lambda_\lambda{}^\nu{}_\mu + S_{\lambda\mu}{}^\nu; \quad \Lambda_\lambda{}^\nu{}_\mu = \Gamma_{(\lambda}{}^\nu{}_{\mu)}; \quad S_{\lambda\mu}{}^\nu = \Gamma_{[\lambda}{}^\nu{}_{\mu]}. \tag{1.8}$$

A connection $\Gamma_\lambda{}^\nu{}_\mu$ is said to be Einstein if it satisfies the following system of Einstein's equations:

$$\partial_\omega g_{\lambda\mu} - \Gamma_\lambda{}^\alpha{}_\omega g_{\alpha\mu} - \Gamma_\omega{}^\alpha{}_\mu g_{\lambda\alpha} = 0. \tag{1.9}$$

or equivalently

$$D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha}. \tag{1.10}$$

where D_ω is the symbolic vector of the covariant derivative with respect to $\Gamma_\lambda{}^\nu{}_\mu$. In order to obtain $g_{\lambda\mu}$ involved in the solution for $\Gamma_\lambda{}^\nu{}_\mu$ in (1.9), certain conditions are imposed. These conditions may be condensed to

$$S_\lambda = S_{\lambda\alpha}{}^\alpha = 0, \quad R_{[\mu\lambda]} = \partial_{[\mu} Y_{\lambda]}, \quad R_{(\mu\lambda)} = 0. \tag{1.11}$$

where Y_λ is an arbitrary vector, and

$$R_{\omega\mu\lambda}{}^{\nu} = 2(\partial_{[\mu}\Gamma_{|\lambda|}{}^{\nu}{}_{\omega]} + \Gamma_{\alpha}{}^{\nu}{}_{[\mu}\Gamma_{|\lambda|}{}^{\alpha}{}_{\omega]}). \tag{1.12}$$

If the system (1.10) admits a solution $\Gamma_\lambda{}^\nu{}_\mu$, it must be of the form (Hlavatý, 1957)

$$\Gamma_\lambda{}^\nu{}_\mu = \left\{ \begin{array}{c} \nu \\ \lambda\mu \end{array} \right\} + S_{\lambda\mu}{}^{\nu} + U^{\nu}{}_{\lambda\mu}. \tag{1.13}$$

where $U^{\nu}{}_{\lambda\mu} = 2h^{\nu\alpha}S_{\alpha(\lambda}{}^{\beta}k_{\mu)\beta}$ and $\left\{ \begin{array}{c} \nu \\ \lambda\mu \end{array} \right\}$ are Christoffel symbols defined by $h_{\lambda\mu}$.

(b) Some notations and results The following quantities are frequently used in our further considerations:

$$g = \frac{\mathfrak{g}}{\mathfrak{h}}, \quad k = \frac{\mathfrak{k}}{\mathfrak{h}}, \tag{1.14}$$

$$K_p = k_{[\alpha_1}{}^{\alpha_1}k_{\alpha_2}{}^{\alpha_2}\cdots k_{\alpha_p]}{}^{\alpha^p}, \quad (p = 0, 1, 2, \cdots), \tag{1.15}$$

$${}^{(0)}k_\lambda{}^\nu = \delta^\nu_\lambda, \ {}^{(p)}k_\lambda{}^\nu = k_\lambda{}^\alpha \ {}^{(p-1)}k_\alpha{}^\nu \quad (p = 1, 2, \cdots). \tag{1.16}$$

In X_n it was proved in [5] that

$$K_0 = 1, \ K_n = k \text{ if } n \text{ is even, and } \mathrm{K_p} = 0 \text{ if p is odd.} \tag{1.17}$$

$$\begin{aligned} \mathfrak{g} &= \mathfrak{h}(1 + K_1 + K_2 + \cdots + K_n) \\ \text{or } g &= 1 + K_1 + K_2 + \cdots + K_n. \end{aligned} \tag{1.18}$$

$$\sum_{s=0}^{n-\sigma} K_s \ {}^{(n-s+p)}k_\lambda{}^\nu = 0 \quad (p = 0, 1, 2, \cdots). \tag{1.19}$$

We also use the following useful abbreviations for an arbitrary vector Y, for $p = 1, 2, 3, \cdots$:

$${}^{(p)}Y_\lambda = {}^{(p-1)} k_\lambda{}^\alpha Y_\alpha, \tag{1.20}$$

$${}^{(p)}Y^\nu = {}^{(p-1)} k^\nu{}_\alpha Y^\alpha. \tag{1.21}$$

(c) n-dimensional ES manifold ESX_n

In this subsection, we display an useful representation of the ES connection in n-g-UFT.

DEFINITION 1.1. *A connection* $\Gamma_\lambda{}^\nu{}_\mu$ *is said to be semi-symmetric if its torsion tensor* $S_{\lambda\mu}{}^\nu$ *is of the form*

$$S_{\lambda\mu}{}^{\nu} = 2\delta^{\nu}_{[\lambda}X_{\mu]} \tag{1.22}$$

for an arbitrary non-null vector X_μ.

A connection which is both semi-symmetric and Einstein is called an ES connection. An n-dimensional generalized Riemannian manifold X_n, on which the differential geometric structure is imposed by $g_{\lambda\mu}$ by means of an ES connection, is called an n-dimensional ES manifold. We denote this manifold by $g - ESX_n$ in our further considerations.

THEOREM 1.2. *Under the condition* (1.22), *the system of equations* (1.10) *is equivalent to*

$$\Gamma_\lambda{}^\nu{}_\mu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + 2k^{\nu}_{(\lambda}X_{\mu)} + 2\delta^{\nu}_{[\lambda}X_{\mu]}. \tag{1.23}$$

Proof. Substituting (1.22) for $S_{\lambda\mu}{}^\nu$ into (1.13), we have the representation (1.23). □

THEOREM 1.3. *In* $g - ESX_n$, *the following relations hold for* $p, q = 1, 2, 3, \cdots$:

$$S_\lambda = (1-n)X_\lambda, \tag{1.24}$$

$$U_\lambda = \frac{1}{2}\partial_\lambda ln\mathfrak{g}, \tag{1.25}$$

$${}^{(p+1)}S_\lambda = (1-n)^{(p)}U_\lambda, \tag{1.26}$$

$${}^{(p)}U_\alpha{}^{(q)}X^\alpha = 0 \qquad \textit{if} \quad p+q-1 \quad \textit{is} \quad \textit{odd}, \tag{1.27}$$

$$D_\lambda X_\mu = \nabla_\lambda X_\mu, \tag{1.28}$$

$$D_{[\lambda}X_{\mu]} = \nabla_{[\lambda}X_{\mu]} = \partial_{[\lambda}X_{\mu]}, \tag{1.29}$$

$$\nabla_{[\lambda}U_{\mu]} = 0, \qquad D_{[\lambda}U_{\mu]} = 2U_{[\lambda}X_{\mu]} = 2^{(2)}X_{[\lambda}X_{\mu]}, \tag{1.30}$$

where ∇_ω *is the symbolic vector of the covariant derivative with respect to the Christoffel symbols defined by* $h_{\lambda\mu}$.

THEOREM 1.4. *In $g - ESX_n$ under the present conditions, the ES curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ may be given by*

$$R_{\omega\mu\lambda}{}^{\nu} = L_{\omega\mu\lambda}{}^{\nu} + M_{\omega\mu\lambda}{}^{\nu} + N_{\omega\mu\lambda}{}^{\nu}, \tag{1.31}$$

where

$$L_{\omega\mu\lambda}{}^{\nu} = 2\left(\partial_{[\mu}\left\{ \begin{matrix} \nu \\ \omega]\lambda \end{matrix} \right\} + \left\{ \begin{matrix} \nu \\ \alpha[\mu \end{matrix} \right\}\left\{ \begin{matrix} \alpha \\ \omega]\lambda \end{matrix} \right\}\right), \tag{1.32}$$

$$M_{\omega\mu\lambda}{}^{\nu} = 2(\delta^{\nu}_{\lambda}\partial_{[\mu}X_{\omega]} + \delta^{\nu}_{[\mu}\nabla_{\omega]}X_{\lambda} + \nabla_{[\mu}U^{\nu}{}_{\omega]\lambda}), \tag{1.33}$$

$$N_{\omega\mu\lambda}{}^{\nu} = 2(\delta^{\nu}_{[\omega}X_{\mu]}X_{\lambda} +^{(2)} X_{\lambda}k_{[\mu}{}^{\nu}X_{\omega]}). \tag{1.34}$$

THEOREM 1.5. *(Generalized Bianchi's identity in $g - ESX_n$) Under the present conditions, the ES curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ of $g - ESX_n$ satisfies the following identity:*

$$D_{[\epsilon}R_{\omega\mu]\lambda}{}^{\nu} = -4X_{[\epsilon}L_{\omega\mu]\lambda}{}^{\nu} + O_{[\epsilon\omega\mu]\lambda}{}^{\nu}, \tag{1.35}$$

where

$$\begin{aligned} \frac{1}{8}O_{\epsilon\omega\mu\lambda}{}^{\nu} &= \delta^{\nu}_{\lambda}X_{\epsilon}\partial_{\omega}X_{\mu} + X_{\epsilon}\delta^{\nu}_{\omega}\nabla_{\mu}X_{\lambda} \\ &+ X_{\epsilon}\nabla_{\omega}U^{\nu}{}_{\mu\lambda} + X_{\epsilon}\delta^{\nu}_{\mu}X_{\omega}X_{\lambda} +^{(2)} X_{\lambda}X_{\epsilon}k_{\omega}{}^{\nu}X_{\mu}. \end{aligned} \tag{1.36}$$

2. The contracted ES curvature tensors in $g - ESX_n$

This section is devoted to the study of the contracted n-dimensional ES curvature tensors, defined by the ES connection in g-UFT under the present conditions, and of some useful identities involving them.

The tensors

$$R_{\mu\lambda} = R_{\alpha\mu\lambda}{}^{\alpha}, \qquad V_{\omega\mu} = R_{\omega\mu\alpha}{}^{\alpha} \tag{2.1}$$

are called *the first and second contracted ES curvature tensors* of the ES connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$, respectively. We see in the following two theorems that they appear as functions of the vectors $X_{\lambda}, S_{\lambda}, U_{\lambda}$, and hence also as functions of $g_{\lambda\mu}$ and its first two derivatives in virtue of (1.24, 25) and (1.31).

Theorem 2.1. *The first contracted ES curvature tensor $R_{\mu\lambda}$ in $g-ESX_n$ may be given by*

$$R_{\mu\lambda} = L_{\mu\lambda} + 2\partial_{[\mu}X_{\lambda]} + \nabla_\mu T_\lambda - \nabla_\alpha U^\alpha{}_{\mu\lambda} + (n-1)X_\mu X_\lambda + U_\mu U_\lambda, \tag{2.2}$$

where

$$L_{\mu\lambda} = L_{\alpha\mu\lambda}{}^\alpha, \tag{2.3}$$

$$T_{\lambda\mu}{}^\nu = S_{\lambda\mu}{}^\nu + U^\nu{}_{\lambda\mu}, \qquad T_\lambda = T_{\lambda\alpha}{}^\alpha = S_\lambda + U_\lambda. \tag{2.4}$$

Proof. Putting $\omega = \nu = \alpha$ in (1.31) and making use of (2.3), we have

$$R_{\mu\lambda} = L_{\mu\lambda} + M_{\alpha\mu\lambda}{}^\alpha + N_{\alpha\mu\lambda}{}^\alpha. \tag{2.5}$$

In virtue of (1.24, 25), it follows from (1.33) that

$$\begin{aligned} M_{\alpha\mu\lambda}{}^\alpha &= 2\partial_{[\mu}X_{\lambda]} + (1-n)\nabla_\mu X_\lambda + \nabla_\mu U_\lambda - \nabla_\alpha U^\alpha{}_{\mu\lambda} \\ &= 2\partial_{[\mu}X_{\lambda]} + \nabla_\mu T_\lambda - \nabla_\alpha U^\alpha{}_{\mu\lambda}. \end{aligned} \tag{2.6}$$

On the other hand, in virtue of (1.25) the relation (1.34) gives

$$\begin{aligned} N_{\alpha\mu\lambda}{}^\alpha &= (n-1)X_\mu X_\lambda + {}^{(2)}X^{(2)}_\mu X_\lambda - {}^{(2)}X_\lambda X_\mu k_\alpha{}^\alpha \\ &= (n-1)X_\mu X_\lambda + U_\mu U_\lambda. \end{aligned} \tag{2.7}$$

Our assertion follows immediately from (2.5), (2.6) and (2.7). □

Theorem 2.2. *The second contracted ES curvature tensor $V_{\omega\mu}$ in $g-ESX_n$ is a curl of the vector S_λ. That is,*

$$V_{\omega\mu} = 2\partial_{[\omega}S_{\mu]}. \tag{2.8}$$

Proof. Putting $\lambda = \nu = \alpha$ in (1.31), we have

$$V_{\omega\mu} = L_{\omega\mu\alpha}{}^\alpha + M_{\omega\mu\alpha}{}^\alpha + N_{\omega\mu\alpha}{}^\alpha. \tag{2.9}$$

In virtue of (1.11) and (1.24, 25, 30), the relations (1.32, 33, 34) give

$$L_{\omega\mu\alpha}{}^\alpha = N_{\omega\mu\alpha}{}^\alpha = 0$$

$$M_{\omega\mu\alpha}{}^\alpha = 2(1-n)\partial_{[\omega}X_{\mu]} + 2\nabla_{[\mu}U_{\omega]} = 2(1-n)\partial_{[\omega}X_{\mu]} = 2\partial_{[\omega}S_{\mu]}$$

which together with (2.9) proves our assertion. □

THEOREM 2.3. *The tensor $R_{\mu\lambda}$ is symmetric when $n = 3$.*

Proof. The relation (2.2) may be written as

$$R_{\mu\lambda} = L_{\mu\lambda} + (3-n)\nabla_\mu X_\lambda - 2\nabla_{(\mu} X_{\lambda)} + \nabla_\mu U_\lambda - \nabla_\alpha U^\alpha{}_{\mu\lambda} + (n-1)X_\mu X_\lambda + U_\mu U_\lambda, \tag{2.10}$$

where use has been made of (1.24, 29) and (2.4). Hence, in virtue of (1.29, 30) we have $R_{[\mu\lambda]} = 0$ if and only if $(3-n)\nabla_{[\mu} X_{\lambda]} = (3-n)\partial_{[\mu} X_{\lambda]} = 0$.

□

REMARK 2.4. *In the proof of the Theorem (2.3), we excluded the case that $\partial_{[\mu} X_{\lambda]} = 0$, because we assumed that X_λ is not a gradient vector in the definition of semi-symmetric connection in (1.22). In fact, the assumption that X_λ is not a gradient vector is essential in the discussions of the field equations in $g - ESX_n$.*

THEOREM 2.5. *The contracted ES curvature tensors in $g - ESX_n$ are related by*

$$2R_{[\mu\lambda]} = 4\partial_{[\mu} X_{\lambda]} + V_{\mu\lambda}. \tag{2.11}$$

Proof. In virtue of (1.24, 29, 30), the relation (2.11) may be proved from (2.10) as in the following way:

$$\begin{aligned} 2R_{[\mu\lambda]} &= 2(3-n)\partial_{[\mu} X_{\lambda]} \\ &= 2(1-n)\partial_{[\mu} X_{\lambda]} + 4\partial_{[\mu} X_{\lambda]} \\ &= 2\partial_{[\mu} S_{\lambda]} + 4\partial_{[\mu} X_{\lambda]} \\ &= V_{\mu\lambda} + 4\partial_{[\mu} X_{\lambda]}. \end{aligned} \tag{2.12}$$

□

Our next task is to obtain a generalization of the classical identity

$$\nabla_\alpha E_\mu{}^\alpha = 0, \tag{2.13}$$

where

$$L = h^{\alpha\beta} L_{\alpha\beta}, \qquad E_\mu{}^\nu = L_\mu{}^\nu - \frac{1}{2}\delta_\mu^\nu L. \tag{2.14}$$

REMARK 2.6. *The tensor $E_\mu{}^\nu$ is called the Einstein tensor. This tensor has a great deal of applications in physics. It is of fundamental importance since its divergence vanishes identically as we see in* (2.13).

In our further considerations, the quantities

$$R = h^{\alpha\beta} R_{\alpha\beta}, \qquad G_\mu{}^\nu = R_\mu{}^\nu - \frac{1}{2}\delta_\mu^\nu R \tag{2.15}$$

will be referred to *ES curvature invariant and ES Einstein tensor of* $g{-}ESX_n$, respectively. The tensor $G_\mu{}^\nu$ is the generalized concept of $E_\mu{}^\nu$. First of all, we need the following two theorems in order to generalize the identity (2.13) in $g - ESX_n$.

THEOREM 2.7. *In $g - ESX_n$, we have*

$$D_\omega h^{\lambda\mu} = 2X^{(\lambda} g_\omega^{\mu)} - 2X_\omega h^{\lambda\mu}. \tag{2.16}$$

Proof. Substituting (1.22) into (1.10) for $S_{\omega\alpha}{}^\nu$ and making use of (1.2) and (1.21), the relations (2.16) follows as in the following way:

$$\begin{aligned} D_\omega h^{\lambda\mu} &= 2S_{\omega(\alpha}{}^\gamma g_{\beta)\gamma} h^{\lambda\alpha} h^{\mu\beta} \\ &= 2(\delta^\gamma_{[\omega} X_{\alpha]} g_{\beta\gamma} + \delta^\gamma_{[\omega} X_{\beta]} g_{\alpha\gamma}) h^{\lambda\alpha} h^{\mu\beta} \\ &= 2(g_{\beta[\omega} X_{\alpha]} + g_{\alpha[\omega} X_{\beta]}) h^{\lambda\alpha} h^{\mu\beta} \\ &= 2X^{(\lambda} g_\omega^{\mu)} - 2X_\omega h^{\lambda\mu}. \end{aligned}$$

□

THEOREM 2.8. *In $g - ESX_n$, we have*

$$\begin{aligned} R = L + (1-n)\nabla_\alpha X^\alpha &+ \nabla_\alpha U^\alpha + (n-1)X \\ &+ U - \nabla_\gamma U^\gamma{}_{\alpha\beta}, \end{aligned} \tag{2.17}$$

$$D_\alpha R_\mu{}^\alpha = \nabla_\alpha R_\mu{}^\alpha + (U_\alpha - nX_\alpha) R_\mu{}^\alpha + RX_\mu - U^\alpha R_{\alpha\mu}, \tag{2.18}$$

where

$$X = X_\alpha X^\alpha, \qquad U = U_\alpha U^\alpha. \tag{2.19}$$

Proof. In virtue of (2.14), (2.15) and (1.24, 29), the representation (2.17) follows from (2.2). On the other hand, the representation (2.18) may be proved as in the following way in virtue of (1.13), (1.22), (1.24), (1.25) and (2.15):

$$\begin{aligned}
D_\alpha R_\mu{}^\alpha &= \partial_\alpha R_\mu{}^\alpha + \Gamma^\alpha_{\beta\alpha} R_\mu{}^\beta - \Gamma^\beta_{\mu\alpha} R_\beta{}^\alpha \\
&= \nabla_\alpha R_\mu{}^\alpha + (S_\beta + U_\beta) R_\mu{}^\beta - S_{\mu\alpha}{}^\beta R_\beta{}^\alpha - U^\beta{}_{\mu\alpha} R_\beta{}^\alpha \\
&= \nabla_\alpha R_\mu{}^\alpha + (1-n) X_\alpha + U_\alpha R_\mu{}^\alpha + 2\delta^\beta_{[\alpha} X_{\mu]} R_\beta{}^\alpha - U^\beta{}_{\mu\alpha} R_\beta{}^\alpha \\
&= \nabla_\alpha R_\mu{}^\alpha + (U_\alpha - nX_\alpha) R_\mu{}^\alpha + R X_\mu - U^\beta{}_{\mu\alpha} R_\beta{}^\alpha \\
&= \nabla_\alpha R_\mu{}^\alpha + (U_\alpha - nX_\alpha) R_\mu{}^\alpha + R X_\mu - U^\alpha R_{\alpha\mu}.
\end{aligned}$$

□

Now we are ready to prove the following generalization of (2.13).

THEOREM 2.9. *(A variation of the generalized Bianchi's identity in $g-ESX_n$). The ES Einstein tensor $G_\mu{}^\nu$ satisfies the following identity in $g-ESX_n$:*

$$D_\alpha G_\mu{}^\alpha = P_\mu - \frac{1}{2}\partial_\mu Q, \tag{2.20}$$

where

$$P_\mu = \nabla_\alpha (R_\mu{}^\alpha - L_\mu{}^\alpha) + (U_\alpha - nX_\alpha) R_\mu{}^\alpha + R X_\mu - U^\alpha R_{\alpha\mu}, \tag{2.21}$$

$$Q = (1-n)\nabla_\alpha X^\alpha + \nabla_\alpha U^\alpha + U + (n-1)X - U_\gamma U^\gamma{}_{\alpha\beta}. \tag{2.22}$$

Proof. The relation (2.15) gives

$$\begin{aligned}
D_\alpha G_\mu{}^\alpha &= D_\alpha (R_\mu{}^\alpha - \frac{1}{2}\delta^\alpha_\mu R) \\
&= \nabla_\alpha (R_\mu{}^\alpha - L_\mu{}^\alpha) + (U_\alpha - nX_\alpha) R_\mu{}^\alpha + R X_\mu - U^\alpha R_{\alpha\mu} \\
&\quad - \frac{1}{2}\partial_\mu [(1-n)\nabla_\alpha X^\alpha + \nabla_\alpha U^\alpha + U + (n-1)X - U_\gamma U^\gamma{}_{\alpha\beta}].
\end{aligned} \tag{2.23}$$

The proof of the identity (2.20) immediately follows by substituting (2.17, 18) into (2.23) and making use of (2.21, 22). □

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