Korean J. Math. **28** (2020), No. 3, pp. 459–481
http://dx.doi.org/10.11568/kjm.2020.28.3.459

RELATIVE L-ORDER OF AN ENTIRE FUNCTION

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Abstract. In this paper we introduce relative $L-$order of a nonconstant entire function f with respect to another nonconstant entire function g. Also we investigate the existence of relative $L-$proximate order of f with respect to g.

1. Introduction

Let f be a nonconstant entire function defined on $\mathbb{C}$. Then the maximum modulus function $M_f(r)$ of f, defined by $M_f(r) = \max_{|z|=r} |f(z)|$ is continuous and strictly increasing. In such case the inverse function $M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is also continuous, strictly increasing and $\lim_{s\to\infty} M_f^{-1}(s) = \infty$. The growth of an entire function f is generally measured by its order and type.

In 1988, Luis Bernal [1] introduced the order of growth of a nonconstant entire function f relative to another entire function g, which is defined by

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu), \text{ for all } r > r_0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1}(M_f(r))}{\log r}. \end{aligned}$$

Received February 14, 2020. Revised July 2, 2020. Accepted August 29, 2020.
2010 Mathematics Subject Classification: 30D20, 30D30, 30D35.
Key words and phrases: entire function, property (A), relative $L-$order, relative $L-$proximate order.
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In 1988, D. Somasundaram and R. Thamizharasi [4] introduced the $L-$ order of an entire function f, defined by

$$\rho_L = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log(rL(r))},$$

where $L(r)$ is a positive continuous function, increasing slowly i.e. $L(ar) \sim L(r)$ as $r \to \infty$, for all $a > 0$, given by Singh and Bekar [3]. The function, $L(r)$ is called slowly increasing function.

In 1923, Valiron [5] initiated the terminology and generalized the concept of proximate order and in 1946, S.M. Shah [2] defined it in more justified form and gave a simple proof of its existence.

In this paper we introduce relative $L-$order of a nonconstant entire function f with respect to another nonconstant entire function g. Also we investigate the existence of relative $L-$proximate order of f with respect to g.

2. Basic definitions and preliminary lemmas

Here we give some definitions and lemmas.

Definition 2.1. Let f be a nonconstant entire function. We say that f satisfies the property (A) if and only if for each $\sigma > 1$,

$$M_f(r)^2 \leq M_f(r^\sigma)$$

exists.

For example, $f(z) = \exp(z)$ satisfies property (A). But no polynomial satisfies property (A). Moreover, there are some transcendental functions which do not satisfy property (A).

Lemma 2.2. [1] *If f is a nonconstant entire function, then f satisfies the property (A) if and only if for each $\sigma > 1$ and positive integer n,*

$$M_f(r)^n \leq M_f(r^\sigma), \text{ for all } r > 0.$$

Lemma 2.3. [1] *Let f be a nonconstant entire function, $\alpha > 1, 0 < \beta < \alpha, s > 1, 0 < \mu < \lambda$ and n be a positive integer. Then*

a) $M_f(\alpha r) > \beta M_f(r)$,

b) *There exists $K = K(s, f) > 0$ such that*

$$f(r)^s \leq K M_f(r^s), \text{ for all } r > 0,$$

c) $\lim_{r\to\infty} \frac{M_f(r^s)}{M_f(r)} = \infty = \lim_{r\to\infty} \frac{M_f(r^\lambda)}{M_f(r^\mu)}$,

d) *If f is transcendental, then* $\lim_{r\to\infty} \frac{M_f(r^s)}{r^n M_f(r)} = \infty = \lim_{r\to\infty} \frac{M_f(r^\lambda)}{r^n M_f(r^\mu)}$.

Lemma 2.4. [1] *Suppose that f and g are entire functions, $f(0) = 0$ and $h = g \circ f$. Then there exists $c \in (0, 1)$, independent of f and g, such that*

$$M_h(r) > M_g\left(cM_f\left(\frac{r}{2}\right)\right), \text{ for all } r > 0.$$

Lemma 2.5. [1] *Let $R > 0, \eta \in \left(0, \frac{3e}{2}\right)$ and f is analytic in $|z| \leq 2eR$ with $f(0) = 1$. Then on the disc $|z| \leq R$, excluding a family of discs the sum of whose radii is not greater than $4\eta R$,*

$$\log|f(z)| > -T(\eta)\log M_f(2eR),$$

where $T(\eta) = 2 + log\left(\frac{3e}{2\eta}\right)$.

Lemma 2.6. [1] *If f is a nonconstant entire function and* $A(r) = \max\{\operatorname{Re} f(z) : |z| = r\}$, *then*

$$M_f(r) < A(145r).$$

Lemma 2.7. [1] *If f is a nonconstant entire function, then*

$$T(r) \leq \log^+ M_f(r) \leq \left(\frac{R+r}{R-r}\right) T(r), \text{ for } 0 < r < R.$$

3. Main Results

In this section we defined relative $L-$order of f with respect to g, relative $L-$lower order of f with respect to g and established some theorems related to these. Also we prove the existence of relative $L-$proximate order of f with respect to g.

Definition 3.1 (Relative L-order of f with respect to g). Let f and g be entire functions and $L(r)$ be a positive slowly increasing function. The relative $L-$order of f with respect to g is given by

$$\begin{aligned}\rho_g^L(f) &= \inf\{\mu > 0 : M_f(r) < M_g((rL(r))^\mu), \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1}(M_f(r))}{\log(rL(r))}.\end{aligned}$$

DEFINITION 3.2 (Relative L-lower order of f with respect to g). Let f and g be entire functions and $L(r)$ be a positive slowly increasing function. The relative $L-$lower order of f with respect to g is given by

$$\lambda_g^L(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1}(M_f(r))}{\log(rL(r))}.$$

THEOREM 3.3. *Let f, g, h be nonconstant entire functions and $L_i(i = 1,2,3,4)$ be nonconstant linear functions, i.e. $L_i(z) = a_i z + b_i$, for all $z \in \mathbb{C}$, with $a_i, b_i \in \mathbb{C}, a_i \neq 0 (i = 1,2,3,4)$. Then*

a) If g is a polynomial and f is a transcendental, $\rho_g^L(f) = \infty$,

b) If g is a transcendental and f is a polynomial, $\rho_g^L(f) = 0$,

c) If f and g are polynomials, $\rho_g^L(f) = \frac{\deg(f)}{\deg(g)}$,

d) If $M_f(r) \leq M_g(r)$, $\rho_h^L(f) \leq \rho_h^L(g)$,

e) If $M_g(r) \leq M_h(r)$, $\rho_g^L(f) \geq \rho_h^L(f)$,

f) $\rho^L_{(L_4\circ g\circ L_3)}(L_2 \circ f \circ L_1) = \rho_g^L(f)$.

Proof. **a**) Let the degree of g be n. Then $M_f(r) > Kr^m$ and $M_g(r) \leq K_1 r^n$, where K, K_1 are constant and $m > 0$ be any real number, for sufficiently large r.

Then,

$$\begin{aligned}
\rho_g^L(f) &= \limsup_{r\to\infty} \frac{\log M_g^{-1}(M_f(r))}{\log(rL(r))} > \limsup_{r\to\infty} \frac{\log M_g^{-1}(Kr^m)}{\log(rL(r))} \\
&\geq \limsup_{r\to\infty} \frac{\log\left(\frac{1}{K_1}(Kr^m)^{\frac{1}{n}}\right)}{\log(rL(r))} = \limsup_{r\to\infty} \frac{\log\left(\frac{K^{\frac{1}{n}}}{K_1} r^{\frac{m}{n}}\right)}{\log(rL(r))} \\
&= \limsup_{r\to\infty} \frac{\log \frac{K^{\frac{1}{n}}}{K_1} + \log r^{\frac{m}{n}}}{\log(rL(r))} = \frac{m}{n} \limsup_{r\to\infty} \frac{\log r}{\log(rL(r))}.
\end{aligned}$$

Hence,

$$\rho_g^L(f) > \frac{m}{n}, \text{ for all real } m.$$

Therefore,

$$\rho_g^L(f) = \infty.$$

b) Let the degree of f be n. Then $M_f(r) \leq Kr^n$ and $M_g(r) > K_1 r^m$, where K, K_1 are constant and $m > 0$ be any real number, for sufficiently large r. Then we have for $\mu > 0$ and for sufficiently large r

$$\begin{aligned} M_g((rL(r))^{\mu}) &> K_1((rL(r))^{\mu})^m \\ &= K_1\,(rL(r))^{\mu m} \\ &> Kr^n, \text{ for choosing suitable large } m \\ &\geq M_f(r). \end{aligned}$$

Which implies,

$$\rho_g^L(f) = 0.$$

c) Let $f(z) = a_0 z^m + a_1 z^{m-1} + ... + a^m, a_0 \neq 0$ and $g(z) = b_0 z^n + b_1 z^{n-1} + ... + b^n, b_0 \neq 0$. Then $M_f(r) \leq K_1 r^m$ and $M_g(r) > \frac{1}{2}\left|b_0\right| r^n$, where K_1 is a constant, for sufficiently large r.

Then,

$$\begin{aligned} \rho_g^L(f) &= \limsup_{r\to\infty} \frac{\log M_g^{-1}(M_f(r))}{\log(rL(r))} \leq \limsup_{r\to\infty} \frac{\log\left(\frac{2K_1 r^m}{|b_0|}\right)^{\frac{1}{n}}}{\log(rL(r))} \\ &= \limsup_{r\to\infty} \frac{\log\left(\frac{2K_1}{|b_0|}\right)^{\frac{1}{n}} + \log r^{\frac{m}{n}}}{\log(rL(r))} = \frac{m}{n}\limsup_{r\to\infty} \frac{\log r}{\log(rL(r))}. \end{aligned}$$

Hence,

$$\rho_g^L(f) \leq \frac{m}{n}.$$

Again we can write, $M_f(r) > \frac{1}{2}\left|a_0\right| r^m$ and $M_g(r) \leq K_2 r^n$, where K_2 is a constant, for sufficiently large r.

Then interchanging the role of $M_f(r)$ and $M_g(r)$, we get

$$\rho_g^L(f) \geq \frac{m}{n}.$$

Thus,

$$\rho_g^L(f) = \frac{m}{n} = \frac{\deg(f)}{\deg(g)}.$$

□

3.1. Relative L-order of composition. The following theorem solves the problem of the relative $L-$order on the composition of entire functions.

THEOREM 3.4. *Let f, f_1, f_2, g and m be nonconstant entire functions and $h = g \circ f$, then*
a) $\rho^L_{g\circ f_2}(g \circ f_1) = \rho^L_{f_2}(f_1)$,
b) $\max\{\rho^L_m(f), \rho^L_m(g)\} \leq \rho^L_m(h)$,

Proof. **a)** Let $h_i = g \circ f_i, (i = 1, 2)$, then h_i is nonconstant entire function.

We can suppose that $f_i(0) = 0$, if not we take $f_i^*(z) = f_i(z) - f_i(0)$ and $g_i^*(z) = g(z+f_i(0))$ and we would have $h_i = g_i^* \circ f_i^*$, and by Theorem 3.3**(f)**, we get $\rho^L_{f_2^*}(f_1^*) = \rho^L_{f_2}(f_1)$.

So, without loss of generality we take $f_i(0) = 0$.

We have by Lemma 2.4, there exists $c \in (0, 1)$ such that

$$M_{h_i}(r) \geq M_g\left(cM_{f_i}\left(\frac{r}{2}\right)\right), \text{ for all } r > 0, i = 1, 2.$$

Again using Lemma 2.3 with $\alpha = \frac{1}{d}, \beta = \frac{1}{c}$ we have

$$M_{f_i}\left(\frac{1}{d}.\frac{dr}{2}\right) > \frac{1}{c}.M_{f_i}\left(\frac{dr}{2}\right)$$

$$\Rightarrow M_{f_i}\left(\frac{r}{2}\right) > \frac{1}{c}M_{f_i}\left(\frac{dr}{2}\right) \text{ for all } d \in (0, c) \text{ since } M_{h_i} \leq M_g \circ M_{f_i}.$$

Then

$$M_{h_i}(r) > M_g\left(M_{f_i}\left(\frac{dr}{2}\right)\right) \geq M_{h_i}\left(\frac{dr}{2}\right), i = 1, 2. \tag{1}$$

Again from (1)

$$M_{h_1}(r) > M_g\left(M_{f_1}\left(\frac{dr}{2}\right)\right)$$

$$\Rightarrow M_{h_2}^{-1}(M_{h_1}(r)) > M_{h_2}^{-1}\left(M_g\left(M_{f_1}\left(\frac{dr}{2}\right)\right)\right)$$

Again

$$M_{h_2}^{-1} \circ M_g(t) \geq M_{f_2}^{-1}(t).$$

Therefore

$$M_{h_2}^{-1}(M_{h_1}(r)) > M_{h_2}^{-1}\left(M_g\left(M_{f_1}\left(\frac{dr}{2}\right)\right)\right) > M_{f2}^{-1}\left(M_{f_1}\left(\frac{dr}{2}\right)\right). \tag{2}$$

In (1), for $i=2$, we put $M_{h_2}(r)=t$. i.e., $r=M_{h_2}^{-1}(t)$ and we get

$$\begin{aligned} t &> M_g\left(M_{f_2}\left(\frac{d}{2}M_{h_2}^{-1}(t)\right)\right) \\ M_{f_2}^{-1}(M_g^{-1}(t)) &> \frac{d}{2}M_{h_2}^{-1}(t) \Rightarrow M_{h_2}^{-1}(t) < \frac{2}{d}M_{f_2}^{-1}(M_g^{-1}(t)). \end{aligned}$$

Putting $t=M_{h_1}(r)$, we have

$$M_{h_2}^{-1}(M_{h_1}(r)) < \frac{2}{d}M_{f_2}^{-1}(M_g^{-1}(M_{h_1}(r))) \le \frac{2}{d}M_{f_2}^{-1}(M_{f_1}(r)). \tag{3}$$

Combining (2) and (3) we have,

$$M_{f_2}^{-1}\left(M_{f_1}\left(\frac{dr}{2}\right)\right) < M_{h_2}^{-1}(M_{h_1}(r)) < \frac{2}{d}M_{f_2}^{-1}(M_{f_1}(r)).$$

Taking logarithm and dividing by $\log(rL(r))$ and taking lim sup as $r\to\infty$, we get

$$\rho_{g\circ f_2}^L(g\circ f_1)=\rho_{f_2}^L(f_1).$$

b) As in part (a), we can assume that $f(0)=0$.

Since f and g are nonconstant, there exists $\alpha>0$ such that $M_f(r) > \alpha r$ and $M_g(r)>\alpha r$.

Applying the Lemma 2.4, there exists $c\in(0,1)$ such that

$$M_h(r) \ge M_g\left(cM_f\left(\frac{r}{2}\right)\right) > M_g\left(\frac{c\alpha r}{2}\right) > M_g(r^\sigma), \tag{4}$$

for each $\sigma\in(0,1)$ and for sufficiently large r.

Again,

$$M_h(r) \ge M_g\left(cM_f\left(\frac{r}{2}\right)\right) > \alpha.c.M_f\left(\frac{r}{2}\right) > M_f(r^\sigma), \tag{5}$$

for each $\sigma\in(0,1)$ and for sufficiently large r.

From (4), we have

$$M_m^{-1}(M_g(r^\sigma)) \le M_m^{-1}(M_h(r)).$$

Taking logarithms and dividing by $\log(rL(r))$, for sufficiently large r, we get

$$\frac{\log M_m^{-1}(M_g(r^\sigma))}{\log(rL(r))} \leq \frac{\log M_m^{-1}(M_h(r))}{\log(rL(r))}$$
$$\Rightarrow \frac{\log M_m^{-1}(M_g(r))}{\log\left((rL(r))^{\frac{1}{\sigma}}\right)} < \frac{\log M_m^{-1}(M_g(r^\sigma))}{\log(rL(r))} \leq \frac{\log M_m^{-1}(M_h(r))}{\log(rL(r))},$$
$$\Rightarrow \frac{\log M_m^{-1}(M_g(r))}{\log(rL(r))} \leq \frac{1}{\sigma}\frac{\log M_m^{-1}(M_h(r))}{\log(rL(r))},$$

since $L(r)^\alpha > L(r^\alpha)$, for $\alpha > 1$.

Now taking lim sup as $r \to \infty$, we get

$$\rho_m^L(g) \leq \frac{1}{\sigma}\rho_m^L(h).$$

Similarly from (5), we get

$$\rho_m^L(f) \leq \frac{1}{\sigma}\rho_m^L(h).$$

From the above two results (b) follows. □

3.2. Relative L-order of sum and product. We know that the classical order of a finite sum of entire functions is generally the highest of the orders of them. This is also true for relative $L-$order. Likewise, the order of a finite product of entire functions is generally the highest of the orders of them. But the same result is not valid for the relative $L-$order. For this, we have to introduce some restriction on the functions.

THEOREM 3.5. *Let f, f_1, f_2 and g are nonconstant entire functions, then*

a) $\rho_g^L(f_1 + f_2) \leq \max\{\rho_g^L(f_1), \rho_g^L(f_2)\}$, equality occurs if $\rho_g^L(f_1) \neq \rho_g^L(f_2)$,

b) if f is a transcendental and P is a polynomial then, $\rho_g^L(Pf) = \rho_g^L(f)$,

c) $\rho_g^L(f) \leq \rho_g^L(f^n) \leq n.\rho_g^L(f)$,

d) if g satisfies property (A), then $\rho_g^L(f_1 f_2) \leq \max\{\rho_g^L(f_1), \rho_g^L(f_2)\}$, equality occurs if $\rho_g^L(f_1) \neq \rho_g^L(f_2)$.

Proof. **a)** Let $h = f_1 + f_2, \rho^L = \rho_g^L(h), \rho_i^L = \rho_g^L(f_i), (i = 1, 2)$.

If h is constant, the case is trivial.

Suppose that h is not a constant.
Without loss of generality we may take $\rho_1^L \le \rho_2^L$.
If $\rho_2^L = \infty$, the case is trivial.
So, we take $\rho_1^L \le \rho_2^L < \infty$.
Given $\varepsilon > 0$,

$$\begin{aligned} M_{f_1}(r) &\le M_g\left(rL(r))^{\rho_1^L+\varepsilon}\right) \le M_g\left(rL(r))^{\rho_2^L+\varepsilon}\right) \\ \text{and } M_{f_2}(r) &\le M_g\left(rL(r))^{\rho_2^L+\varepsilon}\right), \text{ for } r > r_0 \end{aligned}$$

Then,

$$\begin{aligned} M_h(r) &\le M_{f_1}(r) + M_{f_2}(r) \\ &\le 2M_g\left(rL(r))^{\rho_2^L+\varepsilon}\right) \le M_g\left(3(rL(r))^{\rho_2^L+\varepsilon}\right), \text{ using Lemma 2.3(}\mathbf{a}\text{).} \\ &\Rightarrow \frac{\log M_g^{-1}(M_h(r))}{\log(rL(r))} \le \frac{\log 3 + (\rho_2^L+\varepsilon)\log(rL(r))}{\log(rL(r))}. \end{aligned}$$

Now taking lim sup as $r \to \infty$, we get $\rho^L \le \rho_2^L + \varepsilon$, for each $\varepsilon > 0$.
Consequently, $\rho^L \le \rho_2^L = \max\{\rho_1^L, \rho_2^L\}$.
Now suppose that, $\rho_1^L < \rho_2^L$ and let's take λ, μ such that $\rho_1^L < \mu < \lambda < \rho_2^L$.
Then $M_{f_1}(r) \le M_g(rL(r))^{\mu})$ and there is a sequence $\{r_n\}$ tending to infinity with $M_{f_2}(r_n) > M_g(r_nL(r_n))^{\lambda})$, for all n.
Now by Lemma 2.3(**c**)

$$M_g(rL(r))^{\lambda}) > 2M_g(rL(r))^{\mu}), \text{ for all } n.$$

Therefore

$$2M_{f_1}(r_n) < 2M_g(r_nL(r_n))^{\mu}) < M_g(r_nL(r_n))^{\lambda}) < M_{f_2}(r_n) \text{ for sufficiently large } n.$$

and so by Lemma 2.3(**a**)

$$\begin{aligned} M_h(r_n) &\ge M_{f_2}(r_n) - M_{f_1}(r_n) \ge \frac{1}{2}M_{f_2}(r_n) \\ &> \frac{1}{2}M_g(r_nL(r_n))^{\lambda}) > M_g\left(\frac{1}{3}(r_nL(r_n))^{\lambda})\right) \text{ for sufficiently large } n. \\ &\Rightarrow \frac{\log M_g^{-1}(M_h(r_n))}{\log(r_nL(r_n))} > \frac{\log\frac{1}{3} + \lambda\log(r_nL(r_n))}{\log(r_nL(r_n))}. \end{aligned}$$

Now taking lim sup as $r \to \infty$, we get $\rho^L \ge \lambda$, for each $\lambda \in (\rho_1^L, \rho_2^L)$.
So, $\rho^L \ge \rho_2^L = \max\{\rho_1^L, \rho_2^L\}$.
Hence, $\rho^L = \rho_2^L = \max\{\rho_1^L, \rho_2^L\}$ for $\rho_1^L < \rho_2^L$.

b) Since $P(z)$ is a polynomial and $h = Pf$, taking $0 < \alpha < 1$ and $s > 1$, we get

$$\begin{aligned} M_f(\alpha r) &< 2\alpha M_f(r), \text{ using Lemma 2.3}(\mathbf{a}) \\ &< |P(z)|M_f(r), \text{ on } |z| = r \\ &= M_h(r) \\ &< r^n M_f(r) \\ &< M_f(r^s), \text{ using Lemma 2.3}(\mathbf{d}), \text{ for sufficiently large } r. \end{aligned} \tag{6}$$

Consequently,

$$M_g^{-1}(M_f(\alpha r)) < M_g^{-1}(M_h(r)) < M_g^{-1}(M_f(r^s))$$
$$\Rightarrow \frac{\log M_g^{-1}(M_f(\alpha r))}{\log(rL(r))} \le \frac{\log M_g^{-1}(M_h(r))}{\log(rL(r))} \le \frac{\log M_g^{-1}(M_h(r^s))}{\log(rL(r))}$$
$$\Rightarrow \frac{\log M_g^{-1}(M_f(r))}{\log\left(\frac{r}{\alpha}L\left(\frac{r}{\alpha}\right)\right)} \le \frac{\log M_g^{-1}(M_h(r))}{\log(rL(r))} \le \frac{\log M_g^{-1}(M_h(r))}{\log\left(r^{\frac{1}{s}}L\left(r^{\frac{1}{s}}\right)\right)} \le \frac{\log M_g^{-1}(M_h(r))}{\log(rL(r))^{\frac{1}{s}}}.$$

Now taking lim sup as $r \to \infty$, we get

$$\begin{aligned} \rho_g^L(f) &\le \rho_g^L(h) \le s\rho_g^L(f), \text{ for all } s > 1 \\ &\Rightarrow \rho_g^L(f) = \rho_g^L(h). \end{aligned}$$

c) We know that,

$$\max\{|f^n(z)| : |z| = r\} = M_f(r)^n \le KM_f(r^n) < M_f((K+1)r^n),$$

using Lemma 2.3(**b**) and 2.3(**a**).

Therefore,

$$\begin{aligned} \frac{\log M_g^{-1}((M_f(r))^n)}{\log(rL(r))} &\le \frac{\log M_g^{-1}(M_f((K+1)r^n))}{\log(rL(r))} \\ &= \frac{\log M_g^{-1}(M_f(r))}{\log\left(\frac{1}{(K+1)^{\frac{1}{n}}}r^{\frac{1}{n}}L\left(\frac{1}{(K+1)^{\frac{1}{n}}}r^{\frac{1}{n}}\right)\right)} \\ &< \frac{\log M_g^{-1}(M_f(r))}{\log\left(\left(\frac{r}{K+1}\right)^{\frac{1}{n}}L\left(\frac{r}{K+1}\right)^{\frac{1}{n}}\right)} \\ &= n\frac{\log M_g^{-1}(M_f(r))}{\log\left(\left(\frac{r}{K+1}\right)L\left(\frac{r}{K+1}\right)\right)}. \end{aligned}$$

Now taking $\limsup$ as $r \to \infty$, we get

$$\begin{aligned}\limsup_{r\to\infty} \frac{\log M_g^{-1}((M_f(r))^n)}{\log(rL(r))} &\leq n.\limsup_{r\to\infty} \frac{\log M_g^{-1}(M_f(r))}{\log\left(\left(\frac{r}{K+1}\right) L\left(\frac{r}{K+1}\right)\right)} \\ &= n.\limsup_{r\to\infty} \frac{\log M_g^{-1}(M_f(r))}{\log(rL(r))} \\ &\Rightarrow \rho_g^L(f^n) \leq n.\rho_g^L(f).\end{aligned}$$

Again,

$$(M_f(r))n > M_f(r)$$
$$\Rightarrow \limsup_{r\to\infty} \frac{\log M_g^{-1}((M_f(r))^n)}{\log(rL(r))} \geq \limsup_{r\to\infty} \frac{\log M_g^{-1}(M_f(r))}{\log(rL(r))}$$
$$\Rightarrow \rho_g^L(f^n) \geq \rho_g^L(f).$$

d) Let f_1, f_2 are transcendental, otherwise it would be trivial.
Denote $h = f_1 f_2, \rho^L = \rho_g^L(h), \rho_i^L = \rho_g^L(f_i), (i = 1, 2)$.
Without loss of generality we may take $\rho_1^L \leq \rho_2^L$.
If $\rho_2^L = \infty$, the case is trivial.
So, we take $\rho_1^L \leq \rho_2^L < \infty$.
Given $\varepsilon > 0$,

$$M_{f_i}(r) \leq M_g\left(rL(r))^{\rho_2^L + \frac{\varepsilon}{2}}\right) \text{ for sufficiently large } r, (i = 1, 2).$$

Then,

$$M_h(r) \leq M_{f_1}(r) M_{f_2}(r) < M_g\left(rL(r))^{\rho_2^L + \frac{\varepsilon}{2}}\right)^2.$$

Applying property (A), with $\sigma = \frac{\rho_2^L + \varepsilon}{\rho_2^L + \frac{\varepsilon}{2}} > 1$, we get

$$\begin{aligned} M_h(r) &\leq M_g\left(rL(r))^{\rho_2^L + \varepsilon}\right), \text{ for sufficiently large } r \\ &\Rightarrow \rho^L \leq \rho_2^L = \max\{\rho_1^L, \rho_2^L\}.\end{aligned}$$

Now suppose that, $\rho_1^L < \rho_2^L$.

Again we have, the product of f by a factor $\frac{c}{z^n}$ does not alter its order, so we can assume without loss of generality that $f_i(0) = 1$.

Take λ, μ with $\rho_1^L < \mu < \lambda < \rho_2^L$.

Then there is a sequence $\{R_n\}$ tending to ∞ such that

$$M_{f_2}(R_n) > M_g(R_n L(R_n))^{\lambda}),$$

for all n and $M_{f_1}(r) < M_g(rL(r))^{\mu})$, for sufficiently large r.

Now by Lemma 2.5, taking $f = f_1, R = 2R_n, \eta = \frac{1}{16}$, we get,

$$\log|f_1(z)| > -T\left(\frac{1}{16}\right)\log M_{f_1}(4eR_n).$$

where $T\left(\frac{1}{16}\right) = 2 + log\left(\frac{3e}{2.\frac{1}{16}}\right) = 2 + log(24e)$.

Therefore,

$$\log|f_1(z)| > -(2 + log(24e))\log M_{f_1}(4eR_n),$$

on the disc $|Z| \leq 2R_n$, excluding a family of discs, the sum of radii is not greater than $\frac{R_n}{2}$.

Therefore there exists $r_n \in (R_n, 2R_n)$ such that $|z| = r_n$, it does not intersect any of the excluded discs, then

$$\begin{aligned} \log|f_1(z)| &> -7\log M_{f_1}(4eR_n) \text{ in } |z| = r_n \\ &\Rightarrow |f_1(z)| > M_{f_1}(4eR_n)^{-7} \text{ in } |z| = r_n. \end{aligned}$$

Also,

$$M_{f_2}(r_n) > M_{f_2}(R_n) > M_g(R_nL(R_n))^{\lambda}) > M_g\left(\left(\frac{r_n}{2}\right)^{\lambda} L\left(\frac{r_n}{2}\right)^{\lambda}\right).$$

If z_r is a point in $|z| = r$ with $M_{f_2}(r) = |f_2(z_r)|$, we have

$$M_h(r) \geq |f_1(z_r)|M_{f_2}(r).$$

And therefore,

$$\begin{aligned} M_h(r_n) &> M_g\left(\left(\frac{r_n}{2}\right)^{\lambda} L\left(\frac{r_n}{2}\right)^{\lambda}\right).M_{f_1}(4eR_n)^{-7} \\ &> M_g\left(\left(\frac{r_n}{2}\right)^{\lambda} L\left(\frac{r_n}{2}\right)^{\lambda}\right).(M_g(4eR_nL(R_n))^{\mu})^{-7}, \text{ for sufficiently large } n \\ &> M_g\left(\left(\frac{r_n}{2}\right)^{\lambda} L\left(\frac{r_n}{2}\right)^{\lambda}\right).(M_g(4er_nL(r_n))^{\mu})^{-7}, \text{ since } r_n > R_n. \end{aligned}$$

Now taking $\nu \in (\mu, \lambda)$ and applying Lemma 2.2 for $\sigma = \frac{\nu}{\mu} > 1$, $n = 8$ and $r = (4er_nL(r_n))^{\mu}$ we obtain,

$$\begin{aligned} M_h(r_n) &> M_g(4er_nL(r_n))^{\nu}).(M_g(4er_nL(r_n))^{\mu})^{-7} \\ &> M_g(4er_nL(r_n))^{\mu})^8.(M_g(4er_nL(r_n))^{\mu})^{-7}, \\ &= M_g(4er_nL(r_n))^{\mu}) \\ &> M_g(r_nL(r_n))^{\mu}), \text{ for sufficiently large } n. \end{aligned}$$

Consequently,

$$\mu \leq \rho^L \text{ for each } \mu < \rho_2^L.$$

So,

$$\rho^L = \rho_2^L.$$

□

3.3. Relative L-order of derivative. We know that the classical order of an entire function is same as that of its derivative. But in our case, this result is false if both of the entire functions be polynomials.

THEOREM 3.6. *Let f and g be two nonconstant entire functions such that at least one of them is transcendental. Then $\rho_g^L(f') = \rho_g^L(f)$.*

Proof. We can assume that f and g are transcendental, the other cases are trivial.

We can assume that $f(0) = 0$.

Let $\widetilde{M}_f(r) = \max\{|f'(z)| : |z| = r\}$

We know that

$$f(z) = \int_0^z f'(t)dt$$

where we have taken the integral over the segment that joins the origin with z.

$$\begin{aligned} |f(z)| &\leq \int_0^z |f'(t)|\, dt \leq \widetilde{M}_f(r).r \\ &\Rightarrow M_f(r) \leq \widetilde{M}_f(r).r \end{aligned}$$

Using Cauchy's formula,

$$\begin{aligned} f'(z) &= \frac{1}{2\pi i} \oint_c \frac{f(t)}{(t-z)^2} dt, \text{ where } |z| = r, c = \{t : |t-z| = r\} \\ &\Rightarrow |f'(z)| \leq \frac{1}{2\pi} \frac{M_f(r)}{r^2}.2\pi < M_f(2r) \\ &\Rightarrow \widetilde{M}_f(r) < M_f(2r). \end{aligned}$$

Therefore

$$\frac{M_f(r)}{r} < \widetilde{M}_f(r) < M_f(2r), \text{ for each } r > 0$$

Let $\sigma \in (0,1)$, from Lemma 2.3(**d**) and taking $\lambda = 1, \mu = \sigma$, we get

$$\lim_{r\to\infty} \frac{M_f(r)}{rM_f(r^\sigma)} = \infty$$
$$\Rightarrow M_f(r) > rM_f(r^\sigma), \text{ for sufficiently large } r.$$

Therefore

$$M_f(r^\sigma) < \widetilde{M}_f(r) < M_f(2r)$$
$$\Rightarrow M_g^{-1}(M_f(r^\sigma)) < M_g^{-1}(\widetilde{M}_f(r)) < M_g^{-1}(M_f(2r)).$$

Taking logarithm and dividing by $\log(rL(r))$, we get

$$\frac{\log M_g^{-1}(M_f(r^\sigma))}{\log(rL(r))} < \frac{\log M_g^{-1}(\widetilde{M}_f(r))}{\log(rL(r))} < \frac{\log M_g^{-1}(M_f(2r))}{\log(rL(r))}$$
$$\Rightarrow \frac{\log M_g^{-1}(M_f(r))}{\log(rL(r))^{\frac{1}{\sigma}}} < \frac{\log M_g^{-1}(M_f(r))}{\log\left(r^{\frac{1}{\sigma}} L\left(r^{\frac{1}{\sigma}}\right)\right)} < \frac{\log M_g^{-1}(\widetilde{M}_f(r))}{\log(rL(r))} < \frac{\log M_g^{-1}(M_f(r))}{\log\left(\frac{r}{2} L\left(\frac{r}{2}\right)\right)}$$

taking lim sup as $r \to \infty$, we have

$$\begin{aligned} \sigma.\rho_g^L(f) \quad &\leq \quad \rho_g^L(f') \leq \rho_g^L(f) \quad \text{for each } \sigma \in (0,1) \\ &\Rightarrow \quad \rho_g^L(f') = \rho_g^L(f) \end{aligned}$$

□

3.4. Relative L-order of real and imaginary parts. The relative $L-$order is completely determined by the real and imaginary parts of given functions.

THEOREM 3.7. *Let f and g be two nonconstant entire functions. Denote*

$$\begin{aligned} A(r) &= \max\{\operatorname{Re} f(z) : |z| = r\}, \\ B(r) &= \max\{\operatorname{Im} f(z) : |z| = r\}, \\ C(r) &= \max\{\operatorname{Re} g(z) : |z| = r\}, \\ D(r) &= \max\{\operatorname{Im} g(z) : |z| = r\}. \end{aligned}$$

Then

$$\begin{aligned} \rho_g^L(f) &= \inf\{\mu > 0 : M(r) < N((rL(r))^\mu) \\ &= \limsup_{r\to\infty} \frac{\log N^{-1}(M(r))}{\log(rL(r))}, \end{aligned}$$

where M is any of the functions A, $B \circ M_f$ and N is any of the functions C, $D \circ M_g$.

Proof. It is known that A, B, C and D are continuous, strictly increasing, then A^{-1}, B^{-1}, C^{-1} and D^{-1} are exist.

From Lemma 2.6, there exists $\alpha > 0$ such that

$$M(r) \leq M_f(r) \leq M(\alpha r) \text{ and } N(r) \leq M_g(r) \leq N(\alpha r)$$

Let us take $\rho^L = \rho_g^L(f)$ and $\beta = \inf\{\mu > 0 : M(r) < N((rL(r))^\mu)\}$.
Let us first prove that $\beta \leq \rho$.
If $\rho = \infty$, then it is trivial.
If ρ be finite, take λ, μ such that $\rho < \lambda < \mu < \infty$.
Therefore, $M_f(r) < M_g((rL(r))^\lambda)$ and

$$\begin{aligned} M(r) &\leq M_f(r) < M_g((rL(r))^\lambda) < N(\alpha^\lambda (rL(r))^\lambda) \\ &< N((rL(r))^\mu), \text{ for sufficiently large } r \\ &\Rightarrow \mu \geq \beta, \text{ for all } \mu > \rho \\ &\Rightarrow \beta \leq \rho. \end{aligned}$$

Now we prove that $\beta \geq \rho$.
If $\rho = 0$, it is trivial.
If $\rho > 0$, then take λ, μ such that $0 < \mu < \lambda < \rho$.
Then for a sequence of values of r_n tending to ∞ such that

$$M_f(r_n) > M_g((r_n L(r_n))^\lambda), \text{ for all } n.$$

So,

$$\begin{aligned} M(\alpha r_n) &> M_f(r_n) > M_g((r_n L(r_n))^\lambda) > M_g((\alpha r_n L(\alpha r_n))^\mu) \\ &\geq N((\alpha r_n L(\alpha r_n))^\mu), \text{ for sufficiently large } n \\ &\Rightarrow \beta \geq \mu \text{ with for all } \mu < \rho \\ &\Rightarrow \beta \geq \rho. \end{aligned}$$

□

3.5. Relative L-order of Nevanlinna. The following theorem generalizes the concepts of classical order to relative $L-$order determined by $T(r)$.

THEOREM 3.8. *Let f and g be two nonconstant entire functions. Then*

$$\begin{aligned}\rho_g^L(f) &= \inf\{\mu > 0 : T_f(r) < T_g((rL(r))^\mu)\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1}(T_f(r))}{\log(rL(r))}.\end{aligned}$$

Proof. Let $\rho^L = \rho_g^L(f)$ and $\alpha = \inf\{\mu > 0 : T_f(r) < T_g((rL(r))^\mu)\}$
Let us prove that $\alpha \le \rho^L$.
If $\rho^L = \infty$, the case is trivial.
So, we take ρ^L be finite and let's take $\gamma, \delta, \lambda, \mu$ such that $\rho^L < \gamma < \delta < \lambda < \mu < \infty$.
Now for sufficiently large r, it is clear that

$$\frac{\gamma}{\delta} < \frac{(rL(r))^\mu - (rL(r))^\lambda}{(rL(r))^\mu + (rL(r))^\lambda}.$$

By Lemma 2.3(**b**) and 2.3(**c**) applying to M_g, taking $s = \frac{\delta}{\gamma}$, gives

$$M_g(r^\gamma)^s \le KM_g(r^\delta) < M_g(r^\lambda).$$

and

$$\begin{aligned}M_g((rL(r))^\gamma)^s &= M_g((rL(r))^\gamma)^{\frac{\delta}{\gamma}} \le KM_g\left(r^\delta\left(L\left(r^{\frac{\delta}{\gamma}}\right)\right)^\gamma\right) \\ &\le KM_g(r^\delta(L(r))^\delta), \text{ for sufficiently large } r \\ &< M_g((rL(r))^\lambda).\end{aligned}$$

Therefore,

$$\frac{\delta}{\gamma}\log M_g((rL(r))^\gamma) < \log M_g((rL(r))^\lambda).$$

Which implies

$$\begin{aligned}\log M_g((rL(r))^\gamma) &< \frac{\gamma}{\delta}\log M_g((rL(r))^\lambda) \\ &< \frac{(rL(r))^\mu - (rL(r))^\lambda}{(rL(r))^\mu + (rL(r))^\lambda}\log M_g((rL(r))^\lambda) \\ &\le T_g((rL(r))^\mu).\end{aligned}$$

Again from Lemma 2.7

$$\begin{aligned} T_f(r) &\leq \log M_f(r) < \log M_g((rL(r))^{\lambda}) \\ &\Rightarrow T_f(r) < T_g((rL(r))^{\mu}) \\ &\Rightarrow \mu \geq \alpha, \text{ for all } \mu > \rho^L \\ &\Rightarrow \rho^L \geq \alpha. \end{aligned}$$

Next let us prove, $\alpha \geq \rho^L$.
If $\rho^L = 0$, the case is trivial.
So let $\rho^L > 0$, and take γ, δ, μ with $0 < \mu < \lambda < \gamma < \rho^L$.
Then there exist $\{r_n\}$ tending to ∞ such that

$$M_f(r_n) > M_g((r_n L(r_n))^{\gamma}), \text{ for all } n.$$

$c \in \left(\frac{\lambda}{\gamma}, 1\right)$ and $d > \frac{1+c}{1-c}$.
Then

$$\begin{aligned} T_f(dr_n) &> \frac{dr_n - r_n}{dr_n + r_n} \log M_f(r_n) \\ &= \frac{d-1}{d+1} \log M_f(r_n) \\ &> c \log M_f(r_n) \\ &> \log M_g((r_n L(r_n))^{\gamma})^c \\ &> \log \frac{M_g((r_n L(r_n))^{\gamma c})}{K}, \text{ using Lemma 2.3}(\mathbf{b}) \text{ for } c < 1 \\ &> \log M_g((r_n L(r_n))^{\lambda}), \text{ as } c > \frac{\lambda}{\gamma} \\ &\geq \log M_g((dr_n L(dr_n))^{\mu}), \text{ for sufficiently large } n \\ &\geq T_g((dr_n L(r_n))^{\mu}). \end{aligned}$$

Therefore,

$$\begin{aligned} T_f(dr_n) &> T_g((dr_n L(r_n))^{\mu}), \text{ for sufficiently large } n \\ &\Rightarrow \alpha \geq \mu, \text{ for all } \mu < \rho^L \\ &\Rightarrow \alpha \geq \rho^L. \end{aligned}$$

Hence,

$$\rho^L = \alpha = \{\mu > 0 : T_f(r) < T_g((rL(r))^{\mu})\}.$$

$\square$

3.6. Relative L-proximate order. Here we define relative $L-$proximate order of f with respect to g and relative $L-$lower proximate order of f with respect to g and then give simple prove of their existence.

Definition 3.9 (Relative L-proximate order of f with respect to g). Let $f(z)$ be an integral function of finite $L-$order of growth of f relative to g, $\rho_g^L(f)$.

A function $\rho_g^L(f)(r)$ is said to be a $L-$proximate order of growth of f relative to g if the following properties holds:

i) $\rho_g^L(f)(r)$ is differentiable for $r > r_0$ except at isolated points at which $\rho_L'(r-0)$ and $\rho_L'(r+0)$ exist,

ii) $\lim\limits_{r\to\infty} \rho_g^L(f)(r) = \rho_g^L(f)$,

iii) $\lim\limits_{r\to\infty} r.(\rho_g^L(f))'(r).\log\{rL(r)\} = 0$,

iv) $\limsup\limits_{r\to\infty} \frac{M_g^{-1}(M_f(r))}{\{rL(r)\}^{\rho_g^L(f)(r)}} = 1$.

Theorem 3.10 (Existence of Relative L-proximate order of f with respect to g). *For every entire function $f(z)$ of finite $L-$order of growth of f relative to g, $\rho_g^L(f)$, there exists a $L-$proximate order $\rho_g^L(f)(r)$.*

Proof. Let

$$\sigma(r) = \frac{\log M_g^{-1}(M_f(r))}{\log\{rL(r)\}}.$$

Then either $\sigma(r) > \rho_g^L(f)$ for a sequence of r tending to infinity, or $\sigma(r) \leq \rho_g^L(f)$ for all large values of r.

Case A: $\sigma(r) > \rho_g^L(f)$ for a sequence of r tending to infinity.

We define,

$$\phi(r) = \max_{x\geq r}\{\sigma(x)\}.$$

Since, $\sigma(r)$ is continuous, $\limsup\limits_{r\to\infty} \sigma(r) = \rho_g^L(f)$, and $\sigma(r) > \rho_g^L(f)$ for a sequence of values of r tending to infinity.

Therefore, $\phi(r)$ exists. $\phi(r)$ is a non-increasing function of r.

Let $r_1 > 0$ be such that $\phi(r_1) = \sigma(r_1)$. Such values will exist for a sequence of values of r tending to infinity.

Let $\rho_g^L(f)(r_1) = \phi(r_1)$. Let t_1 be the smallest integer not less than $r_1 + 1$ such that $\phi(r_1) > \phi(t_1)$ and let

$$\rho_g^L(f)(r) = \rho_g^L(f)(r_1) = \phi(r_1) \text{ for } r_1 < r \leq t_1.$$

Define u_1 as follows:

$u_1 > t_1$,

$\rho_g^L(f)(r) = \rho_g^L(f)(r_1) - \log\log\log\{rL(r)\} + \log\log\log\{t_1 L(t_1)\}$ for $t_1 \leq r \leq u_1$,

$\rho_g^L(f)(r) = \phi(r)$ for $r = u_1$ but $\rho_g^L(f)(r) > \phi(r)$ for $t_1 \leq r < u_1$.

Let r_2 be the smallest value of r for which $r_2 \geq u_1$ and $\phi(r_2) = \sigma(r_2)$.

If $r_2 > u_1$ then let $\rho_g^L(f)(r) = \phi(r)$ for $u_1 \leq r \leq r_2$. Since $\phi(r)$ is constant for $u_1 \leq r \leq r_2$, therefore $\rho_g^L(f)(r)$ is constant for $u_1 \leq r \leq r_2$.

We repeat the argument and obtain that $\rho_g^L(f)(r)$ is differentiable in adjacent intervals.

Further,

$$(\rho_g^L(f))'(r) = 0 \text{ or } -\frac{1}{\log\log\{rL(r)\}.\log\{rL(r)\}.rL(r)}\{rL'(r) + L(r)\}.$$

Therefore,

$$r.(\rho_g^L(f))'(r).\log\{rL(r)\} = 0 \text{ or } -\frac{1}{\log\log\{rL(r)\}}\left\{\frac{rL'(r)}{L(r)} + 1\right\}.$$

Hence,

$$\lim_{r\to\infty} r.(\rho_g^L(f))'(r)\log\{rL(r)\} = 0.$$

Also note that, $\rho_g^L(f)(r) \geq \phi(r) \geq \sigma(r)$ for $r \geq r_1$.

Further, $\rho_L(r) = \phi(r)$ for $r = r_1, r_2, r_3, \ldots$ and $\rho_L(r)$ is non-increasing and $\lim\limits_{r\to\infty} \phi(r) = \rho_L$.

Hence,

$$\limsup_{r\to\infty} \rho_g^L(f)(r) = \lim_{r\to\infty} \rho_L(r) = \rho_L.$$

Again since, $M_g^{-1}(M_f(r)) = \{rL(r)\}^{\sigma(r)} = \{rL(r)\}^{\rho_g^L(f)(r)}$ for infinitely many values of r and $M_g^{-1}(M_f(r)) < \{rL(r)\}^{\rho_g^L(f)(r)}$ for the remaining r.

Hence,

$$\limsup_{r\to\infty} \frac{M_g^{-1}(M_f(r))}{\{rL(r)\}^{\rho_g^L(f)(r)}} = 1.$$

Case B: $\sigma(r) \leq \rho_g^L(f)$ for all large values of r.

Here, there are two possibilities:

Subcase B.1: $\sigma(r) = \rho_g^L(f)$, for at least a sequence of values of r tending to infinity.

Here, we take $\rho_g^L(f)(r) = \rho_g^L(f)$ for all large r.

Subcase B.2: $\sigma(r) < \rho_g^L(f)$, for all large r.

Let $X > 0$ be such that $\sigma(r) < \rho_g^L(f)$ where $r \geq X$.

We define,

$$\xi(r) = \max_{X \leq x \leq r} \{\sigma(x)\}.$$

Therefore $\xi(r)$ is non-decreasing.

Take a suitable value $r_1 > X$ and let

$\rho_g^L(f)(r_1) = \rho_g^L(f)$,

$\rho_g^L(f)(r) = \rho_g^L(f) + \log\log\log\{rL(r)\} - \log\log\log\{r_1 L(r_1)\}$ for $s_1 \leq r \leq r_1$,

where $s_1 < r_1$ is such that $\xi(s_1) = \rho_L(s_1)$.

If $\xi(s_1) \neq \sigma(s_1)$, then we take $\rho_g^L(f)(r) = \xi(r)$ for $t_1 \leq r \leq s_1$.

where t_1 is the nearest point (with $t_1 < s_1$) at which $\xi(t_1) = \sigma(t_1)$. Therefore $\rho_g^L(f)(r)$ is constant for $t_1 \leq r \leq s_1$.

If $\xi(s_1) = \sigma(s_1)$, then let $t_1 = s_1$.

Choose $r_2 > r_1$ suitable large and let

$\rho_g^L(f)(r_2) = \rho_g^L(f)$,

$\rho_g^L(f)(r) = \rho_g^L(f) + \log\log\log\{rL(r)\} - \log\log\log\{r_2 L(r_2)\}$ for $s_2 \leq r \leq r_2$,

where $s_2 (< r_2)$ is such that $\xi(s_2) = \rho_g^L(f)(s_2)$.

If $\xi(s_2) \neq \sigma(s_2)$, then we take $\rho_g^L(f)(r) = \xi(r)$ for $t_2 \leq r \leq s_2$.

where t_2 is the nearest point (with $t_2 < s_2$) at which $\xi(t_2) = \sigma(t_2)$.

If $\xi(s_2) = \sigma(s_2)$, then let $t_2 = s_2$.

For $r < t_2$, let

$\rho_g^L(f)(r) = \rho_g^L(f)(t_2) + \log\log\log\{t_2 L(t_2)\} - \log\log\log\{rL(r)\}$ for $u_1 \leq r \leq t_2$,

where $u_2 (< t_2)$ is the point of intersection of

$$y = \rho_g^L(f) \text{ with } y = \rho_g^L(f)(t_2) + \log\log\log\{t_2 L(t_2)\} - \log\log\log\{rL(r)\}.$$

Let $\rho_g^L(f)(r) = \rho_g^L(f)$ for $r_1 \leq r \leq u_1$

It is always possible to choose r_2 so large that $r_1 < u_1$.

We repeat the procedure and note that $\rho_g^L(f)(r)$ is differentiable in adjacent intervals

Further,

$$(\rho_g^L(f))'(r) = 0 \text{ or } \pm \frac{1}{\log\log\{rL(r)\}.\log\{rL(r)\}r.L(r)}\{rL'(r) + L(r)\}.$$

Therefore,

$$r.(\rho_g^L(f))'(r).\log\{rL(r)\} = 0 \text{ or } \pm \frac{1}{\log\log\{rL(r)\}}\left\{\frac{rL'(r)}{L(r)} + 1\right\}.$$

Hence,

$$\lim_{r\to\infty} r.(\rho_g^L(f))'(r).\log\{rL(r)\} = 0.$$

Also, $\rho_g^L(f)(r) \geq \xi(r) \geq \sigma(r)$ for all large r and $\rho_g^L(f)(r) = \sigma(r)$ for $r = t_1, t_2, t_3, \ldots$.

Hence,

$$\lim_{r\to\infty} \rho_g^L(f)(r) = \rho_g^L(f).$$

And

$$\limsup_{r\to\infty} \frac{M_g^{-1}(M_f(r))}{\{rL(r)\}^{\rho_g^L(f)(r)}} = 1.$$

□

Definition 3.11 (Relative L-lower proximate order of f with respect to g). Let $f(z)$ be an integral function of finite $L-$lower order of growth of f relative to $g, \lambda_g^L(f)$.

A function $\lambda_g^L(f)(r)$ is said to be a $L-$lower proximate order of growth of f relative to g if the following properties holds:

i) $\lambda_g^L(f)(r)$ is differentiable for $r > r_0$ except at isolated points at which $\rho_L'(r-0)$ and $\rho_L'(r+0)$ exist,

ii) $\lim\limits_{r\to\infty} \lambda_g^L(f)(r) = \lambda_g^L(f)$,

iii) $\lim\limits_{r\to\infty} r.(\lambda_g^L(f))'(r).\log\{rL(r)\} = 0$,

iv) $\liminf\limits_{r\to\infty} \frac{M_g^{-1}(M_f(r))}{\{rL(r)\}^{\lambda_g^L(f)(r)}} = 1$.

Theorem 3.12 (Existence of Relative L-lower proximate order of f with respect to g). *For every entire function $f(z)$ of finite $L-$lower order of growth of f relative to $g, \lambda_g^L(f)$, there exists a $L-$proximate order $\lambda_g^L(f)(r)$.*

The proof of the above theorem is omitted because it can be carried out in the line of the previous theorem.

3.7. Some examples. In the following two examples we shall find out the relative $L-$order when f and g both are polynomial and both are transcendental respectively. The other two cases are trivial by Theorem 3.3**(a)** and 3.3**(b)**.

Example 3.13. Let us consider a slowly increasing function, $L(r) = \log r$.

Let $f(z) = z^2$ and $g(z) = z^3$.

Then $M_f(r) = r^2$, $M_g(r) = r^3$ and $M_g^{-1}(r) = r^{\frac{1}{3}}$.
Hence

$$\begin{aligned}\rho_g^L(f) &= \limsup_{r\to\infty} \frac{\log M_g^{-1}(M_f(r))}{\log(rL(r))}\\ &= \limsup_{r\to\infty} \frac{\log r^{\frac{2}{3}}}{\log(r\log r)}\\ &= \frac{2}{3}\limsup_{r\to\infty} \frac{\log r}{\log r + \log\log r}\\ &= \frac{2}{3}.\end{aligned}$$

Note: Here $\rho_g^L(f) = \frac{2}{3} = \frac{\deg(f)}{\deg(g)}$.

Example 3.14. Let us consider another slowly increasing function, $L(r) = \log\log r$.
Let $f(z) = e^{z^2}$ and $g(z) = e^z$.
Then $M_f(r) = e^{r^2}$, $M_g(r) = e^r$ and $M_g^{-1}(r) = \log r$.
Hence

$$\begin{aligned}\rho_g^L(f) &= \limsup_{r\to\infty} \frac{\log M_g^{-1}(M_f(r))}{\log(rL(r))}\\ &= \limsup_{r\to\infty} \frac{\log\log e^{r^2}}{\log(r\log\log r)}\\ &= 2\limsup_{r\to\infty} \frac{\log r}{\log r + \log\log\log r}\\ &= 2.\end{aligned}$$

In the next example we shall show that Theorem 3.5(**d**) may not hold if g does not satisfy property (A). For this we take g as a polynomial.

Example 3.15. Let us consider the slowly increasing function, $L(r) = \log r$.
Let $f_1(z) = z^2$, $f_2(z) = z^3$ and $g(z) = z$.
Then $M_{f_1}(r) = r^2$, $M_{f_2}(r) = r^3$, $M_g(r) = r$ and $M_g^{-1}(r) = r$.
And also $f_1(z)f_2(z) = z^5$, $M_{f_1f_2}(r) = r^5$.
Now we see that, $\rho_g^L(f_1) = 2$, $\rho_g^L(f_2) = 3$ and $\rho_g^L(f_1f_2) = 5$.
But

$$5 = \rho_g^L(f_1f_2) \nleq \max\{\rho_g^L(f_1), \rho_g^L(f_2)\} = \max\{2, 3\} = 3.$$

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