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ITERATES OF WEIGHTED BEREZIN TRANSFORM UNDER INVARIANT MEASURE IN THE UNIT BALL

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Abstract. We focus on the interations of the weighted Berezin transform T_α on $L^p(\tau)$, where τ is the invariant measure on the complex unit ball B_n. Iterations of T_α on $L^1_R(\tau)$ the space of radial integrable functions played important roles in proving $\mathcal{M}$-harmonicity of bounded functions with invariant mean value property. Here, we introduce more properties on iterations of T_α on $L^1_R(\tau)$ and observe differences between the iterations of T_α on $L^1(\tau)$ and $L^p(\tau)$ for $1 < p < \infty$.

1. Introduction

Let B_n be the unit ball of $\mathbb{C}^n$ with norm $|z| = \langle z, z\rangle^{1/2}$ where $\langle\ ,\ \rangle$ is the Hermitian inner product, and let ν be the Lebesgue measure on $\mathbb{C}^n$ normalized to $\nu(B_n) = 1$.
For $\alpha > -1$, we define a positive measure ν_α on B_n by

$$d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d\nu(z),$$

where

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)}$$

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is a normalizing constant so that $\nu_\alpha(B_n) = 1$. For such α and $f \in L^1(B_n, \nu_\alpha)$, the weighted Berezin transform $T_\alpha f$ on B_n is defined by

$$(T_\alpha f)(z) \; = \; \int_{B_n} f\big(\varphi_z(w)\big)\; d\nu_\alpha(w) \;\text{ for } z \in B_n,$$

where $\varphi_a \in \mathrm{Aut}(B_n)$ is the canonical automorphism given by

$$\varphi_a(z) \; = \; \frac{a - Pz - (1-|a|^2)^{1/2} Qz}{1 - \langle z, a\rangle}$$

where P is the projection into the space spanned by $a \in B_n$ and $Q_z = z - Pz$.
Equivalently we can write

$$(T_\alpha f)(z) \; = \; \int_{B_n} f(w)\; \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z, w\rangle|^{2n+2+2\alpha}}\; d\nu_\alpha(w). \tag{1.1}$$

The invariant Laplacian $\tilde{\Delta}$ is defined for $f \in C^2(B_n)$ by

$$(\tilde{\Delta} f)(z) \; = \; \Delta(f \circ \varphi_z)(0).$$

The $\mathcal{M}$-harmonic functions in B_n are those for which $\tilde{\Delta} f = 0$. If a function $f \in L^1(B_n, \nu_\alpha)$ is $\mathcal{M}$-harmonic, then $f \circ \psi$ is also $\mathcal{M}$-harmonic for every $\psi \in \mathrm{Aut}(B_n)$. Thus for every given $\alpha > -1$, bounded $\mathcal{M}$-harmonic function f satisfies an invariant mean value property

$$\int_{B_n} (f \circ \psi)\; d\nu_\alpha = f(\psi(0)) \;\text{ for every } \psi \in \mathrm{Aut}(B_n),$$

which is equivalent to saying that $(T_\alpha f)(z) = f(z)$ for every $z \in B_n$.

Conversely, Furstenberg ([2],[3]) provided abstract proofs that on any dimensional symmetric domain, a bounded function which is invariant under a weighted Berezin transform is harmonic with respect to the intrinsic metric, which implies that $f \in L^\infty(B_n)$ satisfying $T_\alpha f = f$ is $\mathcal{M}$-harmonic. In 1993, Ahern, Flores and Rudin ([1]) gave an analytic proof that $f \in L^\infty(B_n)$ satisfying $T_0 f = f$ is $\mathcal{M}$-harmonic, and $f \in L^1(B_n, \nu_\alpha)$ satisfying $T_0 f = f$ has to be $\mathcal{M}$-harmonic if and only if $n \le 11$.

To mention some previous works related to weighted Berezin transform and harmonicity, in 2008 ([4]) the author proved that for any $1 \le p < \infty$ and $c_1, c_2 > -1$, a function $f \in L^p(\nu_{c_1} \times \nu_{c_2})$ on the bidisc which is invariant under the weighted Berezin transform; $T_{c_1,c_2} f = f$ needs not be $2-$harmonic. Properties of such functions on the bidisc is mentioned in the recent work [6]. And in 2010, the author([5]) gave an analytic proof that for every given $\alpha > -1$, $f \in L^\infty(B_n)$ satisfying $T_\alpha f = f$ is $\mathcal{M}$-harmonic. In [5], the author used the spectral theory and interation of T_α on the commutative Banach algebra $L^1_R(\tau)$, the space of all radial function f on B_n integrable with respect to the invariant measure τ defined by $d\tau(z) = (1-|z|^2)^{-n-1}\, d\nu(z)$.

This paper, we focus on the iteration of the weighted Berezin transform T_α on $L^p(B_n,\tau)$, which has not been done any previous researches. Our motivation comes from Lemma 2.1 of [5] which plays a crucial role in the proof of the main result of that paper. Here, we develop further theory and results which follow from Lemma 2.1 of [5] and observe the major difference between the iterations of T_α on $L^1(\tau)$ and $L^p(\tau)$ for $1 < p < \infty$.

In section 2, we introduce some preliminaries on weighted Berezin transform T_α and invariant measure τ on B_n. In section 3, we propose a lemma and three new propositions about iterations of T_α on $L^p(\tau)$ for $1 \le p < \infty$. Throughout the paper α is an arbitrarily given real number with $\alpha > -1$.

2. Preliminaries

Here, we introduce some preliminaries on weighted Berezin transform T_α and invariant measure τ on B_n details of which are explained in [5] and [7]. We focus on the invariant measure τ on B_n defined by $d\tau(z) = (1-|z|^2)^{-n-1}\, d\nu(z)$, which satisfies

$$\int_{B_n} f\, d\tau = \int_{B_n} (f \circ \psi)\, d\tau$$

for every $f \in L^1(\tau)$ and $\psi \in \mathrm{Aut}(B_n)$. Even though τ is not a finite measure on B_n so that a non-zero constant does not belong to $L^1(\tau)$, T_α on $L^\infty(B_n)$ is the adjoint of T_α on $L^1(\tau)$ in the sense that

$$\int_{B_n} (T_\alpha f) \cdot g\, d\tau = \int_{B_n} f \cdot (T_\alpha g)\, d\tau \tag{2.1}$$

for $f \in L^1(\tau)$ and $g \in L^\infty(B_n)$. Since $L^\infty(B_n) = L^1(\tau)^*$, the spectrum of T_α on $L^\infty(B_n)$ is the same as the spectrum of T_α on $L^1(B_n,\tau)$. Moreover from the expression (1.1), we can easily see that the operator T_α on $L^\infty(B_n)$ is a positive contraction, which means that T_α is also a positive contraction on $L^1(B_n,\tau)$ so that we can iterate T_α on $L^1(\tau)$.

For $1 \le p \le \infty$, we denote $L^p_R(\tau)$ as the subspace of $L^p(B_n, \tau)$ which consists of radial functions, which means that $f \in L^p_R(\tau)$ if and only if $f \in L^p(\tau)$ and $f(z) = f(|z|)$ for all $z \in B_n$. In this case, T_α is a contraction on $L^1_R(\tau)$ which is a commutative Banach algebra under the convolution

$$\big(f * g\big)(z) = \int_{B_n} f\big(\varphi_z(w)\big) g(w)\ d\tau(w)$$

for $f, g \in L^1_R(\tau)$. Hence if $f \in L^1_R(\tau)$, we can write $T_\alpha f = f * h_\alpha$ where

$$h_\alpha(z) = c_\alpha (1 - |z|^2)^{n+1+\alpha} \in L^1_R(\tau).$$

In [5], the key step to the proof of the main theorem is Lemma 2.1 which states that

$$\lim_{k\to\infty} \| \ T^k_\alpha (I - T_\alpha) \ \| = 0 \quad \text{on} \quad L^1_R(\tau). \tag{2.2}$$

We start section 3 by introducing a lemma on iteration of T_α on $L^1_R(\tau)$ which is a direct result of (2.2). Then we extend this lemma to a more general case.

3. The iterations of T_α

We start this section by introducing Lemma 3.1 on iteration of T_α on $L^1_R(\tau)$ which is an application of (2.2).

Lemma 3.1. *For $f \in L^1_R(\tau)$, we have*

$$\lim_{k\to\infty} \int_{B_n} |\ T^k_\alpha f\ |\ d\tau \ = \ 0 \quad \textit{if and only if} \quad \int_{B_n} f\ d\tau \ = \ 0.$$

Proof. Let $f \in L^1_R(\tau)$. By putting $g = 1$ in (2.1) we get

$$\int_{B_n} T^k_\alpha f\ d\tau = \int_{B_n} f\ d\tau \quad \text{for every } k \ge 0.$$

Hence

$$\lim_{k\to\infty} \int_{B_n} |\ T^k f\ |\ d\tau \ = \ 0 \quad \text{implies} \quad \int_{B_n} f\ d\tau \ = \ 0.$$

To prove the converse, if we define

$$D = \Big\{ f \in L^1_R(\tau) \ \Big| \ \int_{B_n} f\ d\tau \ = \ 0 \Big\}.$$

Then $(I-T)L^1_R(\tau) \subset E$. Now let $\ell \in L^\infty_R(B_n)$ satisfy

$$\int_{B_n} (f - T_\alpha f\ ) \cdot \ell \, d\tau = 0 \quad \text{for every } f \in L^1_R(\tau).$$

Then by (2.1)

$$\int_{B_n} f \cdot (\ell - T_\alpha \ell) \, d\tau = 0 \quad \text{for every } f \in L^1_R(\tau).$$

Hence $T_\alpha \ell = \ell$, which means ℓ is radial $\mathcal{M}$-harmonic so that ℓ is a constant. Hence we get

$$\int_{B_n} g \cdot \ell \, d\tau = 0 \quad \text{for every } g \in D.$$

By the Hahn-Banach theorem, this means $(I-T)L^1_R(\tau)$ is dense in D. Now from (2.2), we have

$$\lim_{k\to\infty} \| \, T^k_\alpha (f - T_\alpha f) \, \|_{L^1(\tau)} = 0 \quad \text{for every } f \in L^1_R(\tau).$$

Therefore, we conclude

$$\lim_{k\to\infty} \int_{B_n} | \, T^k_\alpha f \, | \, d\tau = 0 \quad \text{for every } f \in D.$$

□

Next proposition is a generalization of Lemma 3.1. Since non-zero constant does not belong to $L^1(\tau)$, we can not simply apply Lemma 3.1 to the function $f - \int_{B_n} f \, d\tau$.

Proposition 3.2. *If $f \in L^1_R(\tau)$, then we have*

$$\lim_{k\to\infty} \int_{B_n} \left| \, T^k_\alpha f \, \right| d\tau \ = \left| \int_{B_n} f \, d\tau \right|.$$

Proof. Let's denote $A = \left\{\ell \in L^\infty_R(B_n) \mid \|\ell\|_\infty \le 1\right\}$, $E_k = T^k_\alpha A$ and $E = \bigcap_{k=1}^\infty E_k$. Then for $f \in L^1_R(\tau)$,

$$\begin{aligned} \int_{B_n} | \, T^k_\alpha f \, | \, d\tau &= \sup\left\{ \left| \int_{B_n} (\ T^k_\alpha f\ ) \cdot \ell \, d\tau \right| \ \middle| \ \ell \in A \right\} \\ &= \sup\left\{ \left| \int_{B_n} f \cdot (\ T^k_\alpha \ell\ ) \, d\tau \right| \ \middle| \ \ell \in A \right\}. \end{aligned}$$

Hence

$$\lim_{k\to\infty} \| T_\alpha^k f \|_{L^1(\tau)} \ \geq \ \sup \left\{ \left| \int_{B_n} f \cdot h \ d\tau \right| \ \Big| \ h \in E \right\}. \tag{3.1}$$

On the other hand, for every $\varepsilon > 0$ and $k \geq 1$ there exists $h_k \in A$ with

$$\begin{aligned} \| T_\alpha^k f \|_{L^1(\tau)} &\leq \left| \int_{B_n} \left(T_\alpha^k f\right) \cdot h_k \ d\tau \right| + \varepsilon \\ &= \left| \int_{B_n} f \cdot \left(T_\alpha^k h_k\right) \ d\tau \right| + \varepsilon. \end{aligned}$$

Since E_k is weak $*$ compact and $E_k \downarrow E$, E is also weak $*$ compact. If g is a weak$*$ limit of a subsequence $\{T_\alpha^{k_j}(h_{k_j})\}$ of $\{T_\alpha^k h_k\}$, then $g \in E$ and

$$\begin{aligned} \left| \int_{B_n} f \cdot g \ d\tau \right| &= \lim_{j\to\infty} \left| \int_{B_n} f \left(T_\alpha^{k_j} h_{k_j}\right) \ d\tau \right| \\ &\geq \lim_{j\to\infty} \| T_\alpha^{k_j} f \|_{L^1(\tau)} - \varepsilon. \end{aligned}$$

Hence we have

$$\lim_{k\to\infty} \| T_\alpha^k f \|_{L^1(\tau)} \ \leq \ \sup \left\{ \left| \int_{B_n} f \cdot h \ d\tau \right| \ \Big| \ h \in E \right\}. \tag{3.2}$$

From (3.1), (3.2) we get

$$\lim_{k\to\infty} \| T_\alpha^k f \|_{L^1(\tau)} \ = \ \sup \left\{ \left| \int_{B_n} f \cdot h \ d\tau \right| \ \Big| \ h \in E \right\}. \tag{3.3}$$

From (3.3) and Lemma 3.1, if $u \in L^1_R(\tau)$ then for every $h \in E$

$$\int_{B_n} f \ d\tau = 0 \quad \text{if and only if} \quad \int_{B_n} f \cdot h \ d\tau = 0.$$

Therefore, we conclude

$$E \ = \ \{ \ c \in \mathbb{C} \mid |c| \leq 1 \ \},$$

and we can rewrite (3.3) as

$$\begin{aligned} \lim_{k\to\infty} \| T_\alpha^k f \|_{L^1(\tau)} &= \sup \left\{ \left| \int_{B_n} cf \ d\tau \right| \ \Big| \ |c| \leq 1 \right\} \\ &= \left| \int_{B_n} f \ d\tau \right|. \end{aligned}$$

□

Since T_α is a contraction on $L^1(\tau)$ and $L^\infty(B_n)$, it is also a contraction on $L^p(\tau)$ for $1 < p < \infty$. Next proposition says when $1 < p < \infty$, the iteration of T_α on $L^p(\tau)$ is much simpler in a way that

$$\lim_{k\to\infty} \| T_\alpha^k f \|_{L^p(\tau)} = 0 \quad \text{for every } f \in L^p(\tau).$$

PROPOSITION 3.3. *If $1 < p < \infty$ and $f \in L^p(\tau)$, then*

$$\lim_{k\to\infty} \int_{B_n} | T_\alpha^k f |^p \, d\tau = 0.$$

Proof. Since T_α is a positive contraction on $L^p(\tau)$, by standard approximation, it is enough to prove the proposition when f is a characteristic function χ_K for every compact subset K of B_n .

First, we'll show that $\lim_{k\to\infty} \| T_\alpha^k \chi_K \|_\infty = 0$.
Choose $0 < r < 1$ such that $K \subset rB_n$, and define $u : [0,1] \to \mathbb{R}$ by

$$u(t) = -1 \quad \text{for} \quad 0 \le t \le r$$
$$u(t) = \frac{t-1}{1-r} \quad \text{for} \quad r \le t \le 1.$$

Then $v(z) = u(|z|)$ is sunharmonic in B_n, which implies that $v \circ \varphi_a$ is subharmonic for each $a \in B_n$. Thus from the definition of T_α and submean value property, we get $T_\alpha v \ge v$. Since T_α is a positive operator, $\{T_\alpha^k v\}$ is increasing and uniformly bounded on B_n. Hence $\lim T_\alpha^k v = g$ exists and satisfies $T_\alpha g = g$. Since g is bounded on B_n satisfying $T_\alpha g = g$, g is $\mathcal{M}$-harmonic. Thus we get $g = 0$ on B_n since $g = 0$ on ∂B_n. Therefore, bu Dini's theorem, $\{T_\alpha^k v\}$ converges uniformly to zero, which implies

$$\lim_{k\to\infty} \| T_\alpha^k \chi_K \|_\infty = 0.$$

since $T_\alpha^k v \le -T_\alpha^k \chi_K \le 0$.

Next, let $p = 1 + c$ for some $c > 0$. For a given $\varepsilon > 0$, we define $A_k = \{z \in B_n \mid T_\alpha^k \chi_K > \varepsilon\}$. Then $\| T_\alpha^k \chi_K \|_\infty \le 1$ for every k, and A_k is empty for all k sufficiently large.
Since

$$\int_{B_n} | T_\alpha^k \chi_K |^p \, d\tau = \int_{A_k} (T_\alpha^k \chi_K)(T_\alpha^k \chi_K)^c \, d\tau + \int_{B_n \setminus A_k} (T_\alpha^k \chi_K)(T_\alpha^k \chi_K)^c \, d\tau,$$

we have

$$\int_{B_n} | T_\alpha^k \chi_K |^p \, d\tau \le \tau(A_k) + \tau(K)\varepsilon^c.$$

Therefore, we get the proof of the proposition by taking $k \to \infty$ □

Even though $T_\alpha^k f$ generally does not converges to zero in norm when $f \in L^1(\tau)$, next proposition implies that it converges pointwise to zero in B_n and much more is true.

PROPOSITION 3.4. *If $f \in L^1(\tau)$ and $z \in B_n$, then*

$$\sum_{k=0}^{\infty} \mid T_\alpha^k f(z) \mid < \infty.$$

Proof. First, we prove that the function $u(z) = |z|^2 - 1$ satisfies $T_\alpha u > u$ on B_n.
From the definition of weighted Berezin tranform we get

$$\begin{aligned}(T_\alpha u)(z) &= \int_{B_n} (|w|^2 - 1)\ \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2n+2+2\alpha}}\ d\nu_\alpha(w) \\ (3.4) \qquad &= -(1-|z|^2)^{n+\alpha+1}\frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}\int_{B_n} \frac{(1-|w|^2)^{1+\alpha}}{|1-\langle z,w\rangle|^{2n+2+2\alpha}}\ d\nu(w)\end{aligned}$$

Using the binomial series identity

$$(1-x)^{-\beta} = \sum_{k=0}^{\infty} \frac{\Gamma(k+\beta)}{k!\Gamma(\beta)} x^k$$

for $|x| < 1$, $\beta \geq 0$ and applying integration in polar coordinates (1.4.3 of [7]) together with Proposition 1.4.10 of [7], we get

$$\int_{B_n} \frac{(1-|w|^2)^{1+\alpha}}{|1-\langle z,w\rangle|^{2n+2+2\alpha}}\ d\nu(w) = \frac{n!\Gamma(\alpha+2)}{\Gamma^2(n+\alpha+1)} \sum_{k=0}^{\infty} \frac{\Gamma^2(n+k+\alpha+1)}{k!\Gamma(k+n+\alpha+2)}|z|^{2k}.$$

Therefore, we have

$$\begin{aligned}(T_\alpha u)(z) &= -(1-|z|^2)^{n+\alpha+1}\frac{\alpha+1}{\Gamma(n+\alpha+1)} \sum_{k=0}^{\infty} \frac{\Gamma^2(n+k+\alpha+1)}{k!\Gamma(k+n+\alpha+2)}|z|^{2k} \\ &> -(1-|z|^2)^{n+\alpha+1} \sum_{k=0}^{\infty} \frac{\Gamma(n+k+\alpha+1)}{k!\Gamma(k+n+\alpha+2)}|z|^{2k} \\ (3.5) \qquad &= -(1-|z|^2)^{n+\alpha+1}(1-|z|^2)^{-(n+\alpha)} = u(z).\end{aligned}$$

Next, since u is a uniform limit of a sequence of functions on $C_c(B_n)$ and if $v \in C_c(B_n)$ then we can show exactly the same way as the proof of Proposition 3.3 that

$$\lim_{k\to\infty} \| T_\alpha^k v \|_\infty = 0.$$

Hence we get

$$\lim_{k\to\infty} \| \, T_\alpha^k u \, \|_\infty = 0.$$

Thus if we define $g = Tu - u$, then $g > 0$ and $\|g\|_\infty \leq 2$. Moreover,

$$\sum_{k=0}^{m} T_\alpha^k g$$

converges uniformly to $-u$ as $m \to \infty$. Combining this and (2.1), we get

$$\begin{aligned} \int_{B_n} \left(\sum_{k=0}^{\infty} T_\alpha^k |f| \right) \cdot g \, d\tau &= \int_{B_n} |f| \cdot \left(\sum_{k=0}^{\infty} T_\alpha^k g \right) \, d\tau \\ &= \int_{B_n} |f| \cdot (-u) \, d\tau \leq \|f\|_{L^1(\tau)} \|u\|_\infty < \infty. \end{aligned}$$

Since $g > 0$, the proof is complete. □

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