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OSCILLATION AND NONOSCILLATION CRITERIA FOR DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract. We give necessary and sufficient conditions such that the homogeneous differential equations of the type:

$$(r(t)x'(t))' + q(t)x'(t) + p(t)x(t) = 0$$

are nonoscillatory where $r(t) > 0$ for $t \in I = [\alpha, \infty)$, $\alpha > 0$. Under the suitable conditions we show that the above equation is nonoscillatory if and only if for $\gamma > 0$,

$$(r(t)x'(t))' + q(t)x'(t) + p(t)x(t-\gamma) = 0$$

is nonoscillatory. We obtain several comparison theorems.

1. Introduction

The main purpose of this paper is to find necessary and sufficient conditions for the differential equations of the type:

$$(A_1) \qquad (r(t)x'(t))' + q(t)x'(t) + p(t)x(t) = 0$$

are oscillatory or nonoscillatory where $r(t) > 0$ for $t \in I = [\alpha, \infty)$, $\alpha > 0$. We shall restrict our attention to the solutions of (A_1) that exist on some ray of the form $[t, \infty)$ where $t \geq \alpha$. Throughout of this paper the coefficients $p(t)$, $q(t)$ and $r(t)$ satisfy the conditions

- (A) $p(t)$ is real valued and locally integrable over I and not identically zero in any neighborhood of ∞.
- (B) $q(t)$ is real valued and locally integrable over I.
- (C) For all $t \in I$, $r(t) > 0$ and $\int_\alpha^\infty \frac{1}{r(t)}\, dt = \infty$.

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By a solution to (A_1) we mean a real valued function u that satisfies (A_1) in I and that u and u' are locally absolutely continuous over I. We consider only nontrivial solutions of (A_1). The usual existence theorems hold(see Naimark[9]).

DEFINITION. A solution $x(t)$ of (1) is said to be oscillatory if it has arbitrarily large zeros over I, otherwise it is said to be nonoscillatory.

It is well known (see Reid[10]) that either all the solutions of (1) are nonoscillatory, or all the solutions are oscillatory. In the former case, we call the differential equation (1) nonoscillatory and in the later case, (1) oscillatory.

The investigation of the oscillation for the equation

$$(p(t)x'(t))' + q(t)x(t) = 0 \tag{A_2}$$

may be done in the following many directions([1], [2]-[7], [11]) : among these, an often considered way is to determine "integral tests" involving functions p and q in order to obtain oscillatory criteria. An example is the following well-known Leighton's result(see [8]) : The every solution of (A_2) is oscillatory if

$$\int_0^\infty \frac{1}{p(\sigma)}\, d\sigma = \infty, \qquad \int_0^\infty q(\sigma)\, d\sigma = \infty. \tag{R_1}$$

We note that the equation $x''(t) + q(t)x(t) = 0$ is oscillatory on I if $\int_\alpha^\infty q(\sigma)\, d\sigma = \infty$. The main purpose of this paper is to establish the nonoscillatory characterizations of solutions by using Ricatti transform. Using these nonoscillatory characterizations and a differential inequality, we derive some comparison theorems and give examples.

2. Main results

Consider a Ricatti transform of the equation (A_1). Put

$$W(t) = \frac{r(t)x'(t)}{x(t)}. \tag{1}$$

We have

$$\begin{aligned} W'(t) &= -\frac{W^2(t)}{r(t)} - \frac{q(t)W(t)}{r(t)} - p(t) \\ &= -\frac{1}{r(t)}\left[W(t) + \frac{q(t)}{2}\right]^2 - \left[p(t) - \frac{q^2(t)}{4r(t)}\right]. \end{aligned} \tag{2}$$

Put $E(t) = \exp \int_\alpha^t \frac{q(\sigma)}{r(\sigma)}\, d\sigma$ and $\Phi(t) = \int_t^\infty \left[p(\sigma) - \frac{q^2(\sigma)}{4r(\sigma)} \right] d\sigma \geq 0$ for $t \geq \alpha$. We immediately obtain the following.

THEOREM 2.1. *The equation* (A_1) *is oscillatory if for* $t \geq \alpha$, $p(t) \geq 0$ *and*

$$\int_\alpha^\infty \frac{1}{E(\sigma)r(\sigma)}\, d\sigma = \infty, \tag{3}$$

$$\Phi(\alpha) = \infty. \tag{4}$$

Proof. Since $E'(t) = \frac{q(t)}{r(t)} E(t)$, the equation (A_1) is reduced to

$$(E(t)r(t)x'(t))' + E(t)p(t)x(t) = 0. \tag{5}$$

Note that (A_1) is oscillatory if and only if (5) is oscillatory. Assume that (A_1) is nonoscillatory. Then there exists a nonoscillatory solution $x(t)$ of (A_1). So we may assume that $x(t) > 0$ on $[t_1, \infty)$ for some $t_1 \geq \alpha$. In the case that $x(t) < 0$, put $y(t) = -x(t)$. We show that $x'(t) > 0$ for $t \geq t_1$. Since

$$(E(t)r(t)x'(t))' \leq 0$$

$E(t)r(t)x'(t)$ is not increasing for $t \geq t_1$. Assume that $E(t_2)r(t_2)x'(t_2) < 0$ for some $t_2 \geq t_1$. Put $A := E(t_2)r(t_2)x'(t_2)$. Then for $t \geq t_2$, we have $E(t)r(t)x'(t) \leq A$. Dividing both sides by $E(t)r(t)$ and integrating from t_2 to $t\,(> t_2)$ we obtain

$$x(t) \leq x(t_2) + A \int_{t_2}^t \frac{1}{E(\sigma)r(\sigma)}\, d\sigma.$$

Thus it follows that $x(t) < 0$ for sufficiently large t, which is a contradiction. So we have $x'(t) > 0$ for $t \geq t_1$. We use Ricatti transform (1). Then we have the equality (2). Integrating (2) from t_1 to $t(> t_1)$ we have

$$W(t) - W(t_1) + \int_{t_1}^t \left[p(\sigma) - \frac{q^2(\sigma)}{4r(\sigma)} \right] d\sigma = -\int_{t_1}^t \frac{1}{r(\sigma)} \left[W(\sigma) + \frac{q(\sigma)}{2} \right]^2 d\sigma.$$

By means of (4) there exists a $t_3 \geq t_1$ such that for $t \geq t_3$,

$$W(t) \leq -\int_{t_1}^t \frac{1}{r(\sigma)} \left[W(\sigma) + \frac{q(\sigma)}{2} \right]^2 d\sigma,$$

which is impossible because $W(t) > 0$ for $t \geq t_1$. □

We note that the results of Theorem 1 is more general form than those of Horng–Jaan[5] and Kulenovic[6].

Theorem 2.2. *The equation* (A_1) *is nonoscillatory if and only if there exists a continuously differentiable function W(t) in* $t \in I = [t_1, \infty)$ *for some* $t_1 > \alpha$ *such that a Ricatti differential inequality*

$$W'(t) + \frac{W^2(t)}{r(t)} + \frac{q(t)W(t)}{r(t)} + p(t) \leq 0 \tag{6}$$

is valid.

Proof. Let $x(t)$ be a nonoscillatory solution of (A_1). We may assume that there exists a $t_1 \geq \alpha$ such that $x(t) > 0$ for $t \geq t_1$. In the case of $x(t) < 0$, we can apply the analogous method to $y(t) = -x(t)$. (1) satisfies a Ricatti equation

$$W'(t) + \frac{W^2(t)}{r(t)} + \frac{q(t)W(t)}{r(t)} + p(t) = 0. \tag{A_4}$$

This proves "only if " part of theorem. If there exists a continuously differentiable function W(t) on I satisfying (6), let $p_0(t) = W'(t) + \frac{W^2(t)}{r(t)} + \frac{q(t)W(t)}{r(t)} + p(t)$. Then $p_0(t) \leq 0$ and

$$W'(t) + \frac{W^2(t)}{r(t)} + \frac{q(t)W(t)}{r(t)} + p(t) - p_0(t) = 0$$

which is a Ricatti equation for $(r(t)x'(t))' + q(t)x'(t) + (p(t) - p_0(t))\, x(t) = 0$. It follows that

$$(E(t)r(t)x'(t))' + E(t)(p(t) - p_0(t))x(t) = 0. \tag{7}$$

We compare this equation with (5). Since $p_0(t) \leq 0$, (7) is a Sturm majorant for (5). On the other hand, (7) possesses a positive solution

$$x(t) = \exp \int_\alpha^t \frac{W(\sigma)}{r(\sigma)}\, d\sigma.$$

This shows that (5) is nonoscillatory and thus (A_1) is nonoscillatory. $\square$

Lemma 2.3. *Assume that for* $t \geq \alpha$, $p(t) \geq 0$, $q(t) \geq 0$ *and (3) are valid. If the differential equation* (A_1) *has a positive solution, we have*

$$\lim_{t \to \infty} \frac{r(t)x'(t)}{x(t)} = 0.$$

Proof. Let $x(t) > 0$ be a solution of (A_1). As seen in the proof of Theorem 2.1, it follows from (3) and $p(t) \geq 0$ for $t \geq \alpha$ that there exists a $t_1 \geq \alpha$ such that $x'(t) > 0$ for $t \geq t_1$. Put $W(t) = \frac{r(t)x'(t)}{x(t)} > 0$ for $t \geq t_1$ and consider the Ricatti equation (A_4). It is obvious that

$$-\frac{W'(t)}{W^2(t)} \geq \frac{1}{r(t)}.$$

Integrating the above formula over $[t_1, \infty)$ and considering the condition (C), we see $\lim_{t\to\infty} W(t) = 0$. □

In Theorem 2.1 we proved the equation (A_1) is oscillatory under the assumption $\Phi(\alpha) = \infty$. We consider the remaining case.

THEOREM 2.4. *Assume that for $t \geq \alpha$, $p(t) \geq 0$, $q(t) \geq 0$, $|\Phi(\alpha)| < \infty$ and (3) are valid. The following two statements are equivalent.*

(i) *The equation (A_1) is nonoscillatory on I.*

(ii) *There exist a $T > \alpha$ and a continuously differentiable function $W(t)$ for $t \in I_1 = [T, \infty)$ satisfying*

$$W(t) = \Phi(t) + \int_t^\infty \frac{1}{r(\sigma)} \left[W(\sigma) + \frac{q(\sigma)}{2} \right]^2 d\sigma.$$

Proof. Let the equation (A_1) be nonoscillatory on I. We may assume that there exist a $t_1 \geq \alpha$ and a solution $x(t)$ of (A_1) such that $x(t) > 0$ for $t \geq t_1$. In the case of $x(t) < 0$, we can apply the same method to $y(t) = -x(t)$. From (3) we can deduce that there exists a $t_2 \geq t_1$ such that $x'(t) > 0$ for $t \geq t_2$. Considering a Ricatti transform (1), we obtain (A_4). It follow that

$$-W'(t) = \left[p(t) - \frac{q^2(t)}{r(t)} \right] + \frac{1}{r(t)} \left[W(t) + \frac{q(t)}{2} \right]^2.$$

Integrating this equality over $[t, \infty)$, $t \geq t_2$ and taking account of Lemma 2.3, we obtain (ii). Conversely, if (ii) is valid, immediately we obtain (A_4). Then it follows from theorem 2.2 that the differential equation (A_1) is nonoscillatory on I. □

Set $\phi(t) = \int_t^\infty p(\sigma)\, d\sigma$.

THEOREM 2.5. *Assume that for $t \geq \alpha$, $p(t) \geq 0$, $q(t) \leq 0$, $\phi(\alpha) < \infty$ and $|\Phi(\alpha)| < \infty$ are valid. Two statements in Theorem 2.4 are equivalent.*

Proof. We only prove that (i) $\Rightarrow$ (ii) of Theorem 2.4. We may assume that there exists a $t_0 \geq \alpha$ such that for $t \geq t_0$, $x(t) > 0$ is a nonoscillatory solution of (A_1). Putting $W(t) = \frac{r(t)x'(t)}{x(t)}$ for $t \geq t_0$, we obtain a Ricatti equation (A_4). Our proof is analogous to those of Wintner[12]. Integrating (A_4) on $[t, \xi]$, we have

$$(8) \quad W(\xi) - W(t) + \int_t^\xi \left[p(\sigma) - \frac{q^2(\sigma)}{4r(\sigma)}\right] d\sigma + \int_t^\xi \frac{1}{r(\sigma)} \left[W(\sigma) + \frac{q(\sigma)}{2}\right]^2 d\sigma = 0.$$

Two cases arise:

(a) $\int_t^\infty \frac{1}{r(\sigma)} \left[W(\sigma) + \frac{q(\sigma)}{2}\right]^2 d\sigma < \infty$,

(b) $\int_t^\infty \frac{1}{r(\sigma)} \left[W(\sigma) + \frac{q(\sigma)}{2}\right]^2 d\sigma = \infty$.

Assume that (b) is valid. There exists a $t_1 \geq t_0$ such that

$$W(\xi) + \int_{t_1}^\xi \frac{1}{r(\sigma)} \left[W(\sigma) + \frac{q(\sigma)}{2}\right]^2 d\sigma = W(t) - \int_t^\xi \left[p(\sigma) - \frac{q^2(\sigma)}{4r(\sigma)}\right] d\sigma - \int_t^{t_1} \frac{1}{r(\sigma)} \left[W(\sigma) + \frac{q(\sigma)}{2}\right]^2 d\sigma \leq -1$$

for $\xi \geq t_1$. Thus it follows that

$$(9) \qquad -W(\xi) \geq 1 + \int_{t_1}^\xi \frac{1}{r(\sigma)} \left[W(\sigma) + \frac{q(\sigma)}{2}\right]^2 d\sigma, \ \xi \geq t_1.$$

We see that $W(\xi) < 0$ for $\xi \geq t_1$ and $\lim_{\xi\to\infty} W(\xi) = -\infty$. It follows that

$$\begin{aligned} &\frac{\left[W(\xi) + \frac{q(\xi)}{2}\right]^2}{r(\xi)\left(1 + \int_{t_1}^\xi \frac{1}{r(\sigma)} \left[W(\sigma) + \frac{q(\sigma)}{2}\right]^2 d\sigma\right)} \\ &\geq \frac{W^2(\xi)}{r(\xi)\left(1 + \int_{t_1}^\xi \frac{1}{r(\sigma)} \left[W(\sigma) + \frac{q(\sigma)}{2}\right]^2 d\sigma\right)} \\ &\geq -\frac{W(\xi)}{r(\xi)} = -\frac{x'(\xi)}{x(\xi)}. \end{aligned}$$

Integrating over $[t_1, \xi]$, we have

$$\log\left[1+\int_{t_1}^{\xi}\frac{1}{r(\sigma)}\left[W(\sigma)+\frac{q(\sigma)}{2}\right]^2 d\sigma\right] \geq \log\frac{x(t_1)}{x(\xi)}.$$

Considering (9), we obtain $-W(\xi) = -\frac{r(\xi)x'(\xi)}{x(\xi)} \geq \frac{x(t_1)}{x(\xi)}$. Since $x'(\xi) \leq -\frac{x(t_1)}{r(\xi)}$, by means of the condition (C) we see $x(\xi) < 0$ for sufficiently large ξ, which is a contradiction. Thus (a) holds. Using the condition (C), we have

$$\lim_{\xi\to\infty}\left[W(\xi)+\frac{q(\xi)}{2}\right] = 0.$$

From $\phi(\alpha) < \infty$, $|\Phi(\alpha)| < \infty$ and the condition (C) it follows that $\lim_{\xi\to\infty} q(\xi) = 0$ and $\lim_{\xi\to\infty} W(\xi) = 0$, from which we obtain (ii) of Theorem 2.4. □

THEOREM 2.6. *Under the assumption of theorem 2.4 the following are equivalent.*

(i) *The equation* (A_1) *is nonoscillatory on* I.

(ii) *There exist a* $T > \alpha$ *and a continuously differentiable function* $U(t)$ *for* $t \in I_1 = [T, \infty)$ *satisfying*

$$U(t) \geq \phi(t) + \int_t^{\infty}\frac{U^2(\sigma)+q(\sigma)U(\sigma)}{r(\sigma)}\,d\sigma.$$

Proof. It follows from Theorem 2.4 that (i) implies (ii). We show that "(ii) ⇒ (i)". Set

$$\Gamma = \{v \in C(I)\,|\,0 \leq v(t) \leq U(t),\, t \in I\}.$$

For $t \in I$ and $v \in \Gamma$, we consider an operator T defined by:

$$Tv(t) \;=\; \phi(t) + \int_t^{\infty}\frac{v^2(\sigma)+q(\sigma)v(\sigma)}{r(\sigma)}\,d\sigma.$$

It is obvious that Γ is a nonempty, closed, bounded, convex subset of $C(I)$ with topology of the uniform convergence on every compact subinterval of I. We use the method of successive approximation. Consider

$$v_0(t) = 0, \quad v_n(t) = Tv_{n-1}(t), \quad n \geq 1.$$

Then since $q(\sigma) \geq 0$ for $t \in I$, it is obvious that $T : \Gamma \to \Gamma$ is an increasing operator. In particular, by induction we can show that the sequence $\{v_n\}$ is increasing and bounded from above:

$$v_n(t) \leq v_{n+1}(t) = Tv_n(t), \quad v_n(t) \leq U(t), \quad \text{for all } n.$$

Therefore $\lim_{t\to\infty} v_n(t)$ exists for $t \in I$, call it $v(t)$. By Lebesgue dominated convergence theorem we have

$$v(t) = \lim_{n\to\infty} v_{n+1}(t) = \phi(t) + \lim_{n\to\infty} \int_t^\infty \frac{v_n^2(\sigma) + q(\sigma)v_n(\sigma)}{r(\sigma)}\, d\sigma = Tv(t).$$

i.e., T has a fixed point v in Γ. Thus theorem 2.4 implies that the equation (A_1) is nonoscillatory. $\square$

Consider a Leighton transform $s = \int_\alpha^t 1/r(\sigma)\, d\sigma$. Then the equation (A_1) is reduced to

$$(A_5) \qquad \frac{d^2X}{ds^2} + Q(s)\frac{dX}{ds} + R(s)P(s)X(s) = 0.$$

where $X(s) = x(t(s))$, $P(s) = p(t(s))$, $Q(s) = q(t(s))$, $R(s) = r(t(s))$.

Corollary 2.7. *The equation* (A_1) *is oscillatory if for* $s \geq 0$, $P(s) \geq 0$ *and*

$$\int_0^\infty \exp \int_0^\sigma Q(\tau)\, d\tau d\sigma = \infty,$$
$$\int_0^\infty \left[P(\sigma)R(\sigma) - \frac{Q^2(\sigma)}{4}\right] d\sigma = \infty.$$

Consider the equation (5) and a transform $\rho = \int_\alpha^t \frac{1}{E(\sigma)r(\sigma)}\, d\sigma$. Put $X_1(\rho) = x(t(\rho))$, $P_1(\rho) = p(t(\rho))$, $Q_1(\rho) = q(t(\rho))$, $R_1(\rho) = r(t(\rho))$ and $E_1(\rho) = E(t(\rho))$. Then the equation (A_1) is reduced to the form

$$\frac{d^2X_1}{d\rho^2} + E_1^2(\rho)P_1(\rho)R_1(\rho)X_1(\rho) = 0.$$

Thus we obtain

Corollary 2.8. *The equation* (A_1) *is oscillatory if for* $\rho \geq 0$, $P_1(\rho) \geq 0$ *and*

$$\int_0^\infty E_1^2(\sigma)P_1(\sigma)R_1(\sigma)\, d\sigma = \infty.$$

Compare this result with (R_1).

3. Comparison theorems

Consider the Ricattl differential inequality (6) : $W'(t) + \frac{W^2(t)}{r(t)} + \frac{q(t)W(t)}{r(t)} + p(t) \leq 0$.

THEOREM 3.1. *Consider a differential equation:*

$$(r_1(t)x'(t))' + q_1(t)x'(t) + p_1(t)x(t) = 0. \tag{A_6}$$

Assume that for $t \in I$,

$$0 < r_1(t) \leq r(t), \quad q(t) \leq q_1(t), \quad p(t) \leq p_1(t).$$

The equation (A_1) is nonoscillatory if the differential equation (A_6) is nonoscillatory and (A_6) has a solution $x(t)$ satisfying $x(t) > 0,\ x'(t) \geq 0$ for $t \in I$.

REMARK 3.2. In the above theorem the conditions $p(t) \leq p_1(t)$ and $x'(t) \geq 0$ can be replaced by $p_1(t) \geq p(t) \geq 0$ and the equality

$$\int_\alpha^\infty \frac{1}{E(\sigma)r(\sigma)}\,d\sigma = \infty.$$

THEOREM 3.3. *The equation (A_1) is nonoscillatory if the differential inequality*

$$(r(t)y'(t))' + q(t)y'(t) + p(t)y(t) \leq 0$$

has an eventually positive solution.

Proof. Assume that there exists a $t_1 \geq \alpha$ and that the inequality has a solution $y(t) > 0$ for $t \geq t_1$. Put

$$p_0(t) = (r(t)y'(t))' + q(t)y'(t) + p(t)y(t).$$

Then $y(t) > 0$ for $t \geq t_1$ is a solution of the equation

$$(r(t)x'(t))' + q(t)x'(t) + \left(p(t) - \frac{p_0(t)}{y(t)}\right)x(t) = 0.$$

Considering a Ricatti transform $W(t) = \frac{r(t)x'(t)}{x(t)}$, we obtain

$$W'(t) + \frac{W^2(t)}{r(t)} + \frac{q(t)W(t)}{r(t)} + \left(p(t) - \frac{p_0(t)}{y(t)}\right) = 0.$$

Since $p_0(t) \leq 0$, we have $W'(t) + \frac{W^2(t)}{r(t)} + \frac{q(t)W(t)}{r(t)} + p(t) \leq 0$. Therefore (A_1) is nonoscillatory by means of Theorem 2.2. $\square$

THEOREM 3.4. *Assume that for $t \geq \alpha$, $p(t) \geq 0$, $q(t) + r'(t) \geq 0$, $\gamma > 0$ and (3) are valid. The equation (A_1) is nonoscillatory if and only if the differential equation with a delayed argument*

$$(A_7) \qquad (r(t)x'(t))' + q(t)x'(t) + p(t)x(t-\gamma) = 0.$$

is nonoscillatory.

Proof. Assume that (A_7) is nonoscillatory. We may assume that there exist a $t_1 \geq \alpha$ and a positive solution $x(t) > 0$ and $x(t-\gamma) > 0$ for $t \geq t_1$. By (3) and the similar method to those in proof of Theorem 2.1 we can show that $x'(t) > 0$ for $t \geq t_1$. Put

$$W(t) = \frac{r(t)x'(t)}{x(t-\gamma)} > 0 \quad \text{for } t \geq t_1.$$

Then we have

$$W'(t) = -\frac{q(t)}{r(t)}W(t) - p(t) - r(t)x'(t)\frac{x'(t-\gamma)}{x^2(t-\gamma)}.$$

Since $r(t)x''(t) = -\left(r'(t) + q(t)\right)x'(t) - p(t)x(t) \leq 0$, it follows that $x''(t) \leq 0$ for $t \geq t_1$. Immediately we see that there exists a $t_2 \geq t_1$ such that the inequality $W'(t) + \frac{W^2(t)}{r(t)} + \frac{q(t)W(t)}{r(t)} + p(t) \leq 0$ is valid for $t \geq t_2$, which completes the "only if" part of our theorem. Conversely, assume that (A_1) has a nonoscillatory solution $x(t)$. We may assume that there exists a $t_3 \geq \alpha$ such that $x(t) > 0$ and $x(t-\gamma) > 0$ are valid for $t \geq t_3$. Since the equality (3) is valid, $x'(t) > 0$ and so $x(t) \geq x(t-\gamma)$ for $t \geq t_3$. Thus from (A_1) we obtain

$$(r(t)x'(t))' + q(t)x'(t) + p(t)x(t-\gamma) \leq 0$$

for $t \geq t_3$. Since this differential inequality has a positive solution $x(t)$ for $t \geq t_3$, by Theorem 3.3 (A_7) is nonoscillatory. $\square$

EXAMPLE 3.5. Let a, b and α be positive constants. Consider an Euler differential equation:

$$(10) \qquad x''(t) + \frac{a}{t}\,x'(t) + \frac{b}{t^2}\,x(t) = 0 \quad \text{for} \quad t \geq \alpha,$$

and put $Tv(t) = \phi(t) + \int_t^\infty \left\{v^2(\sigma) + q(\sigma)v(\sigma)\right\}/r(\sigma)\,d\sigma$ where $q(t) = a/t$, $p(t) = b/t^2$ and $\phi(t) = \int_t^\infty p(\sigma)\,d\sigma$. Now we seek a relation between a and b so that (10) is nonoscillatory and $v \in L^2[\alpha, \infty)$. Set

$$v_0(t) = 0, \quad v_n(t) = Tv_{n-1}(t), \quad \text{for} \quad n \geq 1.$$

Immediately we obtain

$$\begin{aligned} v_1(t) &= \phi(t) = \frac{b}{t} = \frac{c_1}{t}, \\ v_2(t) &= \frac{b}{t} + \int_t^\infty \frac{c_1^2 + ac_1}{\sigma^2}\, d\sigma = \frac{b + c_1^2 + ac_1}{t} = \frac{c_2}{t}, \\ v_n(t) &= \frac{b}{t} + \int_t^\infty \frac{c_{n-1}^2 + ac_{n-1}}{\sigma^2}\, d\sigma = \frac{b + c_{n-1}^2 + ac_{n-1}}{t} = \frac{c_n}{t}. \end{aligned}$$

Thus we get

$$c_n = c_{n-1}^2 + ac_{n-1} + b \quad \text{for} \quad n \geq 1.$$

From $a > 0$ it follows that the sequence $\{c_n\}$ is increasing and $c_n > 0$ for all $n \geq 1$. We have two cases.

(a) $\lim\limits_{n\to\infty} c_n = c < 0$,

(b) $\lim\limits_{n\to\infty} c_n = \infty$.

If (b) holds, $v(t) = \lim_{n\to\infty} Tv_{n-1}(t) = \infty$ for any $t \geq \alpha$, which is a contradiction because $v \notin L^2[\alpha, \infty)$. So (a) is valid and then c satisfies the equation $c^2 + (a-1)c + b = 0$. Thus we have $(a-1)^2 - 4b \geq 0$. i.e.,

(c) if $(a-1)^2 - 4b \geq 0$, the differential equation (10) is nonoscillatory,

(d) if $(a-1)^2 - 4b < 0$, the differential equation (10) is oscillatory.

In fact, if we set $t = e^s$, the differential equation (10) is reduced to

$$\frac{d^2x(t(s))}{ds^2} + (a-1)\frac{dx(t(s))}{ds} + b\,x(t(s)) = 0.$$

Therefore either (c) or (d) is valid.

Example 3.6. Consider a differential equation

$$(r(t)x'(t))' + q(t)x'(t) + \frac{\delta x(t)}{E^2(t)r(t)\varphi^2(t)} = 0 \tag{11}$$

where $E(t) = \exp\int_\alpha^t \frac{q(\sigma)}{r(\sigma)}\, d\sigma$ and $\varphi(t) = \int_\alpha^t \frac{d\sigma}{E(\sigma)r(\sigma)}$. Then we have

$$(E(t)r(t)x'(t))' + \frac{\delta x(t)}{E(t)r(t)\varphi^2(t)} = 0.$$

Let $x = \varphi^n(t)$ satisfy the equation (11). Since $\varphi'(t) = \frac{1}{E(t)r(t)}$, we obtain the indicial equation

$$n(n-1) + \delta = 0.$$

Thus it follows that $x = \varphi^{1/2}(t)$ is a nonoscillatory solution when $\delta = \frac{1}{4}$. It is obvious that equation (11) is oscillatory if $\delta > \frac{1}{4}$ and is nonoscillatory if $\delta \leq \frac{1}{4}$.

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