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ON WEIGHTED GENERALIZATION OF OPIAL TYPE INEQUALITIES IN TWO VARIABLES

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Abstract. In this paper, we establish some weighted generalization of Opial type inequalities in two independent variables for two functions. We also obtain weighted Opial type inequalities by using p-norms. Special cases of our results reduce to the inequalities in earlier study.

1. Introduction

In the year 1960, Opial established the following interesting integral inequality [11]:

Theorem 1.1. *Let $x(t) \in C^{(1)}[0,h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then, the following inequality holds*

$$\int_0^h |x(t)x'(t)|\, dt \le \frac{h}{4} \int_0^h (x'(t))^2\, dt. \tag{1}$$

The constant $h/4$ is the best possible

Over the years a large number of papers have been appeared in the literature which deals with the simple proofs, various generalizations and

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discrete analogues of Opial inequality and its generalizations, for some of them please see [5], [6], [9], [12]- [15], [25]- [27].

In [28], Yang proved the following Opial type inequalities in two variables:

THEOREM 1.2. *If $f(t,s)$, $f_1(t,s)$ and $f_{12}(t,s)$ are continuous functions on $[a,b]\times[c,d]$ and if $f(a,s)=f(b,s)=f_1(t,c)=f_1(t,d)=0$ for $a\le t\le b$, $c\le s\le d$, then*

$$\int_a^b\int_c^d |f(t,s)|\,|f_{12}(t,s)|\,dsdt \le \frac{(b-a)(d-c)}{8}\int_a^b\int_c^d |f_{12}(t,s)|^2\,dsdt \tag{2}$$

where

$$f_1(t,s)=\frac{\partial}{\partial t}f(t,s) \text{ and } f_{12}(t,s)=\frac{\partial^2}{\partial t\partial s}f(t,s).$$

On the other hand, B. G. Pachpatte published some papers which focus on the generalizations of the inequality (2). For some of these generalizations, please see [16]- [21]. Moreover using two functions and their partial derivatives, W. S. Cheung established some generalizations of the inequality (2) in [7]. For the other Opial type inequalities in higher dimension, please see [1], [4], [8], [22]- [24].

Recently, Budak and Sarikaya have proved the following generalized Opial type inequalities in [2].

THEOREM 1.3. *Let $f(t,s)$, $f_1(t,s)$, $f_{12}(t,s)$, $g(t,s)$, $g_1(t,s)$ and $g_{12}(t,s)$ be continuous on $[a,b]\times[c,d]$ and let $g_{12}, f_{12}\in L_2([a,b]\times[c,d])$. If $g(a,s)=g(b,s)=g_1(t,c)=g_1(t,d)=0$ for $(t,s)\in[a,b]\times[c,d]$, then for all $(x,y)\in[a,b]\times[c,d]$ we have*

$$\begin{aligned}
&\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,dsdt \\
\le\ & \frac{1}{4}\left(\int_a^b\int_c^d [(b-a)Q(s,y)+(d-c)P(t,x)]\,|f_{12}(t,s)|^2\,dsdt\right)^{\frac{1}{2}} \\
&\times\left(\int_a^b\int_c^d [(b-a)|y-s|+(d-c)|x-t|]\,|g_{12}(t,s)|^2\,dsdt\right)^{\frac{1}{2}}
\end{aligned} \tag{3}$$

$$\leq \frac{1}{8}\left[\int_a^b\int_c^d [(b-a)Q(s,y)+(d-c)P(t,x)]\,|f_{12}(t,s)|^2\,dsdt \right.$$
$$\left. +\int_a^b\int_c^d [(b-a)\,|y-s|+(d-c)\,|x-t|]\,|g_{12}(t,s)|^2\,dsdt\right]$$

where

$$P(t,x)=\begin{cases} t-a, & a\leq t\leq x \\ b-t, & x\leq t\leq b \end{cases} \quad \text{and} \quad Q(s,y)=\begin{cases} s-c, & c\leq s\leq y \\ d-s, & y\leq s\leq d. \end{cases}$$

The aim of this paper is to establish some weighted generalization of Opial type inequalities in two independent variables.

2. Weighted Generalization of Opial type inequalities

In this section, we obtain some weighted Opial type inequalities for two functions. Then we also give a new weighted Opial type inequality involving p-norms.

Theorem 2.1. *Let $w:[a,b]\times[c,d]\to\mathbb{R}$ be an integrable and nonnegative function. Let $f(t,s)$, $f_1(t,s)$, $f_{12}(t,s)$, $g(t,s)$, $g_1(t,s)$ and $g_{12}(t,s)$ be continuous on $[a,b]\times[c,d]$ and let $g_{12},f_{12}\in L_2([a,b]\times[c,d])$. If $g(a,s)=g(b,s)=g_1(t,c)=g_1(t,d)=0$ for $(t,s)\in[a,b]\times[c,d]$, then for all $(x,y)\in[a,b]\times[c,d]$, we have*

$$\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,w(t,s)dsdt \tag{4}$$
$$\leq \frac{1}{4}\left(\int_a^b\int_c^d [(b-a)Q(s,y)+(d-c)P(t,x)]\,|f_{12}(t,s)|^2\,dsdt\right)^{\frac{1}{2}}$$

$$\times \left(\int_a^b \int_c^d \left[\left| \int_a^b \int_s^y w(u,v)dvdu \right| + \left| \int_t^x \int_c^d w(u,v)dvdu \right| \right] |g_{12}(t,s)|^2 dsdt \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{8} \left[\int_a^b \int_c^d [(b-a)Q(s,y)+(d-c)P(t,x)] |f_{12}(t,s)|^2 dsdt + \int_a^b \int_c^d \left[\left| \int_a^b \int_s^y w(u,v)dvdu \right| + \left| \int_t^x \int_c^d w(u,v)dvdu \right| \right] |g_{12}(t,s)|^2 dsdt \right]$$

where $P(t,x)$ and $Q(s,y)$ are defined by as in Theorem 1.3.

Proof. In order to prove Theorem 2.1, we consider the following four cases:

Case I: Let $g(a,s) = g_1(t,c) = 0$ for $(t,s) \in [a,b] \times [c,d]$.

Since $g(a,s) = g_1(t,c) = 0$, we can write

$$g(t,s) = \int_a^t \int_c^s g_{12}(u,v)dvdu.$$

Then, by Cauchy-Schwarz inequality, we have

$$\int_a^b \int_c^d |f_{12}(t,s)g(t,s)| w(t,s)dsdt \tag{5}$$

$$= \int_a^b \int_c^d (t-a)^{\frac{1}{2}} (s-c)^{\frac{1}{2}} |f_{12}(t,s)| \times (t-a)^{-\frac{1}{2}} (s-c)^{-\frac{1}{2}} |g(t,s)| w(t,s)dsdt$$

$$= \int_a^b \int_c^d (t-a)^{\frac{1}{2}} (s-c)^{\frac{1}{2}} |f_{12}(t,s)| (t-a)^{-\frac{1}{2}} (s-c)^{-\frac{1}{2}}$$

$$\times \left| \int_a^t \int_c^s g_{12}(u,v) dv du \right| w(t,s) ds dt$$

$$\leq \left(\int_a^b \int_c^d (t-a)(s-c) |f_{12}(t,s)|^2 w(t,s) ds dt \right)^{\frac{1}{2}}$$

$$\times \left(\int_a^b \int_c^d (t-a)^{-1} (s-c)^{-1} \left| \int_a^t \int_c^s g_{12}(u,v) du dv \right|^2 w(t,s) ds dt \right)^{\frac{1}{2}}.$$

By applying again the Cauchy-Schwarz inequality, we get

$$(t-a)^{-1} (s-c)^{-1} \left| \int_a^t \int_c^s g_{12}(u,v) du dv \right|^2 \tag{6}$$

$$\leq (t-a)^{-1} (s-c)^{-1} \left(\int_a^t \int_c^s du dv \right) \left(\int_a^t \int_c^s |g_{12}(u,v)|^2 du dv \right)$$

$$= \int_a^t \int_c^s |g_{12}(u,v)|^2 du dv.$$

Substituting the inequality (6) in (5), we obtain

$$\int_a^b \int_c^d |f_{12}(t,s) g(t,s)| w(t,s) ds dt \tag{7}$$

$$\leq \left(\int_a^b \int_c^d (t-a)(s-c) |f_{12}(t,s)|^2 w(t,s) ds dt \right)^{\frac{1}{2}}$$

$$\times \left(\int_a^b \int_c^d \left(\int_a^t \int_c^s |g_{12}(u,v)|^2 du dv \right) w(t,s) ds dt \right)^{\frac{1}{2}}.$$

By the integration by parts, one can show that

$$
\begin{aligned}
(8)\qquad & \int_a^b \int_c^d \left(\int_a^t \int_c^s |g_{12}(u,v)|^2 \, dvdu \right) w(t,s) dsdt \\
= & \int_a^b \left[\left(\int_a^t \int_c^s |g_{12}(u,v)|^2 \, dvdu \right) \left(\int_c^s w(t,y) dy \right) \Bigg|_c^d \right. \\
& \left. - \int_c^d \left(\int_a^t |g_{12}(u,s)|^2 \, du \right) \left(\int_c^s w(t,y) dy \right) ds \right] dt \\
= & \int_a^b \left(\int_a^t \int_c^d |g_{12}(u,v)|^2 \, dvdu \right) \left(\int_c^d w(t,y) dy \right) dt \\
& - \int_a^b \int_c^d \left(\int_a^t |g_{12}(u,s)|^2 \, du \right) \left(\int_c^s w(t,y) dy \right) dsdt \\
= & \left(\int_a^t \int_c^d |g_{12}(u,v)|^2 \, dvdu \right) \left(\int_a^t \int_c^d w(x,y) dydx \right) \Bigg|_a^b \\
& - \int_a^b \int_c^d |g_{12}(t,v)|^2 \left(\int_a^t \int_c^d w(x,y) dydx \right) dvdt \\
& - \int_c^d \left[\left(\int_a^t |g_{12}(u,s)|^2 \, du \right) \left(\int_a^t \int_c^s w(x,y) dydx \right) \Bigg|_a^b \right. \\
& \left. - \int_a^b |g_{12}(t,s)|^2 \left(\int_a^t \int_c^s w(x,y) dydx \right) dt \right] ds \\
= & \int_a^b \int_c^d |g_{12}(u,v)|^2 \left(\int_a^b \int_c^d w(x,y) dydx \right) dvdu \\
& - \int_a^b \int_c^d |g_{12}(u,s)|^2 \left(\int_a^b \int_c^s w(x,y) dydx \right) dsdu \\
& - \int_a^b \int_c^d |g_{12}(t,v)|^2 \left(\int_a^t \int_c^d w(x,y) dydx \right) dvdt
\end{aligned}
$$

$$+\int_a^b\int_c^d |g_{12}(t,s)|^2\left(\int_a^t\int_c^s w(x,y)dydx\right)dsdt$$

$$=\int_a^b\int_c^d |g_{12}(t,s)|^2\left(\int_t^b\int_s^d w(u,v)dvdu\right)dsdt.$$

By using the equality (8) in (7), we obtain the following inequality

$$\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\, w(t,s)dsdt \tag{9}$$

$$\leq \left(\int_a^b\int_c^d (t-a)(s-c)|f_{12}(t,s)|^2 w(t,s)dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^b\int_c^d\left(\int_t^b\int_s^d w(u,v)dvdu\right)|g_{12}(t,s)|^2 dsdt\right)^{\frac{1}{2}}.$$

Case II: Let $g(a,s)=g_1(t,d)=0$ for $(t,s)\in[a,b]\times[c,d]$.
We get

$$g(t,s)=-\int_a^t\int_s^d g_{12}(u,v)dvdu$$

for $(t,s)\in[a,b]\times[c,d]$. Then it follows that

$$\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\, w(t,s)dsdt$$

$$=\int_a^b\int_c^d (t-a)^{\frac{1}{2}}(d-s)^{\frac{1}{2}}|f_{12}(t,s)|(t-a)^{-\frac{1}{2}}(d-s)^{-\frac{1}{2}}$$

$$\times\left|\int_a^t\int_s^d g_{12}(u,v)dudv\right| w(t,s)dsdt.$$

By Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\, w(t,s)dsdt \\
\leq\ & \left(\int_a^b\int_c^d (t-a)(d-s)\,|f_{12}(t,s)|^2\, w(t,s)dsdt\right)^{\frac{1}{2}} \\
&\times\left(\int_a^b\int_c^d (t-a)^{-1}(d-s)^{-1}\left|\int_a^t\int_s^d g_{12}(u,v)dvdu\right|^2 w(t,s)dsdt\right)^{\frac{1}{2}} \\
\leq\ & \left(\int_a^b\int_c^d (t-a)(d-s)\,|f_{12}(t,s)|^2\, w(t,s)dsdt\right)^{\frac{1}{2}} \\
&\times\left(\int_a^b\int_c^d \left(\int_a^t\int_s^d |g_{12}(u,v)|^2\, dvdu\right) w(t,s)dsdt\right)^{\frac{1}{2}}.
\end{aligned}$$

By integration by parts, we have

$$\begin{aligned}
&\int_a^b\int_c^d \left(\int_a^t\int_s^d |g_{12}(u,v)|^2\, dudv\right) w(t,s)dsdt \\
&= \int_a^b\int_c^d \left(\int_t^b\int_c^s w(u,v)dvdu\right) |g_{12}(t,s)|^2\, dsdt.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
(10)\qquad &\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\, w(t,s)dsdt \\
\leq\ & \left(\int_a^b\int_c^d (t-a)(d-s)\,|f_{12}(t,s)|^2\, w(t,s)dsdt\right)^{\frac{1}{2}}
\end{aligned}$$

$$\times \left(\int_a^b \int_c^d \left(\int_t^b \int_c^s w(u,v) dv du \right) |g_{12}(t,s)|^2 \, ds dt \right)^{\frac{1}{2}}.$$

Case III: Let $g(b,s) = g_1(t,c) = 0$ for $(t,s) \in [a,b] \times [c,d]$. Then we have

$$g(t,s) = -\int_t^b \int_c^s g_{12}(u,v) dv du.$$

Case IV: Let $g(b,s) = g_1(t,d) = 0$ for $(t,s) \in [a,b] \times [c,d]$. we can write

$$g(t,s) = \int_t^b \int_s^d g_{12}(u,v) dv du.$$

By following similar to those in proof of (9) and (10), but with suitable modifications, we establish the following inequalities in Case III and Case IV:

(11)
$$\int_a^b \int_c^d |f_{12}(t,s) g(t,s)| \, w(t,s) ds dt$$
$$\leq \left(\int_a^b \int_c^d (b-t)(s-c) |f_{12}(t,s)|^2 \, w(t,s) ds dt \right)^{\frac{1}{2}}$$
$$\times \left(\int_a^b \int_c^d \left(\int_a^t \int_s^d w(u,v) dv du \right) |g_{12}(t,s)|^2 \, ds dt \right)^{\frac{1}{2}},$$

(12)
$$\int_a^b \int_c^d |f_{12}(t,s) g(t,s)| \, w(t,s) ds dt$$
$$\leq \left(\int_a^b \int_c^d (b-t)(d-s) |f_{12}(t,s)|^2 \, w(t,s) ds dt \right)^{\frac{1}{2}}$$

$$\times\left(\int_a^b\int_c^d\left(\int_a^t\int_c^s w(u,v)dvdu\right)\left|g_{12}(t,s)\right|^2 dsdt\right)^{\frac{1}{2}},$$

respectively.

Since $g(a,s) = g_1(t,c) = 0$ for $(t,s) \in [a,b] \times [c,d]$, if we write the inequality (9) for the rectangles $[a,b] \times [c,y]$ and $[a,x] \times [c,d]$ for $(x,y) \in [a,b] \times [c,d]$, then we have

$$\int_a^b\int_c^y \left|f_{12}(t,s)g(t,s)\right| w(t,s)dsdt \tag{13}$$

$$\leq \left(\int_a^b\int_c^y (t-a)(s-c)\left|f_{12}(t,s)\right|^2 w(t,s)dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^b\int_c^y\left(\int_t^b\int_s^y w(u,v)dvdu\right)\left|g_{12}(t,s)\right|^2 dsdt\right)^{\frac{1}{2}}$$

and

$$\int_a^x\int_c^d \left|f_{12}(t,s)g(t,s)\right| w(t,s)dsdt \tag{14}$$

$$\leq \left(\int_a^x\int_c^d (t-a)(s-c)\left|f_{12}(t,s)\right|^2 w(t,s)dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^x\int_c^d\left(\int_t^x\int_s^d w(u,v)dvdu\right)\left|g_{12}(t,s)\right|^2 dsdt\right)^{\frac{1}{2}}$$

respectively.

As $g(a,s) = g_1(t,d) = 0$ for $(t,s) \in [a,b] \times [c,d]$, if we apply the inequality (10) for the rectangles $[a,b]\times[y,d]$ and $[a,x]\times[c,d]$ for $(x,y) \in$

$[a,b]\times[c,d]$, then we get

$$\int_a^b\int_y^d |f_{12}(t,s)g(t,s)|\,w(t,s)dsdt \tag{15}$$

$$\leq \left(\int_a^b\int_y^d (t-a)(d-s)\,|f_{12}(t,s)|^2\,w(t,s)dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^b\int_y^d\left(\int_t^b\int_y^s w(u,v)dvdu\right)|g_{12}(t,s)|^2\,dsdt\right)^{\frac{1}{2}}$$

and

$$\int_a^x\int_c^d |f_{12}(t,s)g(t,s)|\,w(t,s)dsdt \tag{16}$$

$$\leq \left(\int_a^x\int_c^d (t-a)(d-s)\,|f_{12}(t,s)|^2\,w(t,s)dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^x\int_c^d\left(\int_t^x\int_c^s w(u,v)dvdu\right)|g_{12}(t,s)|^2\,dsdt\right)^{\frac{1}{2}}.$$

Similarly, since $g(b,s)=g_1(t,c)=0$ for $(t,s)\in[a,b]\times[c,d]$, if we write the inequality (11) for the rectangles $[a,b]\times[c,y]$ and $[x,b]\times[c,d]$ for $(x,y)\in[a,b]\times[c,d]$, then we have

$$\int_a^b\int_c^y |f_{12}(t,s)g(t,s)|\,w(t,s)dsdt \tag{17}$$

$$\leq \left(\int_a^b\int_c^y (b-t)(s-c)\,|f_{12}(t,s)|^2\,w(t,s)dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^b\int_c^y\left(\int_a^t\int_s^y w(u,v)dvdu\right)\left|g_{12}(t,s)\right|^2 dsdt\right)^{\frac{1}{2}}$$

and

$$\int_x^b\int_c^d \left|f_{12}(t,s)g(t,s)\right| w(t,s)dsdt \tag{18}$$

$$\leq \left(\int_x^b\int_c^d (b-t)(s-c)\left|f_{12}(t,s)\right|^2 w(t,s)dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_x^b\int_c^d\left(\int_x^t\int_s^d w(u,v)dvdu\right)\left|g_{12}(t,s)\right|^2 dsdt\right)^{\frac{1}{2}}.$$

Finally, as $g(b,s)=g_1(t,d)=0$ for $(t,s)\in[a,b]\times[c,d]$, if we apply the inequality (12) for the rectangles $[a,b]\times[y,d]$ and $[x,b]\times[c,d]$ for $(x,y)\in[a,b]\times[c,d]$, then we have

$$\int_x^b\int_c^d \left|f_{12}(t,s)g(t,s)\right| w(t,s)dsdt \tag{19}$$

$$\leq \left(\int_x^b\int_c^d (b-t)(d-s)\left|f_{12}(t,s)\right|^2 w(t,s)dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_x^b\int_c^d\left(\int_x^t\int_c^s w(u,v)dvdu\right)\left|g_{12}(t,s)\right|^2 dsdt\right)^{\frac{1}{2}}$$

and

$$\int_a^b\int_y^d \left|f_{12}(t,s)g(t,s)\right| w(t,s)dsdt \tag{20}$$

$$\leq \left(\int_a^b \int_y^d (b-t)(d-s) |f_{12}(t,s)|^2 w(t,s) ds dt \right)^{\frac{1}{2}}$$

$$\times \left(\int_a^b \int_y^d \left(\int_a^t \int_y^s w(u,v) dv du \right) |g_{12}(t,s)|^2 ds dt \right)^{\frac{1}{2}}.$$

Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be reel numbers. Then we have the following Cauchy-Scwarz inequality
(21)

$$a_1 b_1 + a_2 b_2 + ... + a_n b_n \leq \left(a_1^2 + a_2^2 + ... + a_n^2 \right)^{\frac{1}{2}} \left(b_1^2 + b_2^2 + ... + b_n^2 \right)^{\frac{1}{2}}.$$

If we add the inequalities (13)-(20), then by using the Cauchy-Schwarz inequality (21), we obtain

$$4 \int_a^b \int_c^d |f_{12}(t,s) g(t,s)| w(t,s) ds dt$$

$$\leq \left[\int_a^b \int_c^y (b-a)(s-c) |f_{12}(t,s)|^2 w(t,s) ds dt \right.$$

$$+ \int_a^x \int_c^d (t-a)(d-c) |f_{12}(t,s)|^2 w(t,s) ds dt$$

$$+ \int_a^b \int_y^d (b-a)(d-s) |f_{12}(t,s)|^2 w(t,s) ds dt$$

$$\left. + \int_x^b \int_c^d (b-t)(d-c) |f_{12}(t,s)|^2 w(t,s) ds dt \right]^{\frac{1}{2}}$$

$$\times \left[\int_a^b \int_c^y \left(\int_a^b \int_s^y w(u,v) dv du \right) |g_{12}(t,s)|^2 ds dt \right.$$

$$
\begin{aligned}
&+\int_a^x\int_c^d\left(\int_t^x\int_c^d w(u,v)dvdu\right)|g_{12}(t,s)|^2\,dsdt\\
&+\int_a^b\int_y^d\left(\int_a^b\int_y^s w(u,v)dvdu\right)|g_{12}(t,s)|^2\,dsdt\\
&\left.+\int_x^b\int_c^d\left(\int_x^t\int_c^d w(u,v)dvdu\right)|g_{12}(t,s)|^2\,dsdt+\right]^{\frac{1}{2}}\\
=&\left[(b-a)\int_a^b\int_c^d Q(s,y)\,|f_{12}(t,s)|^2\,dsdt\right.\\
&\left.+(d-c)\int_a^b\int_c^d P(t,x)\,|f_{12}(t,s)|^2\,dsdt\right]^{\frac{1}{2}}\\
&\times\left[\int_a^b\int_c^d\left|\int_a^b\int_s^y w(u,v)dvdu\right||g_{12}(t,s)|^2\,dsdt\right.\\
&\left.+\int_a^b\int_c^d\left|\int_t^x\int_c^d w(u,v)dvdu\right||g_{12}(t,s)|^2\,dsdt\right]^{\frac{1}{2}}.
\end{aligned}
$$

This proves the first inequality in (4).

The proof of the second inequality in (4) is obvious from the fact that $\sqrt{pq}\le\frac{1}{2}(p+q)$, for $p,q>0$. □

REMARK 2.2. If we choose $w(t,s)=1$ for all $(t,s)\in[a,b]\times[c,d]$ in Theorem 2.1, then the inequalities (4) reduce to the inequalities (3).

COROLLARY 2.3. *If we choose $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in Theorem 2.1, then we have the following inequality*

$$
\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,w(t,s)dsdt
$$

$$\leq \frac{1}{4}\left(\int_a^b\int_c^d [(b-a)Q(s)+(d-c)P(t)]\left|f_{12}(t,s)\right|^2 dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^b\int_c^d\left[\left|\int_a^b\int_s^{\frac{c+d}{2}} w(u,v)dvdu\right|\right.\right.$$

$$\left.\left.+\left|\int_t^{\frac{a+b}{2}}\int_c^d w(u,v)dvdu\right|\right]\left|g_{12}(t,s)\right|^2 dsdt\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{8}\left[\int_a^b\int_c^d [(b-a)Q(s)+(d-c)P(t)]\left|f_{12}(t,s)\right|^2 dsdt\right.$$

$$\left.+\int_a^b\int_c^d\left[\left|\int_a^b\int_s^{\frac{c+d}{2}} w(u,v)dvdu\right|+\left|\int_t^{\frac{a+b}{2}}\int_c^d w(u,v)dvdu\right|\right]\left|g_{12}(t,s)\right|^2 dsdt\right]$$

where

$$P(t)=\begin{cases} t-a, & a\leq t\leq \frac{a+b}{2}\\ b-t, & \frac{a+b}{2}\leq t\leq b\end{cases} \quad \text{and} \quad Q(s)=\begin{cases} s-c, & c\leq s\leq \frac{c+d}{2}\\ d-s, & \frac{c+d}{2}\leq s\leq d.\end{cases}$$

REMARK 2.4. If we choose $f(t,s)=g(t,s)$ for $(t,s)\in[a,b]\times[c,d]$ in Corollary 2.3, then we have the following inequality

$$\int_a^b\int_c^d \left|f_{12}(t,s)f(t,s)\right| w(t,s)dsdt$$

$$\leq \frac{1}{4}\left(\int_a^b\int_c^d [(b-a)Q(s)+(d-c)P(t)]\left|f_{12}(t,s)\right|^2 dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^b\int_c^d\left[\left|\int_a^b\int_s^{\frac{c+d}{2}} w(u,v)dvdu\right|\right.\right.$$

$$\left.\left.+\left|\int_t^{\frac{a+b}{2}}\int_c^d w(u,v)dvdu\right|\right]|f_{12}(t,s)|^2\,dsdt\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{8}\left[\int_a^b\int_c^d [(b-a)Q(s)+(d-c)P(t)]\,|f_{12}(t,s)|^2\,dsdt\right.$$

$$\left.+\int_a^b\int_c^d\left[\left|\int_a^b\int_s^{\frac{c+d}{2}} w(u,v)dvdu\right|+\left|\int_t^{\frac{a+b}{2}}\int_c^d w(u,v)dvdu\right|\right]|f_{12}(t,s)|^2\,dsdt\right].$$

Theorem 2.5. *Let $w:[a,b]\times[c,d]\to\mathbb{R}$ be an integrable and nonnegative function. Let $f(t,s)$, $f_1(t,s)$, $f_{12}(t,s)$, $g(t,s)$, $g_1(t,s)$ and $g_{12}(t,s)$ be continuous on $[a,b]\times[c,d]$ and let $g_{12}, f_{12}\in L_2([a,b]\times[c,d])$. If $g(a,s)=g(b,s)=g_1(t,c)=g_1(t,d)=0$ for $(t,s)\in[a,b]\times[c,d]$, then for all $(x,y)\in[a,b]\times[c,d]$, we have*

$$\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,w(t,s)dsdt \tag{22}$$

$$\leq \frac{1}{4}\left(\int_a^b\int_c^d [(b-a)Q(s,y)+(d-c)P(t,x)]\,|f_{12}(t,s)|^p\,dsdt\right)^{\frac{1}{p}}$$

$$\times\left(\int_a^b\int_c^d\left[\left|\int_a^b\int_s^{y} w(u,v)dvdu\right|\right.\right.$$

$$\left.\left.+\left|\int_t^{x}\int_c^d w(u,v)dvdu\right|\right]|g_{12}(t,s)|^q\,dsdt\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{4p}\left[\int_a^b\int_c^d [(b-a)Q(s,y)+(d-c)P(t,x)]\,|f_{12}(t,s)|^p\,dsdt\right]$$

$$+\frac{1}{4q}\left[\int_a^b\int_c^d\left[\left|\int_a^b\int_s^y w(u,v)dvdu\right|+\left|\int_t^x\int_c^d w(u,v)dvdu\right|\right]|g_{12}(t,s)|^q\,dsdt\right]$$

where $P(t,x)$ and $Q(s,y)$ are defined by as in Theorem 1.3 and where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Hölder inequality instead of Cauchy-Schwarz inequality and by following similar to those in proof of Theorem 2.1, but with suitable modifications, one can prove the first inequality in (22). The details left to the reader. The proof of the second inequality in (22) is obvious from the Young inequality as defined by

$$a_1^{1/p}a_1^{1/q}\le\frac{1}{p}a_1+\frac{1}{q}a_2,$$

for $a_1,a_2>0$, where $\frac{1}{p}+\frac{1}{q}=1$. □

REMARK 2.6. If we choose $w(t,s)=1$ for all $(t,s)\in[a,b]\times[c,d]$ in Theorem 2.5, then we have the following inequality

$$\begin{aligned}
&\int_a^b\int_c^d|f_{12}(t,s)g(t,s)|\,dsdt\\
\le\ &\frac{1}{4}\left(\int_a^b\int_c^d[(b-a)Q(s,y)+(d-c)P(t,x)]\,|f_{12}(t,s)|^p\,dsdt\right)^{\frac{1}{p}}\\
&\times\left(\int_a^b\int_c^d[(b-a)\,|y-s|+(d-c)\,|x-t|]\,|g_{12}(t,s)|^q\,dsdt\right)^{\frac{1}{q}}\\
\le\ &\frac{1}{4p}\left[\int_a^b\int_c^d[(b-a)Q(s,y)+(d-c)P(t,x)]\,|f_{12}(t,s)|^p\,dsdt\right]\\
&+\frac{1}{4q}\left[\int_a^b\int_c^d[(b-a)\,|y-s|+(d-c)\,|x-t|]\,|g_{12}(t,s)|^q\,dsdt\right]
\end{aligned}$$

which is proved by Budak in [3].

COROLLARY 2.7. *If we choose* $x=\frac{a+b}{2}$ *and* $y=\frac{c+d}{2}$ *in Theorem 2.5, then we have the following inequality*

$$\begin{aligned}
&\int_a^b\int_c^d \left|f_{12}(t,s)g(t,s)\right| w(t,s)dsdt \\
\leq\ & \frac{1}{4}\left(\int_a^b\int_c^d \left[(b-a)Q(s)+(d-c)P(t)\right]\left|f_{12}(t,s)\right|^p dsdt\right)^{\frac{1}{p}} \\
&\times\left(\int_a^b\int_c^d \left[\left|\int_a^b\int_s^{\frac{c+d}{2}} w(u,v)dvdu\right|\right.\right. \\
&\left.\left.+\left|\int_t^{\frac{a+b}{2}}\int_c^d w(u,v)dvdu\right|\right]\left|g_{12}(t,s)\right|^q dsdt\right)^{\frac{1}{q}} \\
\leq\ & \frac{1}{4p}\left[\int_a^b\int_c^d \left[(b-a)Q(s,y)+(d-c)P(t,x)\right]\left|f_{12}(t,s)\right|^p dsdt\right] \\
&+\frac{1}{4q}\left[\int_a^b\int_c^d \left[\left|\int_a^b\int_s^{\frac{c+d}{2}} w(u,v)dvdu\right|\right.\right. \\
&\left.\left.+\left|\int_t^{\frac{a+b}{2}}\int_c^d w(u,v)dvdu\right|\right]\left|g_{12}(t,s)\right|^q dsdt\right]
\end{aligned}$$

where $P(t)$ *and* $Q(s)$ *are defined by as in Corollary 2.3.*

REMARK 2.8. If we choose $f(t,s)=g(t,s)$ for $(t,s)\in[a,b]\times[c,d]$ in Corollary 2.7, then we have the following inequality

$$\int_a^b\int_c^d \left|f_{12}(t,s)f(t,s)\right| w(t,s)dsdt$$

$$\leq \frac{1}{4}\left(\int_a^b\int_c^d [(b-a)Q(s)+(d-c)P(t)]\,|f_{12}(t,s)|^p\,dsdt\right)^{\frac{1}{p}}$$
$$\times\left(\int_a^b\int_c^d\left[\left|\int_a^b\int_s^{\frac{c+d}{2}} w(u,v)dvdu\right|\right.\right.$$
$$\left.\left.+\left|\int_t^{\frac{a+b}{2}}\int_c^d w(u,v)dvdu\right|\right]|f_{12}(t,s)|^q\,dsdt\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{4p}\left[\int_a^b\int_c^d [(b-a)Q(s,y)+(d-c)P(t,x)]\,|f_{12}(t,s)|^p\,dsdt\right]$$
$$+\frac{1}{4q}\left[\int_a^b\int_c^d\left[\left|\int_a^b\int_s^{\frac{c+d}{2}} w(u,v)dvdu\right|\right.\right.$$
$$\left.\left.+\left|\int_t^{\frac{a+b}{2}}\int_c^d w(u,v)dvdu\right|\right]|f_{12}(t,s)|^q\,dsdt\right].$$

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