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A NOTE ON THE MULTIFRACTAL HEWITT-STROMBERG MEASURES IN A PROBABILITY SPACE

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Abstract. In this note, we investigate the multifractal analogues of the Hewitt-Stromberg measures and dimensions in a probability space.

1. Introduction

The notion of dimension is fundamental in the study of fractals. Various definitions of dimension have been proposed, such as the Hausdorff dimension, the packing dimension and the modified lower and upper box dimensions etc. Unlike the Hausdorff and packing dimensions, the modified lower and upper box dimensions are not defined in terms of measures. Hewitt-Stromberg measures were introduced by Hewitt and Stromberg in [21]. Since then, they have been investigated by several authors, highlighting their importance in the study of local properties of fractals and products of fractals. One can cite, for example [2, 3, 19, 20, 22, 27, 28, 34, 35]. In particular, Edgar's textbook [17] provides an excellent and systematic introduction to these measures, which also appears explicitly, for example, in Pesin's monograph [29]

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and implicitly in Mattila's text [26]. The purpose of this paper is to define and study a class of natural multifractal generalizations of the Hewitt-Stromberg measures in a probability space.

A function $g : (0, +\infty) \to (0, +\infty)$ is called a dimension function if g is increasing, right continuous and $\lim_{r\to 0} g(r) = 0$. Let X be a metric space, $E \subseteq X$. The Hausdorff measure associated with a dimension function g is defined, for $\varepsilon > 0$, as follows

$$\mathscr{H}^g_\varepsilon(E) = \inf \left\{ \sum_i g\Big(\operatorname{diam}(E_i)\Big) \;\Big|\; E \subseteq \bigcup_i E_i,\;\; \operatorname{diam}(E_i) < \varepsilon \right\}.$$

This allows to define the g-dimensional Hausdorff measure $\mathscr{H}^g(E)$ of E by

$$\mathscr{H}^g(E) = \sup_{\varepsilon>0} \mathscr{H}^g_\varepsilon(E).$$

The packing measure with a dimension function g is defined, for $\varepsilon > 0$, as follows

$$\overline{\mathscr{P}}^g_\varepsilon(E) = \sup \left\{ \sum_i g\Big(2r_i\Big) \right\},$$

where the supremum is taken over all closed balls $\Big(C(x_i, r_i)\Big)_i$ such that $r_i \le \varepsilon$ and with $x_i \in E$ and $C(x_i, r_i) \cap C(x_j, r_j) = \emptyset$ for $i \neq j$. The g-dimensional packing pre-measure $\overline{\mathscr{P}}^g(E)$ of E is now defined by

$$\overline{\mathscr{P}}^g(E) = \inf_{\varepsilon>0} \overline{\mathscr{P}}^g_\varepsilon(E).$$

This makes us able to define the g-dimensional packing measure $\mathscr{P}^g(E)$ of E as

$$\mathscr{P}^g(E) = \inf \left\{ \sum_i \overline{\mathscr{P}}^g(E_i) \;\Big|\; E \subseteq \bigcup_i E_i \right\}.$$

While Hausdorff and packing measures are defined using coverings and packings by families of sets with diameters less than a given positive number ε, say, the Hewitt-Stromberg measures are defined using packings of balls with the same diameter ε. The Hewitt-Stromberg pre-measures are defined as follows,

$$\overline{\mathscr{U}}^g(E) = \liminf_{r\to 0} M_r(E)\, g(2r)$$

and

$$\overline{\mathscr{V}}^g(E) = \limsup_{r\to 0} M_r(E)\; g(2r),$$

where the packing number $M_r(E)$ of E is given by

$$M_r(E) = \sup\Big\{\sharp\{I\} \;\mid\; \Big(C(x_i,r_i)\Big)_{i\in I} \text{ is a family of closed balls with } x_i\in E \text{ and } C(x_i,r_i)\cap C(x_j,r_j)=\emptyset \text{ for } i\neq j\Big\}.$$

Now, we define the lower and upper g-dimensional Hewitt-Stromberg measures, which we denote respectively by $\mathscr{U}^g(E)$ and $\mathscr{V}^g(E)$, as follows

$$\mathscr{U}^g(E) = \inf\left\{\sum_i \overline{\mathscr{U}}^g(E_i) \;\mid\; E\subseteq \bigcup_i E_i\right\}$$

and

$$\mathscr{V}^g(E) = \inf\left\{\sum_i \overline{\mathscr{V}}^g(E_i) \;\mid\; E\subseteq \bigcup_i E_i\right\}.$$

We recall the basic inequalities satisfied by the Hewitt-Stromberg, the Hausdorff and the packing measure (see [22, 27])

$$\overline{\mathscr{U}}^g(E) \le \overline{\mathscr{V}}^g(E) \le \overline{\mathscr{P}}^g(E)$$

and

$$\mathscr{H}^g(E) \le \mathscr{U}^g(E) \le \mathscr{V}^g(E) \le \mathscr{P}^g(E).$$

In Euclidean space $\mathbb{R}^n$ there is no generally accepted definition of a fractal, even though fractal sets are widely used as models for many physical phenomena. The idea behind these models is that of self-similarity or affineness which is based on the linear structure of $\mathbb{R}^n$. These and other geometrical notions have no obvious meaning in an abstract probability space. Then, Billingsley [5, 6], and Dai et al., in [8], have defined the Hausdorff measure and the packing measure in a probability space. Y. Li et al., in [11, 12] were motivated by those researches. They applied the ideas developed in [5, 6, 8] to generalize the Hausdorff measure and the packing measure and gave the relative multifractal formalism with respect to the relative multifractal Hausdorff measure and packing measure in a probability space. Other works carried in this sense presented many valuable results on the same subject and applications see for example [1, 4, 7, 9, 10, 13–16, 23–25, 30–33].

In this paper, we construct the multifractal analogues of the Hewitt-Stromberg measures lying between relative Hausdorff measure and the relative packing measure in a probability space which determine the modified lower and upper relative box-dimension. We also compare the modified lower and upper relative box-dimension with the relative multifractal dimensions in a probability space. In particular, the relative packing dimension is equal to the modified upper relative box-dimension.

2. Preliminaries

Let we start by defining the relative multifractal Hausdorff and the packing measure in a probability space (see [11–13]). We start with a fixed stochastic process $\Big\{X_n \mid \ n \in \mathbb{N}\Big\}$ on a probability space $\big(\Omega, \mathscr{F}, \nu\big)$ taking values in a finite or countable state space S. The n-cylinder C is defined by

$$C = \Big\{\omega \in \Omega \mid \ \ X_i(\omega) = a_i, \ i = 1, 2, ..., n\Big\}$$

where $a_i \in S$, $i = 1, 2, ..., n$. For each $\omega \in \Omega$ there is a unique n-cylinder set, denoted by $I_n(\omega)$, which contains ω. Thus

$$I_n(\omega) = \Big\{\omega' \in \Omega \mid \ \ X_i(\omega') = X_i(\omega), \ i = 1, 2, ..., n\Big\}.$$

We assume that the process is $\mathscr{F}$-measurable, that is that $\mathfrak{C} \subseteq \mathscr{F}$, where $\mathfrak{C}$ is the class of all cylinder sets. Many details of classical proofs are greatly simplified because $\mathfrak{C}$ is nested, that is, given $C_1, C_2 \in \mathfrak{C}$, then either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$ or $C_1 \cap C_2 = \emptyset$. We use sets in $\mathfrak{C}$ for both covering and packing. It is worth observing that we can use $\mathfrak{C}$ to introduce a pseudo metric in Ω. Given $\omega, \omega' \in \Omega$, let $I_0(\omega) = \Omega$ for all ω, and

$$\rho(\omega, \omega') = 2^{-\sup\big\{k\in\mathbb{N},\ I_k(\omega)=I_k(\omega')\big\}} = 2^{-n}.$$

We allow $n = +\infty$ in the definition, so that $\rho(\omega, \omega') = 0$ if ω, ω' are not distinguished by the sets of $\mathfrak{C}$. The closure of $E \subset \Omega$ is therefore

$$\overline{E} = \Big\{\omega \mid \ \ \rho(\omega, E) = 0\Big\}, \ \ \text{where} \ \ \rho(\omega, E) = \inf\Big\{\rho(\omega, \omega') \mid \ \ \omega' \in E\Big\},$$

then it is easy to check that $E \in \sigma(\mathfrak{C})$, the sigma field generated by the cylinder sets. $\sigma(\mathfrak{C})$ plays the role of Borel sets in the topology generated by the metric ρ. The sets of $\mathfrak{C}$ are both open and closed in this topology, and each $I_n(\omega)$ can be considered as closed ball of radius 2^{-n} centered at ω (see [8–10]).

DEFINITION 2.1. [11, 12] We say that ν is a non-atomic ($\sigma(\mathfrak{C})$-continuous) measure, if

$$\lim_{n\to+\infty} \nu(I_n(\omega)) = 0, \quad \text{for all} \quad \omega \in \Omega.$$

For $E \subseteq \Omega$ and $\delta > 0$, we say that a collection $\Big(C_i\Big)_{i\in\mathbb{N}}$ is a centered δ-packing of E if C_i is of the form $I_{n_i}(\omega_i)$ with $\omega_i \in E$, $\nu(C_i) < \delta$ and $C_i \cap C_j = \emptyset$ for all $i \neq j$. Similarly, we say that $\Big(C_i\Big)_{i\in\mathbb{N}}$ is a centered δ-covering of E if C_i is of the form $I_{n_i}(\omega_i)$ with $\omega_i \in E$, $\nu(C_i) < \delta$ and $E \subseteq \bigcup_i C_i$.

In this paper, we will assume that ν is non-atomic. Let $E \subseteq \Omega$ and $\delta > 0$, suppose μ is a probability measure on $(\Omega, \mathscr{F})$. For $q, t \in \mathbb{R}$, we define

$$\begin{aligned}&\overline{\mathscr{P}}^{q,t}_{\mu,\nu,\delta}(E)\\ &\quad = \sup\left\{\sum_i \mu\big(C_i\big)^q \nu\big(C_i\big)^t \mid \Big(C_i\Big)_i \text{ is a centered } \delta\text{-packing of } E\right\}.\end{aligned}$$

The relative multifractal packing pre-measure is then given by

$$\overline{\mathscr{P}}^{q,t}_{\mu,\nu}(E) = \inf_{\delta>0} \overline{\mathscr{P}}^{q,t}_{\mu,\nu,\delta}(E).$$

In a similar way, we define

$$\begin{aligned}&\overline{\mathscr{H}}^{q,t}_{\mu,\nu,\delta}(E)\\ &\quad = \inf\left\{\sum_i \mu\big(C_i\big)^q \nu\big(C_i\big)^t \mid \Big(C_i\Big)_i \text{ is a centered } \delta\text{-covering of } E\right\}.\end{aligned}$$

The relative multifractal Hausdorff pre-measure is defined by

$$\overline{\mathscr{H}}^{q,t}_{\mu,\nu}(E) = \sup_{\delta>0} \overline{\mathscr{H}}^{q,t}_{\mu,\nu,\delta}(E).$$

with the conventions $0^q = \infty$ for $q \le 0$ and $0^q = 0$ for $q > 0$.

$\overline{\mathscr{H}}^{q,t}_{\mu,\nu}$ is σ-subadditive but not increasing (it is easy to check that if $A \subseteq B$, then a centered δ-covering of B is not necessarily a centered δ-covering of A, thus $\overline{\mathscr{H}}^{q,t}_{\mu,\nu}$ is not necessarily monotone) and $\overline{\mathscr{P}}^{q,t}_{\mu,\nu}$ is increasing but not σ-subadditive. That's why Dai et al. in [11, 12] introduced the following modifications on the relative Hausdorff and packing measures $\mathscr{H}^{q,t}_{\mu,\nu}$ and $\mathscr{P}^{q,t}_{\mu,\nu}$,

$$\mathscr{H}^{q,t}_{\mu,\nu}(E) = \sup_{F\subseteq E} \overline{\mathscr{H}}^{q,t}_{\mu,\nu}(F) \quad \text{and} \quad \mathscr{P}^{q,t}_{\mu,\nu}(E) = \inf_{E\subseteq \bigcup_i E_i} \sum_i \overline{\mathscr{P}}^{q,t}_{\mu,\nu}(E_i).$$

The functions $\mathscr{H}^{q,t}_{\mu,\nu}$ and $\mathscr{P}^{q,t}_{\mu,\nu}$ are outer measures. An important feature of the Hausdorff and packing measures is that

$$\overline{\mathscr{H}}^{q,t}_{\mu,\nu} \le \mathscr{H}^{q,t}_{\mu,\nu} \le \mathscr{P}^{q,t}_{\mu,\nu} \le \overline{\mathscr{P}}^{q,t}_{\mu,\nu}.$$

The measure $\mathscr{H}^{q,t}_{\mu,\nu}$ is of course a multifractal generalization of the Billingsley's Hausdorff measure (see [5,6]), whereas $\mathscr{P}^{q,t}_{\mu,\nu}$ is a multifractal generalization of the packing measure (see [8]) in a probability space. The measures $\mathscr{H}^{q,t}_{\mu,\nu}$ and $\mathscr{P}^{q,t}_{\mu,\nu}$ and the pre-measure $\overline{\mathscr{P}}^{q,t}_{\mu,\nu}$ assign in the usual way a multifractal dimensions to each subset E of $supp\ \mu \cap supp\ \nu$. They are respectively denoted by $b^q_{\mu,\nu}(E)$, $B^q_{\mu,\nu}(E)$ and $\Delta^q_{\mu,\nu}(E)$. More precisely, we have

$$\begin{aligned} b^q_{\mu,\nu}(E) &= \inf\Big\{t\in\mathbb{R} \mid \quad \mathscr{H}^{q,t}_{\mu,\nu}(E)=0\Big\} \\ &= \sup\Big\{t\in\mathbb{R} \mid \quad \mathscr{H}^{q,t}_{\mu,\nu}(E)=+\infty\Big\}, \end{aligned}$$

$$\begin{aligned} B^q_{\mu,\nu}(E) &= \inf\Big\{t\in\mathbb{R} \mid \quad \mathscr{P}^{q,t}_{\mu,\nu}(E)=0\Big\} \\ &= \sup\Big\{t\in\mathbb{R} \mid \quad \mathscr{P}^{q,t}_{\mu,\nu}(E)=+\infty\Big\}, \end{aligned}$$

and

$$\begin{aligned} \Delta^q_{\mu,\nu}(E) &= \inf\Big\{t\in\mathbb{R} \mid \quad \overline{\mathscr{P}}^{q,t}_{\mu,\nu}(E)=0\Big\} \\ &= \sup\Big\{t\in\mathbb{R} \mid \quad \overline{\mathscr{P}}^{q,t}_{\mu,\nu}(E)=+\infty\Big\}. \end{aligned}$$

It is also readily seen that

$$b^q_{\mu,\nu}(E) \le B^q_{\mu,\nu}(E) \le \Delta^q_{\mu,\nu}(E).$$

For convenience, we write

$$b(q) := b_{\mu,\nu}(q) = b^q_{\mu,\nu}(supp\ \mu\cap supp\ \nu),$$

$$B(q) := B_{\mu,\nu}(q) = B^q_{\mu,\nu}(supp\ \mu\cap supp\ \nu)$$

and

$$\Lambda(q) := \Lambda_{\mu,\nu}(q) = \Delta^q_{\mu,\nu}(supp\ \mu\cap supp\ \nu).$$

In fact, it is easily seen that the following holds for $t \geq 0$ and $E \subset \Omega$,

$$L^t_\nu(E) \leq \mathscr{H}^{0,t}_{\mu,\nu}(E), \quad \mathscr{P}^t_\nu(E) = \mathscr{P}^{0,t}_{\mu,\nu}(E) \quad \text{and} \quad \overline{\mathscr{P}}^t_\nu(E) = \overline{\mathscr{P}}^{0,t}_{\mu,\nu}(E)$$

where L^t_ν denotes the t-dimensional Hausdorff measure with respect to ν (see [5, 6]), $\mathscr{P}^t_\nu$ denotes the t-dimensional packing measure and $\overline{\mathscr{P}}^t_\nu$ denotes the t-dimensional pre-packing measure with respect to ν (see [8]). In particular, we have

$$\dim_\nu(E) \leq b^0_{\mu,\nu}(E), \quad \mathrm{Dim}_\nu(E) = B^0_{\mu,\nu}(E) \quad \text{and} \quad \Delta_\nu(E) = \Delta^0_{\mu,\nu}(E).$$

3. Main results

3.1. The multifractal Hewitt-Stromberg measures. The Hewitt-Stromberg measure has recently received some interest in the fractal geometric community and it is both natural and timely to investigate multifractal analogues of this measure. Let $q, t \in \mathbb{R}$, $\mu \in \mathscr{P}(\Omega)$. We will now construct the multifractal analogues of Hewitt-Stromberg measures $H^{q,t}_{\mu,\nu}$ and $P^{q,t}_{\mu,\nu}$ in a probabilistic setting that are analogues to Billingley's classical results for the Hausdorff and packing measures in [5, 6, 8]. For $E \subseteq \Omega$, the pre-measure of E is defined by

$$C^{q,t}_{\mu,\nu}(E) = \limsup_{\delta \to 0} S^q_{\mu,\nu,\delta}(E)\ \delta^t,$$

where

$$S^q_{\mu,\nu,\delta}(E) = \sup\Big\{ \sum_i \mu\big(C_i\big)^q \mid \quad C_i \cap C_i = \emptyset,\ i \neq j, \\ \frac{\delta}{2} \leq \nu(C_i) < \delta,\ C_i = I_n(\omega) \ \text{ with } \ \omega \in E \Big\}.$$

It is readily seen that $C^{q,t}_{\mu,\nu}$ is increasing and $C^{q,t}_{\mu,\nu}(\emptyset) = 0$ but it is not σ-additive. For this we introduce the outer measure $P^{q,t}_{\mu,\nu}$-measure defined by

$$P^{q,t}_{\mu,\nu}(E) = \inf\left\{ \sum_i C^{q,t}_{\mu,\nu}(E_i) \mid \quad E \subseteq \cup_i E_i \right\}.$$

In a similar way we define

$$L^{q,t}_{\mu,\nu}(E) = \liminf_{\delta \to 0} T^q_{\mu,\nu,\delta}(E)\ \delta^t,$$

where

$T^q_{\mu,\nu,\delta}(E) =$

$$\inf\left\{\sum_i \mu\big(C_i\big)^q \mid \ E \subseteq \cup_i C_i,\ \frac{\delta}{2} \leq \nu(C_i) < \delta;\ C_i = I_n(\omega) \text{ with } \omega \in E\right\}.$$

Since $L^{q,t}_{\mu,\nu}$ is not increasing and not countably subadditive, one needs a standard modification to get an outer measure. Hence we modify the definition to

$$\overline{H}^{q,t}_{\mu,\nu}(E) = \sup_{F\subseteq E} \overline{L}^{q,t}_{\mu,\nu}(F)$$

and

$$H^{q,t}_{\mu,\nu}(E) = \inf\left\{\sum_i \overline{H}^{q,t}_{\mu,\nu}(E_i) \mid \ \ E \subseteq \cup_i E_i\ \right\}.$$

Our first main result describes some of the basic properties of the multifractal Hewitt-Stromberg measures including the fact that $H^{q,t}_{\mu,\nu}$ and $P^{q,t}_{\mu,\nu}$ are outer measures and summarises the basic inequalities satisfied by the multifractal Hewitt-Stromberg measures, the relative multifractal Hausdorff measure and the relative multifractal packing measure.

THEOREM 3.1. *Let $q, t \in \mathbb{R}$ and $E \subseteq \Omega$. Then*

1. *the set functions $H^{q,t}_{\mu,\nu}$ and $P^{q,t}_{\mu,\nu}$ are outer measures.*
2. *There exist $\phi^* \geq \phi_* > 0$, such that for any $E \subseteq \Omega$*

$$\phi_* \mathscr{H}^{q,t}_{\mu,\nu}(E) \leq H^{q,t}_{\mu,\nu}(E) \leq P^{q,t}_{\mu,\nu}(E) \leq \phi^* \mathscr{P}^{q,t}_{\mu,\nu}(E).$$

3.2. Relative modified box-counting dimension in a probability space. We will now define the lower and upper relative multifractal box-dimension in a probability space. For any subset E of Ω and $q \in \mathbb{R}$, we define

$$\begin{aligned}\underline{\dim}^{q,B}_{\mu,\nu}(E) &= \inf\left\{t \in \mathbb{R} \mid \quad \overline{H}^{q,t}_{\mu,\nu}(E) = 0\right\}\\ &= \sup\left\{t \in \mathbb{R} \mid \ \overline{H}^{q,t}_{\mu,\nu}(E) = +\infty\right\}\end{aligned}$$

and

$$\begin{aligned}\overline{\dim}^{q,B}_{\mu,\nu}(E) &= \inf\left\{t \in \mathbb{R} \mid \quad C^{q,t}_{\mu,\nu}(E) = 0\right\}\\ &= \sup\left\{t \in \mathbb{R} \mid \quad C^{q,t}_{\mu,\nu}(E) = +\infty\right\}.\end{aligned}$$

REMARK 3.1. It is worth observing that $\underline{\dim}_{\mu,\nu}^{q,B}$ and $\overline{\dim}_{\mu,\nu}^{q,B}$ are monotone, but not σ-stable.

The multifractal Hewitt-Stromberg measures $H_{\mu,\nu}^{q,t}$ and $P_{\mu,\nu}^{q,t}$ assign in the usual way a dimension to each set E of Ω called the modified lower and upper relative box-dimension. They are respectively denoted by $\underline{\dim}_{\mu,\nu}^{q,MB}(E)$ and $\overline{\dim}_{\mu,\nu}^{q,MB}(E)$

$$\begin{aligned}\underline{\dim}_{\mu,\nu}^{q,MB}(E) &= \inf\Big\{t\in\mathbb{R} \mid \quad H_{\mu,\nu}^{q,t}(E)=0\Big\}\\ &= \sup\Big\{t\in\mathbb{R} \mid \ H_{\mu,\nu}^{q,t}(E)=+\infty\Big\}\end{aligned}$$

and

$$\begin{aligned}\overline{\dim}_{\mu,\nu}^{q,MB}(E) &= \inf\Big\{t\in\mathbb{R} \mid \quad P_{\mu,\nu}^{q,t}(E)=0\Big\}\\ &= \sup\Big\{t\in\mathbb{R} \mid \quad P_{\mu,\nu}^{q,t}(E)=+\infty\Big\}.\end{aligned}$$

For convenience, we write

$$\underline{\tau}(q) := \underline{\tau}_{\mu,\nu}(q) = \underline{\dim}_{\mu,\nu}^{q,MB}(supp\ \mu\cap supp\ \nu)$$

and

$$\overline{\tau}(q) := \overline{\tau}_{\mu,\nu}(q) = \overline{\dim}_{\mu,\nu}^{q,MB}(supp\ \mu\cap supp\ \nu).$$

REMARK 3.2. Clearly $\underline{\dim}_{\mu,\nu}^{q,MB}$ and $\overline{\dim}_{\mu,\nu}^{q,MB}$ are monotone and σ-stable in the sense of [5, 8]. It is clear that from Theorem 3.1 one has

$$b_{\mu,\nu}^{q}(E) \le \underline{\dim}_{\mu,\nu}^{q,MB}(E) \le \overline{\dim}_{\mu,\nu}^{q,MB}(E) \le B_{\mu,\nu}^{q}(E) \le \Delta_{\mu,\nu}^{q}(E).$$

In addition, if $q=0$ we deduce that

$$0 \le \dim_{\nu}(E) \le \underline{\dim}_{\mu,\nu}^{0,MB}(E) \le \overline{\dim}_{\mu,\nu}^{0,MB}(E) \le \mathrm{Dim}_{\nu}(E) \le \Delta_{\nu}(E) \le 1.$$

THEOREM 3.2. *For any subset E of Ω and $q\in\mathbb{R}$, we have*

1. $\underline{\dim}_{\mu,\nu}^{q,B}(E) = \sup\limits_{F\subseteq E}\liminf\limits_{\delta\to 0}\dfrac{\log T_{\mu,\nu,\delta}^{q}(F)}{-\log\delta}.$

2. $\overline{\dim}_{\mu,\nu}^{q,B}(E) = \limsup\limits_{\delta\to 0}\dfrac{\log S_{\mu,\nu,\delta}^{q}(E)}{-\log\delta}.$

There are ways of overcoming the difficulties of relative box-dimension outlined in Theorem 3.4. However, they may not at first seem appealing since they re-introduce all the difficulties of calculation associated with the relative multifractal Hausdorff dimension. For E a subset of Ω we can try to decompose E into a countable number of pieces $E_1, E_2, \dots$ in such a way that the largest piece has as small a dimension as possible. This idea leads to the following relative modified box-counting dimensions :

THEOREM 3.3. *For any subset E of Ω and $q \in \mathbb{R}$, we have*

1. $\underline{\dim}_{\mu,\nu}^{q,MB}(E) = \inf \left\{ \sup_i \underline{\dim}_{\mu,\nu}^{q,B}(E_i) \mid \quad E \subseteq \bigcup_i E_i \;\; \text{and} \;\; E_i \subseteq \Omega \right\}.$

2. $\overline{\dim}_{\mu,\nu}^{q,MB}(E) = \inf \left\{ \sup_i \overline{\dim}_{\mu,\nu}^{q,B}(E_i) \mid \quad E \subseteq \bigcup_i E_i \;\; \text{and} \;\; E_i \subseteq \Omega \right\}.$

In the following theorem, we compare the upper relative box-dimension and the modified upper relative box-dimension with the relative multifractal packing and pre-packing dimensions in a probability space.

THEOREM 3.4. *For any subset E of Ω and $q \leq 1$, we have*

1. $\overline{\dim}_{\mu,\nu}^{q,B}(E) = \Delta_{\mu,\nu}^{q}(E).$

2. $\overline{\dim}_{\mu,\nu}^{q,MB}(E) = B_{\mu,\nu}^{q}(E).$

REMARK 3.3. The first assertion of Theorem 3.4 gives the relevant version of the definitions found in [12, Proposition 3.2].

Example : For $\alpha, \beta \geq 0$, let us introduce the fractal sets

$$\overline{E}(\beta) = \left\{ \omega \in supp\ \mu \cap supp\ \nu \mid \quad \limsup_{n\to+\infty} \frac{\log \mu\big(I_n(\omega)\big)}{\log \nu\big(I_n(\omega)\big)} \leq \beta \right\},$$

$$\underline{E}(\alpha) = \left\{ \omega \in supp\ \mu \cap supp\ \nu \mid \quad \liminf_{n\to+\infty} \frac{\log \mu\big(I_n(\omega)\big)}{\log \nu\big(I_n(\omega)\big)} \geq \alpha \right\},$$

and

$$E(\alpha,\beta) = \underline{E}(\alpha) \cap \overline{E}(\beta), \quad E(\alpha) = \underline{E}(\alpha) \cap \overline{E}(\alpha).$$

Let $q \in \mathbb{R}$ and suppose that $\mathscr{H}_{\mu,\nu}^{q,\overline{\tau}(q)}\big(supp\ \mu \cap supp\ \nu\big) > 0$. Then,

$$\dim_\nu \Big(E\big(-B_r'(q), -B_l'(q)\big)\Big) \geq \begin{cases} -qB_r'(q) + \overline{\tau}(q), & \text{if } q \geq 0 \\ \\ -qB_l'(q) + \overline{\tau}(q), & \text{if } q \leq 0. \end{cases}$$

In particular, if B is differentiable at q and $\alpha = -B'(q)$, then we have

$$\mathrm{Dim}_\nu \big(E(\alpha)\big) = \dim_\nu \big(E(\alpha)\big) = B^*(\alpha) = \overline{\tau}^*(\alpha) = \underline{\tau}^*(\alpha) = b^*(\alpha),$$

where $f^*(\alpha) = \inf_{\beta} \big(\alpha\beta + f(\beta)\big)$ denotes the Legendre transform of the function f. For more details, the reader can be referred to [11].

REMARK 3.4. It is instructive also to consider the special case $q = 0$ since the relative multifractal Hausdorff measure is the Billingsley's Hausdorff ϕ-measure [5] and the relative multifractal packing measure is the packing ϕ-measure introduced by Dai and Taylor in [8] where the function
$\phi := \varphi_t : [0, +\infty) \to [0, +\infty]$ is defined by $\varphi_0(x) = 0$ and

$$\varphi_t(x) = \begin{cases} \begin{matrix} \infty & \text{for} & x = 0 \\ x^t & \text{for} & x > 0 \end{matrix} & \text{for} \quad t < 0, \\ \\ \begin{matrix} 0 & \text{for} & x = 0 \\ x^t & \text{for} & x > 0 \end{matrix} & \text{for} \quad t > 0. \end{cases}$$

The following example is constructed in a similar way as in [5, 8].
Example : In this example, we specialize Ω to the unit interval $[0, 1]$ with the Lebesgue measure. Whenever S is a finite set of s elements $0, 1, 2, ..., s-1$ and the process $\{X_n\}$ consists of independent random variables taking each of these values with probability s^{-1}, the obvious mapping using expansions to base s provides a connection between the theories of this paper and the usual definitions in $\mathbb{R}$. We exploit this connection to show that certain exceptional sets are fractals, and we can determine their dimensions. More specifically, given $\Omega = [0, 1]$, and take $\mathscr{F}$ to be the class of Borel subsets, ν to be Lebesgue measure, and μ to be probability measure on $(\Omega, \mathscr{F})$ with s*upp* $\mu = [0, 1]$. For a fixed integer $s \geq 2$, $\omega \in \Omega$, let

$$\omega = \sum_{i=1}^{+\infty} X_i(\omega) s^{-i}$$

be the nonterminating expansion of ω to base s. Then $\{X_1, X_2, ...\}$ becomes a stochastic process taking values in $S = \{0, 1, 2, ..., s-1\}$, and $I_n(\omega)$ becomes a half-open interval of length s^{-n}. We define independent random variables

$$Y_i(\omega) = \begin{cases} 1 & \text{with probability } \dfrac{1}{2}, \\ \\ -1 & \text{with probability } \dfrac{1}{2}, \end{cases}$$

and

$$S_n(\omega) = \begin{cases} \displaystyle\sum_{i=1}^{n} Y_i(\omega) & \text{if } n = 1, 2, ..., \\ \\ 0 & \text{if } n = 0. \end{cases}$$

Then $\{S_n\}_n$ is called a simple random walk on the integer lattice. The strong law of large numbers implies that

$$\lim_{n\to+\infty} \frac{S_n(\omega)}{n} = 0 \quad \text{a.s.}$$

In particular, if A is a subset of

$$B = \left\{ \omega \in supp\ \mu \cap supp\ \nu \mid \lim_{n\to+\infty} \frac{S_n(\omega)}{n} \neq 0 \right\},$$

then A has measure zero and $-1 \leq \dfrac{S_n(\omega)}{n} \leq 1$. Now, we consider the set

$$E(\alpha) = \left\{ \omega \in supp\ \mu \cap supp\ \nu \mid \lim_{n\to+\infty} \frac{S_n(\omega)}{n} = \alpha \right\}.$$

It is clear that $E(\alpha) \subseteq B$ for all $-1 < \alpha < 0$. We therefore conclude that $\nu(E(\alpha)) = 0$ and

$$\dim_\nu(E(\alpha)) = \underline{\dim}_{\mu,\nu}^{0,MB}(E(\alpha)) = \overline{\dim}_{\mu,\nu}^{0,MB}(E(\alpha)) = \mathrm{Dim}_\nu(E(\alpha)) = \psi(\alpha),$$

where

$$\psi(\alpha) = 1 - \frac{1}{2}(1+\alpha)\log_2(1+\alpha) - \frac{1}{2}(1-\alpha)\log_2(1-\alpha).$$

4. Proof of the main results

4.1. Proof of Theorem 3.1.

1. These properties follow easily from the definitions.

2. Let $F \subseteq E$ and $\delta > 0$. Let $\left(C_i\right)_i \subseteq \mathfrak{C}$ be a centered δ-covering of F with $\frac{\delta}{2} \le \nu(C_i) < \delta$. Then

$$\overline{\mathscr{H}}^{q,t}_{\mu,\nu,\delta}(F) \le \sum_i \mu(C_i)^q \nu(C_i)^t$$

and

$$\overline{\mathscr{H}}^{q,t}_{\mu,\nu,\delta}(F) \le \max(1, 2^{-t}) T^q_{\mu,\nu,\delta}(F) \delta^t.$$

It follows immediately that

$$\phi_* \overline{\mathscr{H}}^{q,t}_{\mu,\nu}(F) \le L^{q,t}_{\mu,\nu}(F) \le \overline{H}^{q,t}_{\mu,\nu}(F).$$

Let $(F_i)_i \subseteq \Omega$ such that $F \subseteq \bigcup_i F_i$,

$$\phi_* \overline{\mathscr{H}}^{q,t}_{\mu,\nu}(F) \le \sum_i \overline{H}^{q,t}_{\mu,\nu}(F_i)$$

and

$$\phi_* \overline{\mathscr{H}}^{q,t}_{\mu,\nu}(F) \le H^{q,t}_{\mu,\nu}(F) \le H^{q,t}_{\mu,\nu}(E).$$

It follows immediately from the definitions that $\phi_* \mathscr{H}^{q,t}_{\mu,\nu}(E) \le H^{q,t}_{\mu,\nu}(E)$.

Take $\delta > 0$. Let $\left(C_i\right)_i \subseteq \mathfrak{C}$ be a centered δ-covering of $F \subseteq E$ with $\frac{\delta}{2} \le \nu(C_i) < \delta$. Since $\mathfrak{C}$ is net, we may suppose that $\left(C_i\right)_i$ is disjoint. So $\left(C_i\right)_i$ is a δ-packing of F, then

$$T^q_{\mu,\nu,\delta}(F) \le \sum_i \mu(C_i)^q \le S^q_{\mu,\nu,\delta}(F).$$

Also observe that it follows from the definitions that

$$L^{q,t}_{\mu,\nu}(F) \le C^{q,t}_{\mu,\nu}(F) \le C^{q,t}_{\mu,\nu}(E) \quad \text{and} \quad \overline{H}^{q,t}_{\mu,\nu}(E) \le C^{q,t}_{\mu,\nu}(E).$$

We therefore conclude

$$H^{q,t}_{\mu,\nu}(E) \le P^{q,t}_{\mu,\nu}(E).$$

Now, let $E \subseteq \Omega$, then, we have

$$\overline{\mathscr{P}}^{q,t}_{\mu,\nu}(E) \ge \min(1, 2^{-t}) C^{q,t}_{\mu,\nu}(E) \tag{4.1}$$

and

$$\begin{aligned}\phi^* \mathscr{P}_{\mu,\nu}^{q,t}(E) &= \phi^* \inf\left\{\sum_i \overline{\mathscr{P}}_{\mu,\nu}^{q,t}(E_i) \;|\;\; E \subseteq \bigcup_i E_i\right\}\\ &\geq \inf\left\{\sum_i C_{\mu,\nu}^{q,t}(E_i) \;|\;\; E \subseteq \bigcup_i E_i\right\}\\ &= P_{\mu,\nu}^{q,t}(E).\end{aligned}$$

□

4.2. Proof of Theorem 3.2. Let E be a subset of Ω and $q \in \mathbb{R}$.

1. First, we suppose that

$$\sup_{F\subseteq E} \liminf_{\delta\to 0} \frac{\log T_{\mu,\nu,\delta}^q(F)}{-\log\delta} > \underline{\dim}_{\mu,\nu}^{q,B}(E) + \varepsilon, \quad \text{for some} \ \ \varepsilon > 0.$$

Then there exist $F \subseteq E$ and $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$,

$$\log T_{\mu,\nu,\delta}^q(F) > -(\underline{\dim}_{\mu,\nu}^{q,B}(E) + \varepsilon)\log\delta$$

and

$$T_{\mu,\nu,\delta}^q(F)\delta^{(\underline{\dim}_{\mu,\nu}^{q,B}(E)+\varepsilon)} > 1.$$

Therefore, we obtain

$$0 = \overline{H}_{\mu,\nu}^{q,(\underline{\dim}_{\mu,\nu}^{q,B}(E)+\varepsilon)}(E) \geq \liminf_{\delta\to 0} T_{\mu,\nu,\delta}^q(F)\delta^{(\underline{\dim}_{\mu,\nu}^{q,B}(E)+\varepsilon)} \geq 1,$$

which is a contradiction. Then

$$\sup_{F\subseteq E} \liminf_{\delta\to 0} \frac{\log T_{\mu,\nu,\delta}^q(F)}{-\log\delta} \leq \underline{\dim}_{\mu,\nu}^{q,B}(E) + \varepsilon, \quad \text{for any} \ \ \varepsilon > 0.$$

Now, suppose that

$$\sup_{F\subseteq E} \liminf_{\delta\to 0} \frac{\log T_{\mu,\nu,\delta}^q(F)}{-\log\delta} < \underline{\dim}_{\mu,\nu}^{q,B}(E) - \varepsilon, \quad \text{for some} \ \ \varepsilon > 0.$$

Then for all $F \subseteq E$, and for every $\delta_0 > 0$ there exists $0 < \delta \leq \delta_0$ such that

$$\log T_{\mu,\nu,\delta}^q(F) < -(\underline{\dim}_{\mu,\nu}^{q,B}(E) - \varepsilon)\log\delta$$

and

$$T_{\mu,\nu,\delta}^q(F)\delta^{(\underline{\dim}_{\mu,\nu}^{q,B}(E)-\varepsilon)} < 1.$$

Hence,

$$+\infty = \overline{H}_{\mu,\nu}^{q,(\underline{\dim}_{\mu,\nu}^{q,B}(E)-\varepsilon)}(E) = \sup_{F\subseteq E} \liminf_{\delta\to 0} T_{\mu,\nu,\delta}^{q}(F)\delta^{(\underline{\dim}_{\mu,\nu}^{q,B}(E)-\varepsilon)} \leq 1,$$

which is a contradiction. Therefore

$$\underline{\dim}_{\mu,\nu}^{q,B}(E) - \varepsilon \leq \sup_{F\subseteq E} \liminf_{\delta\to 0} \frac{\log T_{\mu,\nu,\delta}^{q}(F)}{-\log\delta} \leq \underline{\dim}_{\mu,\nu}^{q,B}(E) + \varepsilon,$$

for any $\varepsilon > 0$ and the result follows since ε is arbitrary.

2. We suppose that

$$\limsup_{\delta\to 0} \frac{\log S_{\mu,\nu,\delta}^{q}(E)}{-\log\delta} > \overline{\dim}_{\mu,\nu}^{q,B}(E) + \varepsilon, \quad \text{for some } \ \varepsilon > 0.$$

For every $\delta_0 > 0$ there exists $0 < \delta \leq \delta_0$ such that

$$\log S_{\mu,\nu,\delta}^{q}(E) > -(\overline{\dim}_{\mu,\nu}^{q,B}(E) + \varepsilon)\log\delta$$

and

$$S_{\mu,\nu,\delta}^{q}(E)\delta^{(\overline{\dim}_{\mu,\nu}^{q,B}(E)+\varepsilon)} > 1.$$

Furthermore, we obtain

$$0 = C_{\mu,\nu}^{q,(\overline{\dim}_{\mu,\nu}^{q,B}(E)+\varepsilon)}(E) = \limsup_{\delta\to 0} S_{\mu,\nu,\delta}^{q}(E)\delta^{(\overline{\dim}_{\mu,\nu}^{q,B}(E)+\varepsilon)} \geq 1,$$

which is a contradiction. Then

$$\limsup_{\delta\to 0} \frac{\log S_{\mu,\nu,\delta}^{q}(E)}{-\log\delta} \leq \overline{\dim}_{\mu,\nu}^{q,B}(E) + \varepsilon, \quad \text{for any } \ \varepsilon > 0.$$

Now, we suppose that

$$\limsup_{\delta\to 0} \frac{\log S_{\mu,\nu,\delta}^{q}(E)}{-\log\delta} < \overline{\dim}_{\mu,\nu}^{q,B}(E) - \varepsilon, \quad \text{for some } \ \varepsilon > 0.$$

Then there exists $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$,

$$\log S_{\mu,\nu,\delta}^{q}(E) < -(\overline{\dim}_{\mu,\nu}^{q,B}(E) - \varepsilon)\log\delta$$

and

$$S_{\mu,\nu,\delta}^{q}(E)\delta^{(\overline{\dim}_{\mu,\nu}^{q,B}(E)-\varepsilon)} < 1.$$

It follows that

$$+\infty = C_{\mu,\nu}^{q,(\overline{\dim}_{\mu,\nu}^{q,B}(E)-\varepsilon)}(E) = \limsup_{\delta\to 0} S_{\mu,\nu,\delta}^{q}(E)\delta^{(\overline{\dim}_{\mu,\nu}^{q,B}(E)-\varepsilon)} \leq 1,$$

which is a contradiction. Therefore

$$\overline{\dim}_{\mu,\nu}^{q,B}(E) - \varepsilon \leq \limsup_{\delta\to 0} \frac{\log S_{\mu,\nu,\delta}^{q}(E)}{-\log\delta} \leq \overline{\dim}_{\mu,\nu}^{q,B}(E) + \varepsilon,$$

for any $\varepsilon > 0$. The result now follows by letting $\varepsilon \to 0$. $\square$

4.3. Proof of Theorem 3.3. Let $q \in \mathbb{R}$ and E be a subset of Ω.

1. Suppose that

   $$\underline{\dim}_{\mu,\nu}^{q,MB}(E) > \inf\left\{\sup_i \underline{\dim}_{\mu,\nu}^{q,B}(E_i) \;\middle|\; E \subseteq \bigcup_i E_i \;\text{ and }\; E_i \subseteq \Omega\right\} =: \beta.$$

   Then there exists $t \in (\beta, \underline{\dim}_{\mu,\nu}^{q,MB}(E))$, so there is a sequence $(E_i)_i$ of Ω such that $E = \cup_i(E_i \cap E)$ and $\sup_i \underline{\dim}_{\mu,\nu}^{q,B}(E_i \cap E) < t$. Thus $\overline{H}_{\mu,\nu}^{q,t}(E_i \cap E) = 0$ for any i, implying that $H_{\mu,\nu}^{q,t}(E) = 0$. It is a contradiction.

   Now, we suppose that $\underline{\dim}_{\mu,\nu}^{q,MB}(E) < \beta$, there exists t in $(\underline{\dim}_{\mu,\nu}^{q,MB}(E), \beta)$, thus $H_{\mu,\nu}^{q,t}(E) = 0$. Therefore, there is a sequence $(E_i)_i$ of Ω such that $E = \cup_i(E_i \cap E)$ and $\overline{H}_{\mu,\nu}^{q,t}(E_i \cap E) < +\infty$, for any i. Then, $\underline{\dim}_{\mu,\nu}^{q,B}(E_i) \leq t$ for any i and $\beta \leq t$. It is a contradiction.
2. The proof is similar to the one of assertion (1). $\square$

4.4. Proof of Theorem 3.4.

1. It follows easily from (4.1) that $\overline{\dim}_{\mu,\nu}^{q,B}(E) \leq \Delta_{\mu,\nu}^{q}(E)$. On the other hand, if $t > \overline{\dim}_{\mu,\nu}^{q,B}(E)$, by using Theorem 3.2, we have

   $$t > \limsup_{\delta\to 0} \frac{\log S_{\mu,\nu,\delta}^{q}(E)}{-\log\delta} \geq 0.$$

   There exists $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$,

   $$S_{\mu,\nu,\delta}^{q}(E) < \delta^{-t}.$$

   Let $\Big(C_i\Big)_i$ be a centered δ-packing of E. Thus

   $$\sum_i \mu(C_i)^q \nu(C_i)^t \leq S_{\mu,\nu,\delta}^{q}(E)\delta^t < \delta^t\delta^{-t} = 1.$$

Then, we have $\overline{\mathscr{P}}^{q,t}_{\mu,\nu,\delta}(E) < 1$ and $\overline{\mathscr{P}}^{q,t}_{\mu,\nu}(E) = \inf_{\delta>0} \overline{\mathscr{P}}^{q,t}_{\mu,\nu,\delta}(E) \leq 1$. Hence, $\Delta^q_{\mu,\nu}(E) \leq t$. It follows that $\Delta^q_{\mu,\nu}(E) \leq \overline{\dim}^{q,B}_{\mu,\nu}(E)$.

2. If $E \subseteq \bigcup_i E_i$, then

$$B^q_{\mu,\nu}(E) \leq \sup_i B^q_{\mu,\nu}(E_i) \leq \sup_i \Delta^q_{\mu,\nu}(E_i) = \sup_i \overline{\dim}^{q,B}_{\mu,\nu}(E_i).$$

Theorem 3.3 now gives that $B^q_{\mu,\nu}(E) \leq \overline{\dim}^{q,MB}_{\mu,\nu}(E)$.

Conversely, if $t > B^q_{\mu,\nu}(E)$ then $\mathscr{P}^{q,t}_{\mu,\nu}(E) = 0$, so that $E \subseteq \cup_i E_i$ for a collection of sets E_i with $\overline{\mathscr{P}}^{q,t}_{\mu,\nu}(E_i) < 1$ for each i. Hence, for each i, if δ is small enough, then $\overline{\mathscr{P}}^{q,t}_{\mu,\nu,\delta}(E_i) < +\infty$ and so by (4.1), $S^q_{\mu,\nu,\delta}(E_i)\delta^t < +\infty$ as $\delta \to 0$. Therefore, $\overline{\dim}^{q,B}_{\mu,\nu}(E_i) \leq t$ for each i, giving from Theorem 3.3 that $\overline{\dim}^{q,MB}_{\mu,\nu}(E) \leq t$. $\square$

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References

[1] N. Attia and B. Selmi, *Relative multifractal box-dimensions*, Filomat **33** (2019), 2841–2859.

[2] N. Attia and B. Selmi. *Regularities of multifractal Hewitt-Stromberg measures*, Commun. Korean Math. Soc. **34** (2019), 213–230.

[3] N. Attia and B. Selmi, *A multifractal formalism for Hewitt−Stromberg measures*, Journal of Geometric Analysis, (to appear). https://doi.org/10.1007/s12220-019-00302-3

[4] N. Attia and B. Selmi, *On the multifractal analysis of Birkhoff sums on a non-compact symbolic space*, Comptes rendus Mathematique., (to appear).

[5] P. Billingsley, *Hausdorff dimension in probability theory I*, Illinois. J. Math. **4** (1960), 187–209.

[6] P. Billingsley, *Hausdorff dimension in probability theory II*, Illinois. J. Math. **5** (1961), 291–298.

[7] J. Cole, *Relative multifractal analysis*, Choas, Solitons & Fractals. **11** (2000), 2233–2250.

[8] C. Dai and S.J. Taylor, *Defining fractal in a probability space*, Illinois. J. Math. **38** (1994), 480–500.
[9] C. Dai, *Frostman lemma for Hausdorff measure and Packing measure in probability spaces*, J Nanjing University, Mathematical Biquarterly **12** (1995), 191–203.
[10] C. Dai and Y. Hou, *Frostman lemmas for Hausdorff measure and packing measure in a product probability space and their physical application*, Chaos, Solitons and Fractals **24** (2005), 733–744.
[11] C. Dai and Y. Li, *A multifractal formalism in a probability space*, Chaos, Solitons and Fractals **27** (2006), 57–73.
[12] C. Dai and Y. Li, *Multifractal dimension inequalities in a probability space*, Chaos, Solitons and Fractals **34** (2007), 213–223.
[13] M. Dai, X. Peng and W. Li, *Relative multifractal analysis in a probability space*, Interna Journal of Nonlinear Science **10** (2010), 313–319.
[14] M. Das, *Billingsley's packing dimension*, Proceedings of the American Mathematical Society **136** (2008), 273–278.
[15] M. Das, *Hausdorff measures, dimensions and mutual singularity*, Transactions of the American Mathematical Society **357** (2005), 4249–4268.
[16] Z. Douzi and B. Selmi, *Multifractal variation for projections of measures*, Chaos, Solitons and Fractals **91** (2016), 414–420.
[17] G. A. Edgar, *Integral, probability, and fractal measures*, Springer-Verlag, New York, (1998).
[18] K. J. Falconer, *Fractal geometry: mathematical foundations and applications*, Chichester (1990). Wiley.
[19] H. Haase, *A contribution to measure and dimension of metric spaces*, Math. Nachr. **124** (1985), 45–55.
[20] H. Haase, *Open-invariant measures and the covering number of sets*, Math. Nachr. **134** (1987), 295–307.
[21] E. Hewitt and K. Stromberg. *Real and abstract analysis. A modern treatment of the theory of functions of a real variable*, Springer-Verlag, New York, (1965).
[22] S. Jurina, N. MacGregor, A. Mitchell, L. Olsen and A. Stylianou, *On the Hausdorff and packing measures of typical compact metric spaces*, Aequationes Mathematicae **92** (2018), 709–735.
[23] M. Khelifi, H. Lotfi, A. Samti and B. Selmi, *A relative multifractal analysis*, Chaos, Solitons and Fractals, (to appear).
[24] Q. Liu, *Local dimensions of the measure on a Galtion-Watson tree*, Ann. Inst. H. Poincarè, Probabilités et Statistiques **37** (2001), 195–222.
[25] Q. Liu, *Exact packing measure on a Galton-Watson tree*, Stochastic Processes and their Applications **85** (2000), 19–28.
[26] P. Mattila, *Geometry of sets and measures in euclidian spaces: Fractals and Rectifiability*, Cambridge University Press (1995).
[27] A. Mitchell and L. Olsen, *Coincidence and noncoincidence of dimensions in compact subsets of* $[0,1]$, arXiv:1812.09542v1, (2018).
[28] L. Olsen, *On average Hewitt-Stromberg measures of typical compact metric spaces*, Mathematische Zeitschrift **293** (2019), 1201–1225.

[29] Y. Pesin, *Dimension theory in dynamical systems, Contemporary views and applications*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, (1997).

[30] C. Qing-Jiang and Q. Xiao-Gang, *Characteristics of a class of vector-valued non-separable higher-dimensional wavelet packet bases*, Chaos, Solitons and Fractals **41** (2009), 1676–1683.

[31] J. M. Rey, *The role of Billingsley dimensions in computing fractal dimensions on Cantor-like spaces*, Proceedings of the American Mathematical Society **128** (1999), 561–572.

[32] B. Selmi, *On the strong regularity with the multifractal measures in a probability space*, Anal. Math. Phys. **9** (2019), 1525–1534

[33] B. Selmi, *On the effect of projections on the Billingsley dimensions.* Asian-European Journal of Mathematics **13** (2020), 2050128/1–17.

[34] B. Selmi, *On the projections of the multifractal Hewitt−Stromberg dimension functions*, arXiv:1911.09643v1

[35] O. Zindulka, *Packing measures and dimensions on Cartesian products*, Publ. Mat. **57** (2013), 393–420.

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