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REMARKS ON A GOLDBACH PROPERTY

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Abstract. In this paper, we study Noetherian Boolean rings. We show that if R is a Noetherian Boolean ring, then R is finite and $R \simeq (\mathbb{Z}_2)^n$ for some integer $n \geq 1$. If R is a Noetherian ring, then R/J is a Noetherian Boolean ring, where J is the intersection of all ideals I of R with $|R/I| = 2$. Thus R/J is finite, and hence the set of ideals I of R with $|R/I| = 2$ is finite. We also give a short proof of Hayes's result : For every polynomial $f(x) \in \mathbb{Z}[x]$ of degree $n \geq 1$, there are irreducible polynomials $g(x)$ and $h(x)$, each of degree n, such that $g(x) + h(x) = f(x)$.

All rings are assumed to be commutative rings with identity. We use the term dimension of R, denoted $dimR$, to refer to the Krull dimension of R. A ring R is called von Neumann regular if for each x in R, there exists y in R such that $x = xyx$. It is well known that R is von Neumann regular if and only if R is zero-dimensional and reduced if and only if R_P is a field for each $P \in Spec(R)$ if and only if each ideal of R is a radical ideal if and only if each principal ideal of R is idempotent [4, Theorem 3.1]. In particular, $dimR = 0$ if and only if $R/nil(R)$ is von Neumann regular (where $nil(R)$ is the nilradical of R) if and only if a power of each principal ideal of R is idempotent - that is, if and only if, for each $x \in R$, there exists $n(x) \in \mathbb{Z}^+$ and $y \in R$ such that $x^{n(x)} = yx^{n(x)+1}$ [4, Theorem 3.4]. The class of von Neumann regular rings is closed under taking homomorphic images, quotient rings, and arbitrary products [4, Result 3.2].

R is called a Boolean ring if every element is idempotent. It is well known that R is a Boolean ring if and only if R_M is isomorphic to $\mathbb{Z}_2$ for each maximal ideal M of R. A Boolean ring is a von Neumann regular

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ring with $x = x1x$. It is known that $R/nil(R)$ is Boolean if and only if $dimR = 0$ and for each maximal ideal M of R, $R/M \simeq \mathbb{Z}_2$ if and only if given $x \in R$, there exists a natural number n with $x^n(1+x)^n = 0$ [1, Theorem 5].

In this paper, we study Noetherian Boolean rings. We show that if R is a Noetherian Boolean ring, then R is finite and $R \simeq (\mathbb{Z}_2)^n$ for some integer $n \geq 1$. If R is a Noetherian ring, then R/J is a Noetherian Boolean ring, where J is the intersection of all ideals of $\mathcal{I}_2 = \{I \mid I \text{ is an ideal of } R \text{ and } |R/I| = 2\}$. Thus R/J is finite, and hence the set $\mathcal{I}_2$ is finite. We also give a short proof of Hayes's result using Chinese Remainder Theorem for rings.

For future reference, we include a result from [4, Theorem 3.1(4)].

LEMMA 1. *If R is a Boolean ring, then $M^2 = M$ for each ideal M of R.*

Proof. If $x \in M$, then $x = x^2 \in M^2$. Hence $M \subseteq M^2$, and thus $M = M^2$. □

Let I and J be ideals of R. Recall that I and J are comaximal if $I + J = R$. Suppose that I and J are comaximal. Then there exist $a \in I$ and $b \in J$ such that $a + b = 1$. For any integer $m, n(\geq 1)$, $1 = (a+b)^{m+n}$ and $(a+b)^{m+n} \in I^m + J^n$; so $I^m + J^n = R$, and hence I^m and J^n are also comaximal [7, Lemma 4].

For future reference, we include the Chinese Remainder Theorem [3, Section 7.6].

LEMMA 2. *(Chinese Remainder Theorem) Let $I_1, I_2, ..., I_n$ be ideals of R. The map $R \to R/I_1 \times R/I_2 \times \cdots \times R/I_n$ defined by $r \mapsto (r+I_1, r+I_2, ..., r+I_n)$ is a ring homomorphism with kernel $I_1 \cap I_2 \cap \cdots \cap I_n$. If each ideals I_i, I_j $(i \neq j)$ are comaximal, then the map is surjective and $I_1 \cap I_2 \cap \cdots \cap I_n = I_1 I_2 \cdots I_n$, so $R/(I_1 I_2 \cdots I_n) \simeq R/(I_1 \cap I_2 \cap \cdots \cap I_n) \simeq R/I_1 \times R/I_2 \times \cdots \times R/I_n$.*

If R is a finite Boolean ring, then $R \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (c.f. [3, Exercise 2, p. 267]). We next show that a Noetherian Boolean ring R is finite with $R \simeq (\mathbb{Z}_2)^n$ for some integer $n \geq 1$.

THEOREM 3. *Let R be a Boolean ring.*

(1) R is zero-dimensional reduced and for each maximal ideal M of R, $R/M \simeq \mathbb{Z}_2$.

(2) If R is Noetherian, then R is a finite ring with $R \simeq (\mathbb{Z}_2)^n$ for some integer $n \geq 1$.

Proof. (1) Suppose that $dim R > 0$. Then there are primes $P \subsetneq M$. Let $x \in M - P$. Then $x = x^2$, and so $x(x-1) = 0 \in P$. Since $x \notin P$, we have $x - 1 \in P \subseteq M$. But $x \in M$, so $1 \in M$, a contradiction. Clearly $nil(R) = \{0\}$. Let M be a maximal ideal of R. Then R/M is a field and a Boolean ring; so $R/M \simeq \mathbb{Z}_2$.

(2) Suppose that R is Noetherian. Then since, each ideal of R contains a product of prime ideals of R [3, Corollary 22, p. 685], we have $(0) = {P_1}^{r_1}{P_2}^{r_2} \cdots {P_n}^{r_n}$. By Lemma 1, each ${P_i}^{r_i} = P_i$; so $(0) = P_1 P_2 \cdots P_n$ and $P_1, P_2, ..., P_n$ are distinct. Since the ideals P_i and P_j with $i \neq j$ are comaximal, the map $R \to R/P_1 \times R/P_2 \cdots \times R/P_n$, $r \mapsto (r + P_1, r + P_2, ..., r + P_n)$ is an epimorphism with kernel $P_1 \cap P_2 \cap \cdots \cap P_n = P_1 P_2 \cdots P_n = \{0\}$ by Lemma 2. Hence $R \simeq R/\{0\} \simeq R/P_1 \times R/P_2 \times \cdots \times R/P_n$. Now, each $R/P_i \simeq \mathbb{Z}_2$ by (1). Hence $R \simeq (\mathbb{Z}_2)^n$ for some integer $n \geq 1$. □

Corollary 4. *(c.f., [7, Lemma 7], [9, Proposition 13]) Let R be a ring and let*

$$\mathcal{I}_2 = \{I \mid I \textit{ is an ideal of } R \textit{ and } |R/I| = 2\}.$$

Let J be the intersection of all ideals in $\mathcal{I}_2$. Then R/J is a Boolean ring. Moreover, if R is Noetherian, then R/J is a finite ring with $|R/J| = 2^n$ for some integer $n \geq 1$ and $\mathcal{I}_2$ is finite.

Proof. Let $x \in R$. For each $I \in \mathcal{I}_2$, we have $x^2 - x \in I$. Thus for each $x \in R$, $x^2 - x \in \bigcap\{I | I \in \mathcal{I}_2\} = J$. Therefore R/J is a Boolean ring. In particular, if R is Noetherian, then R/J is Noetherian, and so by Theorem 3, R/J is a finite ring with $|R/J| = 2^n$ for some integer $n \geq 1$. Hence $\{I/J || R/I| = 2\}$ is finite. Since the map $I \to I/J$ is injective, $\mathcal{I}_2$ is finite. □

Let R be a Noetherian ring and let $\mathcal{I}_n = \{I_a\}_{a \in \Lambda}$, where $|R/I_a| = n$. Define $J = \bigcap_{a \in \Lambda} I_a$. Then R/J can be imbedded in $\prod_{a \in \Lambda}(R/I_a)$. Then R/J is zero-dimensional Noetherian and hence Artinian. Hence

$J = \bigcap I_a$ has a finite subintersections, so R/J is imbedded in $\prod_{i=1}^{k}(R/I_{a_i})$, a ring of cardinality n^k. Therefore R/J is finite and hence $\{I_a/J\}_{a\in\Lambda}$ is finite. Since the map $I_a \to I_a/J$ is injective, $\mathcal{I}_n = \{I_a\}_{a\in\Lambda}$ is finite [5, Result 3].

D. Hayes [6] was the first to observe and prove the following polynomial analogue of the celebrated Goldbach conjecture:

Theorem 5. *For every polynomial $f(x) \in \mathbb{Z}[x]$ of degree $n \geq 1$, there are irreducible polynomials $g(x)$ and $h(x)$, each of degree n, such that $g(x) + h(x) = f(x)$.*

To prove Theorem 5, Hayes used the following [6, Lemma]: if p and q are distinct odd primes, then there exist integers c and d such that $pc + qd = 1$, $p \nmid c$, and $q \nmid d$. Also, Hayes pointed out that more general theorem whenever R is a principal ideal domain with infinitely many maximal ideals. In [7], P. Pollack showed the case that R is a Noetherian domain with infinitely many maximal ideals: Suppose that R is an integral domain which is Noetherian and has infinitely many maximal ideals. Then every element of $R[x]$ of degree $n \geq 1$ can be written as the sum of two irreducibles of degree n. He used distinct maximal ideals P and Q such that (1) $P^2 \neq P$ and $Q^2 \neq Q$, (2) $|R/P|, |R/Q| > 2$ [7, Theorem 5]. Noetherian condition guarantees that $\mathcal{I}_2 = \{I \mid I \text{ is an ideal of } R \text{ and } |R/I| = 2\}$ is finite by Corollary 4, and if M is maximal, then $M^2 \neq M$ [7, Lemma 6]. Also, in [8], F. Saidak gives a short proof of Hayes's result.

In order to prove Theorem 5, we recall the remarkable criterion of Eisenstein [2].

Lemma 6. *(Eisenstein's criterion) If, in the integral polynomial $a_0x^n + a_1x^{n-1} + \cdots + a_n$, all of the coefficients except a_0 are divisible by a prime p, but a_n is not divisible by p^2, then the polynomial is irreducible.*

Proof of Theorem 5. Write $f(x) = m_0x^n + m_1x^{n-1} + \cdots + m_n$. Choose distinct odd primes p and q which do not divide either of m_0 and m_n. Let $R = \mathbb{Z}$, $pR = P$, and $qR = Q$. Since P and Q are comaximal, P^2 and Q^2 are also comaximal. Therefore the two maps $R \to R/P \times R/Q$, $r \mapsto (r + P, r + Q)$ and $R \to R/P^2 \times R/Q^2$, $r \mapsto (r + P^2, r + Q^2)$ are surjective homomorphisms by Lemma 2. Choose $\alpha \notin P$ and $\beta \notin Q$. Let a_0 be a preimage of $(\alpha + P, m_0 - \beta + Q)$

under $R \to R/P \times R/Q$. Set $b_0 = m_0 - a_0$. Then $a_0 \notin P$ and $b_0 \notin Q$. Also, for i $(0 < i < n)$, let a_i be a preimage of $(0 + P, m_i + Q)$ under $R \to R/P \times R/Q$. Set $b_i = m_i - a_i$. Then $a_i \in P$ and $b_i \in Q$. Finally, let a_n be a preimage of $(p+P^2, m_n - q+Q^2)$ under $R \to R/P^2 \times R/Q^2$. Set $b_n = m_n - a_n$. Then we have $a_n \in P, a_n \notin P^2, b_n \in Q$, and $b_n \notin Q^2$. If $g(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ and if $h(x) = b_0x^n + b_1x^{n-1} + \cdots + b_n$, then $f(x) = g(x) + h(x)$. Lemma 6 says that $g(x)$ and $h(x)$ are irreducible polynomials. □

REMARK 7. *(cf. [6, Theorem 1]) As the same notation above, Hayes choose* ${a_n}'$ *and* ${b_n}'$ *such that* $p{a_n}' + q{b_n}' = m_n$*, but* $p \nmid {a_n}'$ *and* $q \nmid {b_n}'$ *by [6, Lemma]. Set* $a_n = p{a_n}'$ *and* $b_n = q{b_n}'$*. Then* $m_n = a_n + b_n$*,* $p|a_n$*,* $p^2 \nmid a_n$*,* $q|b_n$*, and* $q^2 \nmid b_n$*.*

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