Korean J. Math. **28** (2020), No. 3, pp. 489–507
http://dx.doi.org/10.11568/kjm.2020.28.3.489

SOME GROWTH ESTIMATIONS BASED ON (p, q)-φ RELATIVE GOL'DBERG TYPE AND (p, q)-φ RELATIVE GOL'DBERG WEAK TYPE OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Abstract. In this paper we discussed some growth properties of entire functions of several complex variables on the basis of (p, q)-φ relative Gol'dberg type and (p, q)-φ relative Gol'dberg weal type where p , q are positive integers and $\varphi(R) : [0, +\infty) \to (0, +\infty)$ is a non-decreasing unbounded function.

1. Introduction, Definitions and Notations

The complex and real n-spaces are usually denoted by the respective symbols $\mathbb{C}^n$ and R^n. Further, if we assume that the points $(z_1, z_2, \cdots, z_n)$, $(m_1, m_2, \cdots, m_n)$ of $\mathbb{C}^n$ or I^n are represented by their corresponding unsuffixed symbols z, m respectively where I denotes the set of non-negative integers, then the modulus of z, denoted by $|z|$, is defined as $|z| = \left(|z_1|^2 + \cdots + |z_n|^2\right)^{\frac{1}{2}}$. If the coordinates of the vector m are non-negative integers $m_1, m_2, ..., m_n$, then by the symbol z^m we may present the expression $z_1^{m_1} \cdots z_n^{m_n}$ where $\|m\| = m_1 + \cdots + m_n$.

Received March 5, 2020. Revised July 28, 2020. Accepted July 29, 2020.
2010 Mathematics Subject Classification: 32A15.
Key words and phrases: Entire functions of several complex variables, (p, q)-φrelative Gol'dberg order, (p, q) -φ relative Gol'dberg lower order, (p, q)-φ relative Gol'dberg type, (p, q)-φ relative Gol'dberg weak type.
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Consider $D \subseteq \mathbb{C}^n$ as an arbitrary bounded complex n-circular domain having the center at the origin. Then for any entire function $f(z)$ of n complex variables and $R > 0$, the maximum modulus function $M_{f,D}(R)$ may be defined to be $M_{f,D}(R) = \sup_{z \in D_R} |f(z)|$ where a point $z \in D_R$ implies and is implied by $\frac{z}{R} \in D$. It is quite obvious that $M_{f,D}(R)$ is strictly increasing if $f(z)$ is non-constant and the inverse of $M_{f,D}(R)$ i.e., $M_{f,D}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists such that $\lim_{R\to\infty} M_{f,D}^{-1}(R) = \infty$.

In order to introduce the notion of various growth indicators in the theory of entire functions, the following well known notations are frequently used: $\exp^{[k]} R = \exp\left(\exp^{[k-1]} R\right)$ and $\log^{[k]} R = \log\left(\log^{[k-1]} R\right)$, for $k \in \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers. We also denote $\log^{[0]} R = R$, $\log^{[-1]} R = \exp R$, $\exp^{[0]} R = R$ and $\exp^{[-1]} R = \log R$. Further we assume that throughout the present paper p, q and m always denote positive integers. Also throughout the paper an entire function $f(z)$ of n-complex variables will stand for an entire function $f(z)$ for any bounded complete n-circular domain D with center at origin in $\mathbb{C}^n$. Taking this into consideration, we remind that Datta et al. [7] introduced the concept of (p, q)-th Gol'dberg order and (p, q)-th Gol'dberg lower order of an entire function $f(z)$ of n-complex variables where $p \geq q$ in the following way:

$$\begin{matrix} \rho_D^{(p,q)}(f) \\ \lambda_D^{(p,q)}(f) \end{matrix} = \lim_{R\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} R}.$$

It is clear from the above definitions that for $p = 2$ and $q = 1$, the symbols $\rho_D^{(2,1)}(f)$ and $\lambda_D^{(2,1)}(f)$ actually present the classical growth indicators (see e.g. [10, 11]) which are denoted by $\rho_D(f)$ and $\lambda_D(f)$ respectively. However in the line of Gol'dberg (see e.g. [10,11]), it may be easily established that $\rho_D^{(p,q)}(f)$ and $\lambda_D^{(p,q)}(f)$ are independent of the choice of the domain D, and therefore one can write $\rho^{(p,q)}(f)$ and $\lambda^{(p,q)}(f)$ instead of $\rho_D^{(p,q)}(f)$ and $\lambda_D^{(p,q)}(f)$ respectively.

In [14], Shen et al. set up the definition of (p, q)-φ order of an entire function. One may see [14] to collect the details about (p, q)-φ order of an entire function. Consequently, the definition of (p, q)-φ Gol'dberg order and (p, q)-φ Gol'dberg lower order of an entire function $f(z)$ of n-complex variables are established in [4] which are as follows:

DEFINITION 1. [4] Let $\varphi(R) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. Then the (p,q)-φ Gol'dberg order $\rho_D^{(p,q)}(f,\varphi)$ and (p,q)-φ Gol'dberg lower order $\lambda_D^{(p,q)}(f,\varphi)$ of an entire function $f(z)$ of n-complex variables are defined as

$$\begin{matrix} \rho_D^{(p,q)}(f,\varphi) \\ \lambda_D^{(p,q)}(f,\varphi) \end{matrix} = \lim_{R\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} \varphi(R)}.$$

The above definition avoids the restriction $p \geq q$. For any non-decreasing unbounded function $\varphi(R) : [0,+\infty) \to (0,+\infty)$, if it is assumed that $\lim\limits_{R\to+\infty} \frac{\log^{[q]}\varphi(\alpha R)}{\log^{[q]}\varphi(R)} = 1$ for all $\alpha > 0$, then $\rho_D^{(p,q)}(f,\varphi)$ and $\lambda_D^{(p,q)}(f,\varphi)$ are independent of the choice of the domain D (see e.g. [4]), and therefore one can use the symbols $\rho^{(p,q)}(f,\varphi)$ and $\lambda^{(p,q)}(f,\varphi)$ instead of $\rho_D^{(p,q)}(f,\varphi)$ and $\lambda_D^{(p,q)}(f,\varphi)$ respectively.

Keeping this in mind we just state the following definition:

DEFINITION 2. [4] An entire function $f(z)$ of n-complex variables is said to have index-pair (p,q)-φ if $b < \rho^{(p,q)}(f,\varphi) < \infty$ and $\rho^{(p-1,q-1)}(f,\varphi)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < \rho^{(p,q)}(f,\varphi) < \infty$, then

$$\begin{cases} \rho^{(p-m,q)}(f,\varphi) = \infty & \text{for} \quad m < p, \\ \rho^{(p,q-m)}(f,\varphi) = 0 & \text{for} \quad m < q, \\ \rho^{(p+m,q+m)}(f,\varphi) = 1 & \text{for} \quad m = 1,2,\cdots \end{cases}.$$

Similarly for $0 < \lambda^{(p,q)}(f,\varphi) < \infty$,

$$\begin{cases} \lambda^{(p-m,q)}(f,\varphi) = \infty & \text{for} \quad m < p, \\ \lambda^{(p,q-m)}(f,\varphi) = 0 & \text{for} \quad m < q, \\ \lambda^{(p+m,q+m)}(f,\varphi) = 1 & \text{for} \quad m = 1,2,\cdots \end{cases}.$$

If $\varphi(R) = R$ and $p \geq q$, then definition 1 coincides with the definition of (p,q)-th Gol'dberg order and (p,q)-th Gol'dberg lower order introduced by Datta et al. [7]. Consequently for $\varphi(R) = R$, Definition 2 reduces to the the definition of index-pair (p,q) of an entire function $f(z)$ of n-complex variables. For detail about index-pair (p,q) of an entire function $f(z)$ of n-complex variables, one may see [2,3].

Now, for the development of such growth indicators, one may introduce (p,q)-φ Gol'dberg type and (p,q)-φ Gol'dberg weak type in the following way:

Definition 3. Let $\varphi(R) : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function. Let $f(z)$ be an entire functions of n-complex variables such that $0 < \rho_D^{(p,q)}(f, \varphi) < \infty$. Then the (p,q)-φ Gol'dberg type $\sigma_D^{(p,q)}(f, \varphi)$ and the (p,q)-φ Gol'dberg lower type $\overline{\sigma}_D^{(p,q)}(f, \varphi)$ of $f(z)$ are defined as:

$$\begin{array}{c} \sigma_D^{(p,q)}(f, \varphi) \\ \overline{\sigma}_D^{(p,q)}(f, \varphi) \end{array} = \lim_{R\to\infty} \begin{array}{c} \sup \\ \inf \end{array} \frac{\log^{[p-1]}(M_{f,D}(R))}{\left(\log^{[q-1]} \varphi(R)\right)^{\rho_D^{(p,q)}(f,\varphi)}}.$$

Definition 4. Let $\varphi(R) : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function. Let $f(z)$ be an entire functions of n-complex variables such that $0 < \lambda_D^{(p,q)}(f, \varphi) < \infty$. Then the (p,q)-φ Gol'dberg weak type $\tau_D^{(p,q)}(f, \varphi)$ and (p,q)-φ Gol'dberg upper weak type $\overline{\tau}_D^{(p,q)}(f, \varphi)$ of $f(z)$ are defined as:

$$\begin{array}{c} \overline{\tau}_D^{(p,q)}(f, \varphi) \\ \tau_D^{(p,q)}(f, \varphi) \end{array} = \lim_{R\to\infty} \begin{array}{c} \sup \\ \inf \end{array} \frac{\log^{[p-1]}(M_{f,D}(R))}{\left(\log^{[q-1]} \varphi(R)\right)^{\lambda_D^{(p,q)}(f,\varphi)}}.$$

Gol'dberg has shown that (see [10,11]) Gol'dberg type depends on the domain D, so in general all the growth indicators defined in Definition 3 and Definition 4 also depend on D.

If $\varphi(R) = R$, then Definition 3 and Definition 4 reduce to the following definitions.

Definition 5. [2,3] Let $0 < \rho^{(p,q)}(f) < \infty$. The (p,q)-th Gol'dberg type and (p,q)-th Gol'dberg lower type respectively denoted by $\sigma_D^{(p,q)}(f)$ and $\overline{\sigma}_D^{(p,q)}(f)$ of an entire function $f(z)$ of n-complex variables with respect to any bounded complete n-circular domain D with center at all the origin $\mathbb{C}^n$ are defined as follows:

$$\begin{array}{c} \sigma_D^{(p,q)}(f) \\ \overline{\sigma}_D^{(p,q)}(f) \end{array} = \lim_{R\to\infty} \begin{array}{c} \sup \\ \inf \end{array} \frac{\log^{[p-1]} M_{f,D}(R)}{\left[\log^{[q-1]} R\right]^{\rho^{(p,q)}(f)}}.$$

Definition 6. [2,3] Let $0 < \lambda^{(p,q)}(f) < \infty$. The (p,q)-th Gol'dberg weak type denoted by $\tau_D^{(p,q)}(f)$ of an entire function $f(z)$ of n-complex variables with respect to any bounded complete n-circular domain D

with center at all the origin $\mathbb{C}^n$ is defined as follows:

$$\tau_D^{(p,q)}(f) = \liminf_{R\to\infty} \frac{\log^{[p-1]} M_{f,D}(R)}{\left[\log^{[q-1]} R\right]^{\lambda^{(p,q)}(f)}}.$$

Also one may define the (p,q)-th Gol'dberg upper weak type denoted by $\overline{\tau}_D^{(p,q)}(f)$ in the following way:

$$\overline{\tau}_D^{(p,q)}(f) = \limsup_{R\to\infty} \frac{\log^{[p-1]} M_{f,D}(R)}{\left[\log^{[q-1]} R\right]^{\lambda^{(p,q)}(f)}}.$$

However, Mondal et al. [12] introduced the concept of relative Gol'dberg order of $f(z)$ with respect to $g(z)$ for any two entire functions $f(z)$ and $g(z)$ of n-complex variables. In the case of relative Gol'dberg order, it therefore seems reasonable to define suitably the (p,q)-th relative Gol'dberg order. With this in view Biswas [3] introduced the following definition in the light of index-pair.

Definition 7. [3] Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pairs (m,q) and (m,p) respectively. Then the (p,q)-th relative Gol'dberg order $\rho_{g,D}^{(p,q)}(f)$ and (p,q)-th relative Gol'dberg lower order $\lambda_{g,D}^{(p,q)}(f)$ of $f(z)$ with respect to $g(z)$ are defined as

$$\begin{matrix} \rho_{g,D}^{(p,q)}(f) \\ \lambda_{g,D}^{(p,q)}(f) \end{matrix} = \lim_{R\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]} R}.$$

Definition 7 avoids the restriction $p \geq q$ of Definition 1.3 of [1]. In view of Theorem 2.1 of [1] one can easily prove that $\rho_{g,D}^{(p,q)}(f)$ and $\lambda_{g,D}^{(p,q)}(f)$ are independent of the choice of the domain D, and therefore one can write $\rho_g^{(p,q)}(f)$ and $\lambda_g^{(p,q)}(f)$ instead of $\rho_{g,D}^{(p,q)}(f)$ and $\lambda_{g,D}^{(p,q)}(f)$.

In [3], Biswas has introduced the notion of relative (p,q)-th Gol'dberg type $\Delta_{g,D}^{(p,q)}(f)$, relative (p,q)-th Gol'dberg lower type $\overline{\Delta}_{g,D}^{(p,q)}(f)$, relative (p,q)-th Gol'dberg weak type $\tau_{g,D}^{(p,q)}(f)$ and relative (p,q)-th Gol'dberg upper weak type $\overline{\tau}_{g,D}^{(p,q)}(f)$ considering the notion of index-pair which are as follows:

Definition 8. [3] Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pairs (m,q) and (m,p) respectively, such that $0 < \rho_g^{(p,q)}(f) < +\infty$. Then the relative (p,q)-th Gol'dberg type $\Delta_{g,D}^{(p,q)}(f)$, relative (p,q)-th Gol'dberg lower type $\overline{\Delta}_{g,D}^{(p,q)}(f)$ of $f(z)$ with respect to $g(z)$ are defined as:

$$\begin{matrix} \Delta_{g,D}^{(p,q)}(f) \\ \overline{\Delta}_{g,D}^{(p,q)}(f) \end{matrix} = \lim_{R\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{\left(\log^{[q-1]} R\right)^{\rho_g^{(p,q)}(f)}}.$$

Definition 9. [3] Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pairs (m,q) and (m,p) respectively, such that $0 < \lambda_g^{(p,q)}(f) < +\infty$. Then the relative (p,q)-th Gol'dberg weak type $\tau_{g,D}^{(p,q)}(f)$, and relative (p,q)-th Gol'dberg upper weak type $\overline{\tau}_{g,D}^{(p,q)}(f)$ of $f(z)$ with respect to $g(z)$ are defined as:

$$\begin{matrix} \overline{\tau}_{g,D}^{(p,q)}(f) \\ \tau_{g,D}^{(p,q)}(f) \end{matrix} = \lim_{R\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{\left(\log^{[q-1]} R\right)^{\lambda_g^{(p,q)}(f)}}.$$

Since Gol'dberg has shown that (see [10, 11]) Gol'dberg type depends on the domain D, therefore, in general, all the growth indicators defined in Definition 8 and Definition 9 also depend on D (cf. [3]).

In order to make some progress in the study of relative Gol'dberg order, in [4], the definition of (p,q)-φ relative Gol'dberg order and the (p,q)-φ relative Gol'dberg lower order are given which are as follows:

Definition 10. [4] Let $\varphi(R) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. Also let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables. The (p,q)-φ relative Gol'dberg order and the (p,q)-φ relative Gol'dberg lower order of $f(z)$ with respect to $g(z)$ are defined as

$$\begin{matrix} \rho_{g,D}^{(p,q)}(f,\varphi) \\ \lambda_{g,D}^{(p,q)}(f,\varphi) \end{matrix} = \lim_{R\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]} \varphi(R)}.$$

In this paper, we assume that the nondecreasing unbounded function $\varphi(R) : [0,+\infty) \to (0,+\infty)$ always satisfies $\lim_{R\to+\infty} \frac{\log^{[q]} \varphi(\alpha R)}{\log^{[q]} \varphi(R)} = 1$ for all $\alpha > 0$. Since, Biswas et al. [4] have already shown that $\rho_{g,D}^{(p,q)}(f,\varphi)$ and $\lambda_{g,D}^{(p,q)}(f,\varphi)$ are independent of the choice of the domain D when $\varphi(R)$: $[0,+\infty) \to (0,+\infty)$ is a nondecreasing unbounded function and satisfies

$\lim\limits_{R\to+\infty} \frac{\log^{[q]}\varphi(\alpha R)}{\log^{[q]}\varphi(R)} = 1$ for all $\alpha > 0$, so here we shall always use the notations $\rho_g^{(p,q)}(f,\varphi)$ and $\lambda_g^{(p,q)}(f,\varphi)$ instead of $\rho_{g,D}^{(p,q)}(f,\varphi)$ and $\lambda_{g,D}^{(p,q)}(f,\varphi)$ respectively.

For the development of such growth indicators, it is then natural for Biswas et al. [6] to define the (p,q)-φ relative Gol'dberg type, (p,q)-φ relative Gol'dberg lower type, (p,q)-φ relative Gol'dberg weak type and (p,q)-φ relative Gol'dberg upper weak type which are as follows.

Definition 11. [6] Let $\varphi(R) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables such that $0 < \rho_{g,D}^{(p,q)}(f,\varphi) < \infty$. Then the (p,q)-φ relative Gol'dberg type $\sigma_{g,D}^{(p,q)}(f,\varphi)$ and the (p,q)-φ relative Gol'dberg lower type $\overline{\sigma}_{g,D}^{(p,q)}(f,\varphi)$ of $f(z)$ with respect to $g(z)$ are defined as:

$$\begin{matrix} \Delta_{g,D}^{(p,q)}(f,\varphi) \\ \overline{\Delta}_{g,D}^{(p,q)}(f,\varphi) \end{matrix} = \lim_{R\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M_{g,D}^{-1}\left(M_{f,D}(R)\right)}{\left(\log^{[q-1]}\varphi(R)\right)^{\rho_{g,D}^{(p,q)}(f,\varphi)}}.$$

Definition 12. [6] Let $\varphi(R) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables such that $0 < \lambda_{g,D}^{(p,q)}(f,\varphi) < \infty$. Then the (p,q)-φ relative Gol'dberg weak type $\tau_{g,D}^{(p,q)}(f,\varphi)$ and (p,q)-φ relative Gol'dberg upper weak type $\overline{\tau}_{g,D}^{(p,q)}(f,\varphi)$ of $f(z)$ with respect to $g(z)$ are defined as:

$$\begin{matrix} \overline{\tau}_{g,D}^{(p,q)}(f,\varphi) \\ \tau_{g,D}^{(p,q)}(f,\varphi) \end{matrix} = \lim_{R\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M_{g,D}^{-1}\left(M_{f,D}(R)\right)}{\left(\log^{[q-1]}\varphi(R)\right)^{\lambda_{g,D}^{(p,q)}(f,\varphi)}}.$$

As earlier it was discussed that the relative (p,q)-th Gol'dberg type, relative (p,q)-th Gol'dberg lower type and all the other growth indicators mentioned in Definition 8 and Definition 9 depend on the particular choice of the domain D, naturally the growth indicators introduced in Definition 11 and Definition 12, in general, also depend upon the domain D.

During the past decades, several authors {cf. [1] to [13]} made closed investigations on the growth properties of entire functions of several complex variables using different growth indicator such as Gol'dberg order, (p,q)-th Gol'dberg order, relative Gol'dberg order, etc. In this paper we discussed some growth properties using the growth indicators such as (p,q)-φ relative Gol'dberg type, (p,q)-φ relative Gol'dberg weak type etc. of entire functions of several complex variables which significantly extend some earlier results.

2. Main Results

First of all, we recall one related known property which will be needed in order to prove our results, as we see in the following theorem.

THEOREM 1. *[5] Let $f(z)$ and $g(z)$ be any two entire functions of n complex variables. Also let $0 < \lambda^{(m,q)}(f,\varphi) \leq \rho^{(m,q)}(f,\varphi) < \infty$ and $0 < \lambda^{(m,p)}(g) \leq \rho^{(m,p)}(g) < \infty$. Then*

$$\begin{aligned}\frac{\lambda^{(m,q)}(f,\varphi)}{\rho^{(m,p)}(g)} &\leq \lambda_g^{(p,q)}(f,\varphi) \leq \min\left\{\frac{\lambda^{(m,q)}(f,\varphi)}{\lambda^{(m,p)}(g)}, \frac{\rho^{(m,q)}(f,\varphi)}{\rho^{(m,p)}(g)}\right\} \\ &\leq \max\left\{\frac{\lambda^{(m,q)}(f,\varphi)}{\lambda^{(m,p)}(g)}, \frac{\rho^{(m,q)}(f,\varphi)}{\rho^{(m,p)}(g)}\right\} \leq \rho_g^{(p,q)}(f,\varphi) \leq \frac{\rho^{(m,q)}(f,\varphi)}{\lambda^{(m,p)}(g)}.\end{aligned}$$

THEOREM 2. *Let $f(z)$ and $g(z)$ be any two entire functions of n complex variables. Also let $0 < \rho^{(m,q)}(f,\varphi) < \infty$ and $0 < \lambda^{(m,p)}(g) \leq \rho^{(m,p)}(g) < \infty$. Then*

$$\begin{aligned}&\max\left\{\left[\frac{\overline{\Delta}_D^{(m,q)}(f,\varphi)}{\tau_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}, \left[\frac{\Delta_D^{(m,q)}(f,\varphi)}{\overline{\tau}_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}\right\} \\ \leq\ & \Delta_{g,D}^{(p,q)}(f,\varphi) \leq \left[\frac{\Delta_D^{(m,q)}(f,\varphi)}{\overline{\Delta}_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}.\end{aligned}$$

Proof. From the definitions of $\Delta_D^{(m,q)}(f,\varphi)$ and $\overline{\Delta}_D^{(m,q)}(f,\varphi)$, we have for all sufficiently large values of R that

$$M_{f,D}(R) \leq \exp^{[m-1]}\left[\left(\Delta_D^{(m,q)}(f,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(R)\right]^{\rho^{(m,q)}(f,\varphi)}\right], \tag{1}$$

$$M_{f,D}(R) \geq \exp^{[m-1]}\left[\left(\overline{\Delta}_D^{(m,q)}(f,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(R)\right]^{\rho^{(m,q)}(f,\varphi)}\right] \tag{2}$$

and also for a sequence of values of R tending to infinity, we get that

$$(3)\quad M_{f,D}\left(R\right) \geq \exp^{[m-1]}\left[\left(\Delta_D^{(m,q)}(f,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi\left(R\right)\right]^{\rho^{(m,q)}(f,\varphi)}\right],$$

$$(4)\quad M_{f,D}\left(R\right) \leq \exp^{[m-1]}\left[\left(\overline{\Delta}_D^{(m,q)}(f,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi\left(R\right)\right]^{\rho^{(m,q)}(f,\varphi)}\right].$$

Similarly from the definitions of $\Delta_D^{(m,p)}(g)$ and $\overline{\Delta}_D^{(m,p)}(g)$, it follows for all sufficiently large values of R that

$$\begin{aligned} M_{g,D}\left(R\right) &\leq \exp^{[m-1]}\left[\left(\Delta_D^{(m,p)}(g)+\varepsilon\right)\left[\log^{[p-1]}R\right]^{\rho^{(m,p)}(g)}\right] \\ i.e.,\ R &\leq M_{g,D}^{-1}\left[\exp^{[m-1]}\left[\left(\Delta_D^{(m,p)}(g)+\varepsilon\right)\left[\log^{[p-1]}R\right]^{\rho^{(m,p)}(g)}\right]\right] \end{aligned}$$

$$(5)\qquad i.e.,\ M_{g,D}^{-1}\left(R\right) \geq \left[\exp^{[p-1]}\left(\frac{\log^{[m-1]}R}{\left(\Delta_D^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\rho^{(m,p)}(g)}}\right]$$

and

$$(6)\qquad M_{g,D}^{-1}\left(R\right) \leq \left[\exp^{[p-1]}\left(\frac{\log^{[m-1]}R}{\left(\overline{\Delta}_D^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho^{(m,p)}(g)}}\right].$$

Also for a sequence of values of R tending to infinity, we obtain that

$$(7)\qquad M_{g,D}^{-1}\left(R\right) \leq \left[\exp^{[p-1]}\left(\frac{\log^{[m-1]}R}{\left(\Delta_D^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho^{(m,p)}(g)}}\right]$$

and

$$(8)\qquad M_{g,D}^{-1}\left(R\right) \geq \left[\exp^{[p-1]}\left(\frac{\log^{[m-1]}R}{\left(\overline{\Delta}_D^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\rho^{(m,p)}(g)}}\right].$$

From the definitions of $\overline{\tau}_D^{(m,q)}(f,\varphi)$ and $\tau_D^{(m,q)}(f,\varphi)$, we have for all sufficiently large values of R that

$$(9)\quad M_{f,D}\left(R\right) \leq \exp^{[m-1]}\left[\left(\overline{\tau}_D^{(m,q)}(f,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi\left(R\right)\right]^{\lambda^{(m,q)}(f,\varphi)}\right],$$

$$(10)\quad M_{f,D}(R) \geq \exp^{[m-1]}\left[\left(\tau_D^{(m,q)}(f,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(R)\right]^{\lambda^{(m,q)}(f,\varphi)}\right]$$

and also for a sequence of values of R tending to infinity, we get that

(11)

$$M_{f,D}(R) \geq \exp^{[m-1]}\left[\left(\overline{\tau}_D^{(m,q)}(f,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(R)\right]^{\lambda^{(m,q)}(f,\varphi)}\right],$$

(12)

$$M_{f,D}(R) \leq \exp^{[m-1]}\left[\left(\tau_D^{(m,q)}(f,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(R)\right]^{\lambda^{(m,q)}(f,\varphi)}\right].$$

Similarly from the definitions of $\overline{\tau}_D^{(m,p)}(g)$ and $\tau_D^{(m,p)}(g)$, it follows for all sufficiently large values of R that

$$M_{g,D}(R) \leq \exp^{[m-1]}\left[\left(\overline{\tau}_D^{(m,p)}(g)+\varepsilon\right)\left[\log^{[p-1]}R\right]^{\lambda^{(m,p)}(g)}\right]$$

$$i.e.,\ R \leq M_{g,D}^{-1}\left[\exp^{[m-1]}\left[\left(\overline{\tau}_D^{(m,p)}(g)+\varepsilon\right)\left[\log^{[p-1]}R\right]^{\lambda^{(m,p)}(g)}\right]\right]$$

$$(13)\qquad i.e.,\ M_{g,D}^{-1}(R) \geq \left[\exp^{[p-1]}\left(\frac{\log^{[m-1]}R}{\left(\overline{\tau}_D^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda^{(m,p)}(g)}}\right]$$

and

$$(14)\qquad M_{g,D}^{-1}(R) \leq \left[\exp^{[p-1]}\left(\frac{\log^{[m-1]}R}{\left(\tau_D^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\lambda^{(m,p)}(g)}}\right].$$

Also for a sequence of values of R tending to infinity, we obtain that

$$(15)\qquad M_{g,D}^{-1}(R) \leq \left[\exp^{[p-1]}\left(\frac{\log^{[m-1]}R}{\left(\overline{\tau}_D^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\lambda^{(m,p)}(g)}}\right]$$

and

$$(16)\qquad M_{g,D}^{-1}(R) \geq \left[\exp^{[p-1]}\left(\frac{\log^{[m-1]}R}{\left(\tau_D^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda^{(m,p)}(g)}}\right].$$

Now from (3) and in view of (13), we get for a sequence of values of R tending to infinity that

$$M_{g,D}^{-1}\left(M_{f,D}\left(R\right)\right)\geq$$

$$M_{g,D}^{-1}\left[\exp^{[m-1]}\left[\left(\Delta_{D}^{(m,q)}(f,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi\left(R\right)\right]^{\rho^{(m,q)}(f,\varphi)}\right]\right]$$

$i.e.,$ $M_{g,D}^{-1}\left(M_{f,D}\left(R\right)\right)\geq$

$$\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left[\left(\Delta_{D}^{(m,q)}(f,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi\left(R\right)\right]^{\rho^{(m,q)}(f,\varphi)}\right]}{\left(\overline{\tau}_{D}^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda^{(m,p)}(g)}}$$

$i.e.,$ $\log^{[p-1]}M_{g,D}^{-1}\left(M_{f,D}\left(R\right)\right)\geq$

$$\left[\frac{\left(\Delta_{D}^{(m,q)}(f,\varphi)-\varepsilon\right)}{\left(\overline{\tau}_{D}^{(m,p)}(g)+\varepsilon\right)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}\cdot\left[\log^{[q-1]}\varphi\left(R\right)\right]^{\frac{\rho^{(m,q)}(f,\varphi)}{\lambda^{(m,p)}(g)}}.$$

Since in view of Theorem 1, $\frac{\rho^{(m,q)}(f,\varphi)}{\lambda^{(m,p)}(g)}\geq\rho_{g}^{(p,q)}\left(f,\varphi\right)$ and as $\varepsilon\left(>0\right)$ is arbitrary, therefore it follows from above that

$$\limsup_{R\to+\infty}\frac{\log^{[p-1]}M_{g,D}^{-1}\left(M_{f,D}\left(R\right)\right)}{\left[\log^{[q-1]}\varphi\left(R\right)\right]^{\rho_{g}^{(p,q)}(f,\varphi)}}\geq\left[\frac{\Delta_{D}^{(m,q)}(f,\varphi)}{\overline{\tau}_{D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}$$

$$i.e.,\ \Delta_{g,D}^{(p,q)}\left(f,\varphi\right)\geq\left[\frac{\Delta_{D}^{(m,q)}(f,\varphi)}{\overline{\tau}_{D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}. \tag{17}$$

Similarly from (2) and in view of (16), it follows for a sequence of values of R tending to infinity that

$$M_{g,D}^{-1}\left(M_{f,D}\left(R\right)\right)\geq$$

$$M_{g,D}^{-1}\left[\exp^{[m-1]}\left[\left(\overline{\Delta}_{D}^{(m,q)}(f,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi\left(R\right)\right]^{\rho^{(m,q)}(f,\varphi)}\right]\right]$$

$i.e.,$ $M_{g,D}^{-1}\left(M_{f,D}\left(R\right)\right)\geq$

$$\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left[\left(\overline{\Delta}_D^{(m,q)}(f,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi\left(R\right)\right]^{\rho^{(m,q)}(f,\varphi)}\right]}{\left(\tau_D^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda^{(m,p)}(g)}}$$

$i.e.,\ \log^{[p-1]} M_{g,D}^{-1}\left(M_{f,D}\left(R\right)\right) \geq$

$$\left[\frac{\left(\overline{\Delta}_D^{(m,q)}(f,\varphi)-\varepsilon\right)}{\left(\tau_D^{(m,p)}(g)+\varepsilon\right)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}} \cdot \left[\log^{[q-1]}\varphi\left(R\right)\right]^{\frac{\rho^{(m,q)}(f,\varphi)}{\lambda^{(m,p)}(g)}}.$$

Since in view of Theorem 1, it follows that $\frac{\rho^{(m,q)}(f,\varphi)}{\lambda^{(m,p)}(g)} \geq \rho_g^{(p,q)}\left(f,\varphi\right)$. Also $\varepsilon\left(>0\right)$ is arbitrary, so we get from above that

$$\limsup_{R\to+\infty}\frac{\log^{[p-1]} M_{g,D}^{-1}\left(M_{f,D}\left(R\right)\right)}{\left[\log^{[q-1]}\varphi\left(R\right)\right]^{\rho_g^{(p,q)}(f,\varphi)}} \geq \left[\frac{\overline{\Delta}_D^{(m,q)}(f,\varphi)}{\tau_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}$$

$$i.e.,\ \Delta_{g,D}^{(p,q)}\left(f,\varphi\right) \geq \left[\frac{\overline{\Delta}_D^{(m,q)}(f,\varphi)}{\tau_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}. \tag{18}$$

Again in view of (6), we have from (1) for all sufficiently large values of R that

$M_{g,D}^{-1}\left(M_{f,D}\left(R\right)\right) \leq$

$$M_{g,D}^{-1}\left[\exp^{[m-1]}\left[\left(\Delta_D^{(m,q)}(f,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi\left(R\right)\right]^{\rho^{(m,q)}(f,\varphi)}\right]\right]$$

$i.e.,\ M_{g,D}^{-1}\left(M_{f,D}\left(R\right)\right) \leq$

$$\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left[\left(\Delta_D^{(m,q)}(f,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi\left(R\right)\right]^{\rho^{(m,q)}(f,\varphi)}\right]}{\left(\overline{\Delta}_D^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho^{(m,p)}(g)}}$$

$$i.e.,\ \log^{[p-1]}\ M_{g,D}^{-1}\left(M_{f,D}\left(R\right)\right)\ \leq \tag{19}$$

$$\left[\frac{\left(\Delta_D^{(m,q)}(f,\varphi)+\varepsilon\right)}{\left(\overline{\Delta}_D^{(m,p)}(g)-\varepsilon\right)}\right]^{\frac{1}{\rho^{(m,p)}(g)}} \cdot \left[\log^{[q-1]}\varphi\left(R\right)\right]^{\frac{\rho^{(m,q)}(f,\varphi)}{\rho^{(m,p)}(g)}}.$$

As in view of Theorem 1, it follows that $\frac{\rho^{(m,q)}(f,\varphi)}{\rho^{(m,p)}(g)} \leq \rho_g^{(p,q)}(f,\varphi)$. Since $\varepsilon\,(>0)$ is arbitrary, we get from (19) that

$$\limsup_{R\to+\infty} \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{\left[\log^{[q-1]}\varphi(R)\right]^{\rho_g^{(p,q)}(f,\varphi)}} \leq \left[\frac{\Delta_D^{(m,q)}(f,\varphi)}{\overline{\Delta}_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}$$

$$\text{(20)} \qquad i.e.,\ \Delta_{g,D}^{(p,q)}(f,\varphi) \leq \left[\frac{\Delta_D^{(m,q)}(f,\varphi)}{\overline{\Delta}_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}.$$

Thus the theorem follows from (17), (18) and (20). □

The conclusion of the following corollary can be carried out from (6) and (9); (9) and (14) respectively after applying the same technique of Theorem 2 and with the help of Theorem 1. Therefore its proof is omitted.

Corollary 1. *Let $f(z)$ and $g(z)$ be any two entire functions of n complex variables. Also let $0 < \lambda^{(m,q)}(f,\varphi) < \infty$ and $0 < \lambda^{(m,p)}(g) \leq \rho^{(m,p)}(g) < \infty$. Then*

$$\Delta_{g,D}^{(p,q)}(f,\varphi) \leq \min\left\{\left[\frac{\overline{\tau}_D^{(m,q)}(f,\varphi)}{\tau_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}, \left[\frac{\overline{\tau}_D^{(m,q)}(f,\varphi)}{\overline{\Delta}_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}\right\}.$$

Similarly in the line of Theorem 2 and with the help of Theorem 1, one may easily carry out the following theorem from pairwise inequalities (10) and (13); (7) and (9); (6) and (12) respectively and therefore its proof is omitted:

Theorem 3. *Let $f(z)$ and $g(z)$ be any two entire functions of n complex variables. Also let $0 < \lambda^{(m,q)}(f,\varphi) \leq \rho^{(m,q)}(f,\varphi) < \infty$ and $0 < \lambda^{(m,p)}(g) \leq \rho^{(m,p)}(g) < \infty$. Then*

$$\begin{aligned}\left[\frac{\tau_D^{(m,q)}(f,\varphi)}{\overline{\tau}_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}} &\leq \tau_{g,D}^{(p,q)}(f,\varphi)\\ &\leq \min\left\{\left[\frac{\tau_D^{(m,q)}(f,\varphi)}{\overline{\Delta}_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}, \left[\frac{\overline{\tau}_D^{(m,q)}(f,\varphi)}{\Delta_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}\right\}.\end{aligned}$$

Corollary 2. *Let $f(z)$ and $g(z)$ be any two entire functions of n complex variables. Also let $0 < \rho^{(m,q)}(f, \varphi) < \infty$ and $0 < \lambda^{(m,p)}(g) \leq \rho^{(m,p)}(g) < \infty$. Then*

$$\tau_{g,D}^{(p,q)}(f, \varphi) \geq \max\left\{ \left[\frac{\overline{\Delta}_D^{(m,q)}(f, \varphi)}{\Delta_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}, \left[\frac{\overline{\Delta}_D^{(m,q)}(f, \varphi)}{\overline{\tau}_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}} \right\}.$$

With the help of Theorem 1, the conclusion of the above corollary can be carried out from (2) , (5) and (2) , (13) respectively after applying the same technique of Theorem 2 and therefore its proof is omitted.

Theorem 4. *Let $f(z)$ and $g(z)$ be any two entire functions of n complex variables. Also let $0 < \rho^{(m,q)}(f, \varphi) < \infty$ and $0 < \lambda^{(m,p)}(g) \leq \rho^{(m,p)}(g) < \infty$. Then*

$$\left[\frac{\overline{\Delta}_D^{(m,q)}(f, \varphi)}{\overline{\tau}_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}} \leq \overline{\Delta}_{g,D}^{(p,q)}(f, \varphi)$$

$$\leq \min\left\{ \left[\frac{\overline{\Delta}_D^{(m,q)}(f, \varphi)}{\overline{\Delta}_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}, \left[\frac{\Delta_D^{(m,q)}(f, \varphi)}{\Delta_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}} \right\}.$$

Proof. From (2) and in view of (13), we get for all sufficiently large values of R that

$$M_{g,D}^{-1}(M_{f,D}(R)) \geq$$
$$M_{g,D}^{-1}\left[\exp^{[m-1]}\left[\left(\overline{\Delta}_D^{(m,q)}(f, \varphi) - \varepsilon\right)\left[\log^{[q-1]}\varphi(R)\right]^{\rho^{(m,q)}(f,\varphi)}\right]\right]$$

$$i.e., \ M_{g,D}^{-1}(M_{f,D}(R)) \geq$$
$$\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left[\left(\overline{\Delta}_D^{(m,q)}(f, \varphi) - \varepsilon\right)\left[\log^{[q-1]}\varphi(R)\right]^{\rho^{(m,q)}(f,\varphi)}\right]}{\left(\overline{\tau}_D^{(m,p)}(g) + \varepsilon\right)}\right)^{\frac{1}{\lambda^{(m,p)}(g)}}$$

$$i.e., \ \log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R)) \geq$$
$$\left[\frac{\left(\overline{\Delta}_D^{(m,q)}(f, \varphi) - \varepsilon\right)}{\left(\overline{\tau}_D^{(m,p)}(g) + \varepsilon\right)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}} \cdot \left[\log^{[q-1]}\varphi(R)\right]^{\frac{\rho^{(m,q)}(f,\varphi)}{\lambda^{(m,p)}(g)}}.$$

Now in view of Theorem 1, it follows that $\frac{\rho^{(m,q)}(f,\varphi)}{\lambda^{(m,p)}(g)} \geq \rho_g^{(p,q)}(f,\varphi)$. Since $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\liminf_{R\to+\infty} \frac{\log^{[p-1]} M_{g,D}^{-1}\left(M_{f,D}(R)\right)}{\left[\log^{[q-1]}\varphi(R)\right]^{\rho_g^{(p,q)}(f,\varphi)}} \geq \left[\frac{\overline{\Delta}_D^{(m,q)}(f,\varphi)}{\overline{\tau}_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}$$

$$\text{(21)} \qquad i.e.,\ \overline{\Delta}_{g,D}^{(p,q)}(f,\varphi) \geq \left[\frac{\overline{\Delta}_D^{(m,q)}(f,\varphi)}{\overline{\tau}_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}.$$

Further in view of (7), we get from (1) for a sequence of values of R tending to infinity that

$$M_{g,D}^{-1}\left(M_{f,D}(R)\right) \leq$$

$$M_{g,D}^{-1}\left[\exp^{[m-1]}\left[\left(\Delta_D^{(m,q)}(f,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(R)\right]^{\rho^{(m,q)}(f,\varphi)}\right]\right]$$

$$i.e.,\ M_{g,D}^{-1}\left(M_{f,D}(R)\right) \leq$$

$$\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left[\left(\Delta_D^{(m,q)}(f,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(R)\right]^{\rho^{(m,q)}(f,\varphi)}\right]}{\left(\Delta_D^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho^{(m,p)}(g)}}$$

$$\text{(22)} \qquad i.e.,\ \log^{[p-1]} M_{g,D}^{-1}\left(M_{f,D}(R)\right) \leq$$

$$\left[\frac{\left(\Delta_D^{(m,q)}(f,\varphi)+\varepsilon\right)}{\left(\Delta_D^{(m,p)}(g)-\varepsilon\right)}\right]^{\frac{1}{\rho^{(m,p)}(g)}} \cdot \left[\log^{[q-1]}\varphi(R)\right]^{\frac{\rho^{(m,q)}(f,\varphi)}{\rho^{(m,p)}(g)}}.$$

Again as in view of Theorem 1, $\frac{\rho^{(m,q)}(f,\varphi)}{\rho^{(m,p)}(g)} \leq \rho_g^{(p,q)}(f,\varphi)$ and $\varepsilon\,(>0)$ is arbitrary, therefore we get from (22) that

$$\liminf_{R\to+\infty} \frac{\log^{[p-1]} M_{g,D}^{-1}\left(M_{f,D}(R)\right)}{\left[\log^{[q-1]}\varphi(R)\right]^{\rho_g^{(p,q)}(f,\varphi)}} \leq \left[\frac{\Delta_D^{(m,q)}(f,\varphi)}{\Delta_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}$$

$$\text{(23)} \qquad i.e.,\ \overline{\Delta}_{g,D}^{(p,q)}(f,\varphi) \leq \left[\frac{\Delta_D^{(m,q)}(f,\varphi)}{\Delta_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}.$$

Likewise from (4) and in view of (6), it follows for a sequence of values of R tending to infinity that

$$M_{g,D}^{-1}\left(M_{f,D}\left(R\right)\right) \leq$$

$$M_{g,D}^{-1}\left[\exp^{[m-1]}\left[\left(\overline{\Delta}_{D}^{(m,q)}(f,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi\left(R\right)\right]^{\rho^{(m,q)}(f,\varphi)}\right]\right]$$

$$i.e., \; M_{g,D}^{-1}\left(M_{f,D}\left(R\right)\right) \leq$$

$$\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left[\left(\overline{\Delta}_{D}^{(m,q)}(f,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi\left(R\right)\right]^{\rho^{(m,q)}(f,\varphi)}\right]}{\left(\overline{\Delta}_{D}^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho^{(m,p)}(g)}}$$

$$\text{(24)} \qquad i.e., \; \log^{[p-1]} M_{g,D}^{-1}\left(M_{f,D}\left(R\right)\right) \leq$$

$$\left[\frac{\left(\overline{\Delta}_{D}^{(m,q)}(f,\varphi)+\varepsilon\right)}{\left(\overline{\Delta}_{D}^{(m,p)}(g)-\varepsilon\right)}\right]^{\frac{1}{\rho^{(m,p)}(g)}} \cdot \left[\log^{[q-1]}\varphi\left(R\right)\right]^{\frac{\rho^{(m,q)}(f,\varphi)}{\rho^{(m,p)}(g)}}.$$

Analogously, we get from (24) that

$$\liminf_{R\to+\infty}\frac{\log^{[p-1]} M_{g,D}^{-1}\left(M_{f,D}\left(R\right)\right)}{\left[\log^{[q-1]}\varphi\left(R\right)\right]^{\rho_{g}^{(p,q)}(f,\varphi)}} \leq \left[\frac{\overline{\Delta}_{D}^{(m,q)}(f,\varphi)}{\overline{\Delta}_{D}^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}$$

$$\text{(25)} \qquad i.e., \; \overline{\Delta}_{g,D}^{(p,q)}\left(f,\varphi\right) \leq \left[\frac{\overline{\Delta}_{D}^{(m,q)}(f,\varphi)}{\overline{\Delta}_{D}^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}},$$

since in view of Theorem 1, $\frac{\rho^{(m,q)}(f,\varphi)}{\rho^{(m,p)}(g)} \leq \rho_{g}^{(p,q)}\left(f,\varphi\right)$ and $\varepsilon\,(>0)$ is arbitrary.

Thus the theorem follows from (21), (23) and (25). $\square$

Corollary 3. *Let $f\left(z\right)$ and $g\left(z\right)$ be any two entire functions of n complex variables. Also let $0<\lambda^{(m,q)}(f,\varphi)<\infty$ and $0<\lambda^{(m,p)}(g)\leq$*

$\rho^{(m,p)}(g) < \infty$. *Then*

$$\overline{\Delta}_{g,D}^{(p,q)}(f,\varphi) \leq \min\left\{ \left[\frac{\tau_D^{(m,q)}(f,\varphi)}{\tau_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}, \left[\frac{\overline{\tau}_D^{(m,q)}(f,\varphi)}{\overline{\tau}_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}, \right.$$
$$\left. \left[\frac{\overline{\tau}_D^{(m,q)}(f,\varphi)}{\sigma_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}, \left[\frac{\tau_D^{(m,q)}(f,\varphi)}{\overline{\sigma}_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}} \right\}.$$

The conclusion of the above corollary can be carried out from pairwise inequalities (6) and (12) ; (7) and (9) ; (12) and (14); (9) and (15) respectively after applying the same technique of Theorem 4 and with the help of Theorem 1. Therefore its proof is omitted.

Similarly in the line of Theorem 2 and with the help of Theorem 1, one may easily carry out the following theorem from pairwise inequalities (11) and (13) ; (10) and (16); (6) and (9) respectively and therefore its proof is omitted:

Theorem 5. *Let $f(z)$ and $g(z)$ be any two entire functions of n complex variables. Also let $0 < \lambda^{(m,q)}(f,\varphi) < \infty$ and $0 < \lambda^{(m,p)}(g) \leq \rho^{(m,p)}(g) < \infty$. Then*

$$\max\left\{ \left[\frac{\overline{\tau}_D^{(m,q)}(f,\varphi)}{\tau_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}, \left[\frac{\tau_D^{(m,q)}(f,\varphi)}{\tau_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}} \right\}$$
$$\leq \overline{\tau}_{g,D}^{(p,q)}(f,\varphi) \leq \left[\frac{\overline{\tau}_D^{(m,q)}(f,\varphi)}{\overline{\Delta}_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}.$$

Corollary 4. *Let $f(z)$ and $g(z)$ be any two entire functions of n complex variables. Also let $0 < \lambda^{(m,q)}(f,\varphi) \leq \rho^{(m,q)}(f,\varphi) < \infty$ and $0 < \lambda^{(m,p)}(g) \leq \rho^{(m,p)}(g) < \infty$. Then*

$$\overline{\tau}_{g,D}^{(p,q)}(f,\varphi) \geq \max\left\{ \left[\frac{\overline{\Delta}_D^{(m,q)}(f,\varphi)}{\overline{\Delta}_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}, \left[\frac{\Delta_D^{(m,q)}(f,\varphi)}{\Delta_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}, \right.$$
$$\left. \left[\frac{\Delta_D^{(m,q)}(f,\varphi)}{\overline{\tau}_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}, \left[\frac{\overline{\Delta}_D^{(m,q)}(f,\varphi)}{\tau_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}} \right\}.$$

The conclusion of the above corollary can be carried out from pairwise inequalities (3) and (5); (2) and (8); (3) and (13); (2) and (16) respectively after applying the same technique of Theorem 4 and with the help of Theorem 1. Therefore its proof is omitted.

Acknowledgement

The authors are extremely grateful and express their gratitude to the reviewer for his/her valuable suggestions and positive remarks towards the improvement of the paper.

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