Korean J. Math. **29** (2021), No. 3, pp. 467–472
http://dx.doi.org/10.11568/kjm.2021.29.3.467

ON SOME COPSON-TYPE INTEGRAL INEQUALITY

Bouharket Benaissa

Abstract. In this paper, we give some new Copson-type integral inequality with a sharp constant.

1. Introduction

In 1976, E.T. Copson proved the following inequalities [5, Theorem1, Theorem2] and [5, Theorem3, Theorem4].
Let f , ϕ non-negative measurable functions on $[0,\infty)$,

$$\Phi(x)=\int_0^x \phi(t)dt, \quad F(x)=\begin{cases} \displaystyle\int_0^x f(t)\phi(t)dt \quad for \quad c>1, \\ \displaystyle\int_x^\infty f(t)\phi(t)dt \quad for \quad c<1, \end{cases}$$

then for $k\geq 1$,

$$\int_0^b F^k(x)\Phi^{-c}(x)\phi(x)dx \leq \left(\frac{k}{|c-1|}\right)^k \int_0^b f^k(x)\Phi^{k-c}(x)\phi(x)dx, \tag{1}$$

for $0<k\leq 1$,

$$\int_0^b F^k(x)\Phi^{-c}(x)\phi(x)dx \geq \left(\frac{k}{|c-1|}\right)^k \int_0^b f^k(x)\Phi^{k-c}(x)\phi(x)dx. \tag{2}$$

Inequalities (1) and (2) can be easily rewritten in the following forms

$$F(x)=\begin{cases} \displaystyle\int_0^x f(t)\phi(t)dt \quad for \quad \alpha<k-1, \\ \displaystyle\int_x^\infty f(t)\phi(t)dt \quad for \quad \alpha>k-1, \end{cases}$$

Received April 8, 2020. Revised June 15, 2021. Accepted June 24, 2021.
2010 Mathematics Subject Classification: 26D10, 26D15.
Key words and phrases: Hölder's inequality, Copson-Type Integral Inequality, weight function.
This work was supported by DG-RSDT. Algeria.
© The Kangwon-Kyungki Mathematical Society, 2021.
This is an Open Access article distributed under the terms of the Creative commons Attribution Non-Commercial License (http://creativecommons.org/licenses/by-nc/3.0/) which permits unrestricted non-commercial use, distribution and reproduction in any medium, provided the original work is properly cited.

then for $k \geq 1$,

$$\int_0^b F^k(x)\Phi^{\alpha-k}(x)\phi(x)dx \leq \left(\frac{k}{|k-1-\alpha|}\right)^k \int_0^b f^k(x)\Phi^{\alpha}(x)\phi(x)dx, \tag{3}$$

for $0 < k \leq 1$,

$$\int_0^b F^k(x)\Phi^{\alpha-k}(x)\phi(x)dx \geq \left(\frac{k}{|k-1-\alpha|}\right)^k \int_0^b f^k(x)\Phi^{\alpha}(x)\phi(x)dx. \tag{4}$$

Copson's type inequalities have been studied by many authors during the twentieth century and has motivated some important lines of study which are currently active. Several papers have been appeared in the literature which deals with the simple proofs.

We give the reverse Hölder's inequality [1] which will be used frequently in the proof.

Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $p < 0$, we suppose that f, g are measurable on Ω. If $f \in L_p(\Omega)$ and $g \in L_{p'}(\Omega)$ (p' is the conjugate parameter), then

$$\int_\Omega |f.g|dt \geq \|f\|_{L_p}\|g\|_{L_{p'}}. \tag{5}$$

The objective of this paper is to obtain new types of the Copson-type integral inequality which will be useful in applications by using some elementary methods of analysis with negative parameter $k < 0$. We adopt the usual convention $\frac{0}{0} = 0$.

2. Main Results

Let $0 < b \leq +\infty$. Throughout the paper, we will assume that the functions are non-negative integrable and the integrals throughout are assumed to exist and are finite (i.e, convergent).

Lemma 2.1. *Let $k < 0$, $c < 1$ and $f, \phi > 0$ be non-negative measurable functions on $[0, +\infty)$. Let*

$$\Phi(x) = \int_0^x \phi(t)dt, \quad F(x) = \int_0^x f(t)\phi(t)dt,$$

the inequality

$$\int_0^b F^k(x)\Phi^{-c}(x)\phi(x)dx \leq \left(\frac{k}{c-1}\right)^k \int_0^b f^k(x)\Phi^{k-c}(x)\phi(x)dx \tag{6}$$

hold if the right-hand side is finite.

Proof. Let

$$V(x) = F^k(x)\Phi^{1-c}(x),$$

we have

$$\frac{dV}{dx}(x) = kF^{k-1}(x)f(x)\phi(x)\Phi^{1-c}(x) + (1-c)F^k(x)\Phi^{-c}(x)\phi(x),$$

then

$$(1-c)\int_0^b F^k(x)\Phi^{-c}(x)\,\phi(x)\,dx = -k\int_0^b F^{k-1}(x)f(x)\Phi^{1-c}(x)\phi(x)\,dx + F^k(b)\Phi^{1-c}(b)$$
$$\geq -k\int_0^b F^{k-1}(x)f(x)\Phi^{1-c}(x)\,\phi(x)\,dx.$$

Using the reverse Hölder inequality for $\dfrac{1}{k}+\dfrac{1}{k'}=1$, we deduce that

$$\begin{aligned}
\int_0^b F^k(x)\Phi^{-c}(x)\,\phi(x)\,dx &\geq \frac{k}{c-1}\int_0^b F^{k-1}(x)f\Phi^{1-c}(x)\,\phi(x)\,dx\\
&= \frac{k}{c-1}\int_0^b \left(f(x)\Phi^{\frac{k-c}{k}}(x)\phi^{\frac{1}{k}(x)}\right)\times\left(F^{k-1}(x)\Phi^{\frac{-c}{k'}}(x)\phi^{\frac{1}{k'}}(x)\right)dx\\
&\geq \frac{k}{c-1}\left(\int_0^b f^k(x)\Phi^{k-c}(x)\,\phi(x)\,dx\right)^{\frac{1}{k}}\\
&\quad\times\left(\int_0^b F^k(x)\Phi^{-c}(x)\,\phi(x)\,dx\right)^{\frac{1}{k'}},
\end{aligned}$$

so

$$\left(\int_0^b F^k(x)\Phi^{-c}(x)\,\phi(x)\,dx\right)^{\frac{1}{k}} \geq \frac{k}{c-1}\left(\int_0^b f^k(x)\Phi^{k-c}(x)\,\phi(x)\,dx\right)^{\frac{1}{k}}.$$

Thus for $k<0$, we get

$$\int_0^b F^k(x)\Phi^{-c}(x)\,\phi(x)\,dx \leq \left(\frac{k}{c-1}\right)^k\int_0^b f^k(x)\Phi^{k-c}(x)\,\phi(x)\,dx.$$

□

REMARK 2.2. The constant factor $\left(\frac{k}{c-1}\right)^k$ is the best possible.

Proof. Putting $\phi(x)=1$ and for $0<\theta<1-c$, we take

$$f_\theta(x) = \begin{cases} x^{\frac{c-1+\theta}{k}-1}, & x\in(0,1]\\ 0, & x\in(1,b)\end{cases},$$

using the assumption on f_θ in L_1 the left-hand side of (6), we have

$$\begin{aligned}
L_1 &= \int_0^b x^{-c}\left(\int_0^x f_\theta(t)dt\right)^k dx\\
&= \int_0^1 x^{-c}\left(\int_0^x t^{\frac{c-1+\theta}{k}-1}dt\right)^k dx\\
&= \left(\frac{k}{c-1+\theta}\right)^k\int_0^1 x^{\theta-1}dx\\
&= \left(\frac{k}{c-1+\theta}\right)^k\frac{1}{\theta}.
\end{aligned}$$

Let R_1 the right-hand side of (6), we get

$$\begin{aligned} R_1 &= \left(\frac{k}{c-1}\right)^k \int_0^b x^{-c}(xf_\theta(x))^k dx \\ &= \left(\frac{k}{c-1}\right)^k \int_0^1 x^{\theta-1} dx \\ &= \left(\frac{k}{c-1}\right)^k \frac{1}{\theta}. \end{aligned}$$

For $\theta \longrightarrow 0$, we get the constant factor $\left(\frac{k}{c-1}\right)^k$ is the best possible in (6). $\square$

The following Corollary give us some particular cases.

Corollary 2.3. *(i) For $c = k$, one has*

$$\int_0^b F^k(x)\Phi^{-k}(x)\phi(x)dx \le \left(\frac{k}{k-1}\right)^k \int_0^b f^k(x)\phi(x)dx. \tag{7}$$

(ii) For $c = 0$, one has

$$\int_0^b F^k(x)\phi(x)dx \le (-k)^k \int_0^b f^k(x)\Phi^k(x)\phi(x)dx. \tag{8}$$

Lemma 2.4. *Let $k < 0,\ c > 1$ and $f, \phi > 0$ be non-negative measurable functions on $[0, +\infty)$. Let*

$$\Phi(x) = \int_0^x \phi(t)dt, \quad F(x) = \int_x^\infty f(t)\phi(t)dt,$$

the inequality

$$\int_0^b F^k(x)\Phi^{-c}(x)\phi(x)dx \le \left(\frac{k}{1-c}\right)^k \int_0^b f^k(x)\Phi^{k-c}(x)\phi(x)dx \tag{9}$$

hold if the right-hand side is finite.

Proof. Let

$$U(x) = F^k(x)\Phi^{1-c}(x),$$

then

$$\frac{dU}{dx}(x) = -kF^{k-1}(x)f(x)\phi(x)\Phi^{1-c}(x) + (1-c)F^k(x)\Phi^{-c}(x)\phi(x) < 0,$$

we get

$$U(b) - U(0) < 0,$$

therefore

$$\begin{aligned} (1-c)\int_0^b F^k\Phi^{-c}\,\phi\,dx =& k\int_0^b F^{k-1}f\Phi^{1-c}\phi\,dx + U(b) - U(0) \\ &\le k\int_0^b F^{k-1}f\Phi^{1-c}\,\phi\,dx. \end{aligned}$$

Using the reverse Hölder inequality for $\dfrac{1}{k}+\dfrac{1}{k'}=1$, we deduce that

$$\begin{aligned}\int_0^b F^k\Phi^{-c}\,\phi\,dx &\geq \frac{k}{1-c}\int_0^b F^{k-1}f\Phi^{1-c}\,\phi\,dx\\ &\geq \frac{k}{1-c}\left(\int_0^b f^k\Phi^{k-c}\,\phi\,dx\right)^{\frac{1}{k}}\left(\int_0^b F^k\Phi^{-c}\,\phi\,dx\right)^{\frac{1}{k'}},\end{aligned}$$

then

$$\left(\int_0^b F^k\Phi^{-c}\,\phi\,dx\right)^{\frac{1}{k}} \geq \frac{k}{1-c}\left(\int_0^b f^k\Phi^{k-c}\,\phi\,dx\right)^{\frac{1}{k}},$$

hence for $k<0$, we get

$$\int_0^b F^k\Phi^{-c}\,\phi\,dx \leq \left(\frac{k}{1-c}\right)^k\int_0^b f^k\Phi^{k-c}\,\phi\,dx.$$

□

REMARK 2.5. The constant factor $\left(\frac{k}{1-c}\right)^k$ is the best possible.

Proof. Putting $\phi(x)=1$ and for $0<\theta<c-1$, we take

$$f_\theta(x)=\begin{cases} x^{\frac{c-1-\theta}{k}-1}, & x\in[1,b)\\ 0, & x\in(0,1)\end{cases}$$

using the assumption on f_θ in L_2 the left-hand side of (9), we get

$$\begin{aligned}L_2&=\int_0^b x^{-c}\left(\int_x^\infty f_\theta(t)dt\right)^k dx\\ &=\int_1^b x^{-c}\left(\int_x^\infty t^{\frac{c-1-\theta}{k}-1}dt\right)^k dx\\ &=\left(\frac{k}{1-c+\theta}\right)^k\int_1^b x^{-\theta-1}dx.\end{aligned}$$

Let R_2 the integral in right-hand side of (9), we have

$$\begin{aligned}R_2&=\int_0^b x^{-c}(xf_\theta(x))^k dx\\ &=\int_1^b x^{-\theta-1}dx,\end{aligned}$$

for $\theta\longrightarrow 0$, we get the constant factor $\left(\frac{p}{1-c}\right)^p$ is the best possible in (9). □

THEOREM 2.6. *Let $k<0$ and $f,\phi>0$ be non-negative measurable functions on $[0,+\infty)$. Let*

$$\Phi(x)=\int_0^x\phi(t)dt,\quad F(x)=\begin{cases}\displaystyle\int_0^x f(t)\phi(t)dt \quad for \quad c<1,\\[2ex] \displaystyle\int_x^\infty f(t)\phi(t)dt \quad for \quad c>1,\end{cases}$$

then the inequality

$$\int_0^b F^k(x)\Phi^{-c}(x)\phi(x)dx \le \left(\frac{k}{|c-1|}\right)^k \int_0^b f^k(x)\Phi^{k-c}(x)\phi(x)dx \tag{10}$$

hold if the right-hand side is finite.

Proof. By the inequalities (6) and (9), we have the inequality (10). □

The Theorem (2.6)can be easily rewritten in the following form.

THEOREM 2.7. *Let $k < 0$ and $f, \phi > 0$ be non-negative measurable functions on $[0, +\infty)$. Let*

$$\Phi(x) = \int_0^x \phi(t)dt, \quad F(x) = \begin{cases} \displaystyle\int_0^x f(t)\phi(t)dt & for \quad \alpha > k-1, \\ \displaystyle\int_x^\infty f(t)\phi(t)dt & for \quad \alpha < k-1, \end{cases}$$

then the inequality

$$\int_0^b F^k(x)\Phi^{\alpha-k}(x)\phi(x)dx \le \left(\frac{k}{|k-1-\alpha|}\right)^k \int_0^b f^k(x)\Phi^{\alpha}(x)\phi(x)dx \tag{11}$$

hold if the right-hand side is finite.

References

[1] B. Benaissa, *On the Reverse Minkowski's Integral Inequality*, Kragujevac. J. Math. Vol.**46** (3) (2022), 407–416.

[2] B. Benaissa, H. Budak, *More on reverse of Holder's integral inequality*, Korean J. Math. **28** (1) (2020), 9-15. doi:10.11568/kjm.2020.28.1.9

[3] E.T. Copson, *Note on series of positive terms*, Journal London Math. Soc. **2** (1927), 9–12

[4] E.T. Copson, *Note on series of positive terms*, Journal London Math. Soc. **3** (1928), 49–51.

[5] E.T. Copson, *Some Integral Inequalities*, Proceedings of the Royal Society of Edinburgh,**75A, 13**, (1975/76)

[6] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, (1952).

[7] G. H. Hardy, *Notes on a theorem of Hilbert*, Math. Z, **6** (1920), 314–317.

[8] K. Khatri, V.N. Mishra, *Degree of Approximation by the $(T.E^1)$ Means of Conjugate Series of Fourier Series in the Hölder Metric*, Iran. J. Sci. Technol. Trans. A Sci. **43** (4) (2019), 1591–1599. DOI:10.1007/s40995-017-0272-3.

[9] V.N. Mishra, K. Khatri, *Degree of approximation of functions $\widetilde{f} \in H_w$ class by the (Np.E1) means in the Hölder metric*, Int. J. Math. Math. Sci, (2014), ID 837408, 9 pages

[10] V.N. Mishra, L.N. Mishra, *Trigonometric Approximation of Signals (Functions) in L_p- norm*, Int. J. Contemp. Math. Sci. **7** (19) (2012), 909–918.

Bouharket Benaissa
Faculty of Material Sciences, Laboratory of Informatics and Mathematics,
University of Tiaret, Algeria
E-mail: bouharket.benaissa@univ-tiaret.dz