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SOLUTION AND STABILITY OF AN n-VARIABLE ADDITIVE FUNCTIONAL EQUATION

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Abstract. In this paper, we investigate the general solution and the Hyers-Ulam stability of n-variable additive functional equation of the form

$$\Im\left(\sum_{i=1}^{n}(-1)^{i+1}x_i\right)=\sum_{i=1}^{n}(-1)^{i+1}\Im(x_i),$$

where n is a positive integer with $n \geq 2$, in Banach spaces by using the direct method.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [16] concerning the stability of group homeomorphisms. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [2] for additive mappings and by Rassias [13] for linear mappings by considering an unbounded Cauchy difference(see [1, 4, 6, 10, 12, 14, 15]). A generalization of the Rassias theorem was obtained by Gavruta by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach (see [3, 5, 7–9, 11]).

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The Cauchy additive functional equation is of the form

$$\Im(x+y) = \Im(x) + \Im(y). \tag{1.1}$$

In this section, we introduce and investigate the general solution and the Hyers-Ulam stability of the additive functional equation of the form

$$\Im\left(\sum_{i=1}^{n}(-1)^{i+1}x_i\right) = \sum_{i=1}^{n}(-1)^{i+1}\Im(x_i), \tag{1.2}$$

where n is a positive integer with $n \geq 2$, in Banach spaces by using the direct method. Here after, throughout this paper, let us consider X and Y to be a normed space and a Banach space, respectively. Assume that n is a positive integer with $n \geq 2$. For convience,

$$D\Im(x_1, x_2, \cdots, x_n) := \Im\left(\sum_{i=1}^{n}(-1)^{i+1}x_i\right) - \sum_{i=1}^{n}(-1)^{i+1}\Im(x_i)$$

for all $x_1, x_2, \cdots, x_n$.

2. Solution of the additive functional equation (1.2)

In this section, we investigate a general solution of the additive functional equation (1.2).

Lemma 2.1. *If a mapping $\Im : X \to Y$ satisfies the functional equation (1.1) if and only if $\Im : X \to Y$ satisfies the functional equation (1.2) under the assumption that if n is odd then $\Im(0) = 0$.*

Proof. Setting $(x, y) = (0, 0)$ in (1.1), we get $\Im(0) = 0$. Replacing (x, y) by $(x, -x)$ in (1.1), we have $\Im(-x) = -\Im(x)$ for all $x \in X$. So

$$\Im(x-y) = \Im(x) + \Im(-y) = \Im(x) - \Im(y) \tag{2.3}$$

for all $x, y \in X$. It follows from (1.1) and (2.3) that (1.2) holds for $n \geq 2$.

Assume that n is even. Letting $x_1 = x_2 = \cdots = x_n = 0$ in (1.2), we get $\Im(0) = 0$. Letting $x_1 = x_3 = x_4 = \cdots = x_n = 0$ in (1.2), we get $\Im(-x_2) = -\Im(x_2)$ for all $x_2 \in X$. Letting $x_3 = x_4 = \cdots = x_n = 0$ in (1.2), we get

$$\Im(x_1 - x_2) = \Im(x_1) - \Im(x_2) = \Im(x_1) + \Im(-x_2) \tag{2.4}$$

for all $x_1, x_2 \in X$. Replacing (x_1, x_2) by $(x, -y)$ in (2.4), we get

$$\Im(x+y) = \Im(x) + \Im(y)$$

for all $x, y \in X$.

Assume that n is odd. Letting $x_1 = x_3 = x_4 = \cdots = x_n = 0$ in (1.2), we get $\Im(-x_2) = -\Im(x_2)$ for all $x_2 \in X$. So

$$\Im(x_1 - x_2) = \Im(x_1) - \Im(x_2) = \Im(x_1) + \Im(-x_2) \tag{2.5}$$

for all $x_1, x_2 \in X$. Replacing (x_1, x_2) by $(x, -y)$ in (2.5), we get

$$\Im(x + y) = \Im(x) + \Im(y)$$

for all $x, y \in X$. □

3. Stability results for even positive integers in (1.2)

In this section, we present the Hyers-Ulam stability of the functional equation (1.2) for even positive integers n. Assume that n is even.

THEOREM 3.1. *Let $\theta : X^n \to [0, \infty)$ be a function such that*

$$\Phi(x_1, \cdots, x_n) := \sum_{k=0}^{\infty} \frac{\theta\left(n^k x_1, n^k x_2, \cdots, n^k x_n\right)}{n^k} < \infty \tag{3.6}$$

for all $x_1, x_2, \cdots, x_n \in X$. Let $\Im : X \to Y$ be a mapping satisfying the inequality

$$\left\| \Im\left(\sum_{i=1}^{n} (-1)^{i+1} x_i\right) - \sum_{i=1}^{n} (-1)^{i+1} \Im(x_i) \right\| \leq \theta(x_1, x_2, \ldots, x_n) \tag{3.7}$$

for all $x_1, x_2, \ldots, x_n \in X$. There exists a unique additive mapping $A : X \to Y$ which satisfies

$$\|\Im(x) - A(x)\| \leq \frac{1}{n} \Phi(x, -x, x, -x, \cdots, \cdots, x, -x) \tag{3.8}$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{k \to \infty} \frac{\Im\left(n^k x\right)}{n^k}$$

for all $x \in X$.

Proof. Letting $(x_1, x_2, \ldots, x_{n-1}, x_n) = (x, -x, x, -x, \cdots, x, -x)$ in (3.7), we have

$$\|\Im(nx) - n\Im(x)\| \leq \theta(x, -x, x, -x, \ldots, x, -x) \tag{3.9}$$

for all $x \in X$. It follows from (3.9) that

$$\left\| \frac{\Im(nx)}{n} - \Im(x) \right\| \leq \frac{1}{n}\theta(x, -x, x, -x, \cdots, x, -x) \tag{3.10}$$

for all $x \in X$. Replacing x by $n^{l-1}x$ in (3.10) and dividing by n^{l-1}, we obtain

$$\begin{aligned} &\left\| \frac{\Im(n^l x)}{n^l} - \frac{\Im(n^{l-1}x)}{n^{l-1}} \right\| \\ &\leq \frac{1}{n^l}\theta(n^{l-1}x, -n^{l-1}x, n^{l-1}x, -n^{l-1}x, \cdots, n^{l-1}x, -n^{l-1}x) \end{aligned} \tag{3.11}$$

for all $x \in X$. It follows from (3.11) and the triangle inequality that

$$\left\| \frac{\Im(n^k x)}{n^k} - \Im(x) \right\| \leq \frac{1}{n} \sum_{l=0}^{k-1} \frac{1}{n^l} \theta\left(n^l x, -n^l x, n^l x, -n^l x, \cdots, n^l x, -n^l x\right) \tag{3.12}$$

for all $x \in X$.

Replacing x by $n^m x$ and dividing n^m in (3.12), we obtain that

$$\begin{aligned} &\left\| \frac{\Im(n^{k+m}x)}{n^{k+m}} - \frac{\Im(n^m x)}{n^m} \right\| \\ &\leq \frac{1}{n} \sum_{l=0}^{k-1} \frac{1}{n^{l+m}} \theta\left(n^{l+m}x, -n^{l+m}x, n^{l+m}x, -n^{l+m}x, \cdots, n^{l+m}x, -n^{l+m}x\right) \end{aligned}$$

for all $x \in X$. Hence the sequence $\left\{ \frac{\Im(n^m x)}{n^m} \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A : X \to Y$ defined by $A(x) = \lim_{m\to\infty} \frac{\Im(n^m x)}{n^m}$ for all $x \in X$. Letting $k \to \infty$ in (3.12), we get that (3.8) holds for $x \in X$.

To prove that A satisfies (1.2), replacing $(x_1, x_2, \ldots, x_n)$ by $\underbrace{(x, x, \quad 0, \ldots, 0)}_{n-2-\text{times} \quad n-2-\text{times}}$ and dividing $(n-2)^n$ in (3.7), we obtain

$$\frac{1}{n^k} \left\| D\Im\left(n^k x_1, n^k x_2, \cdots, n^k x_n\right) \right\| \leq \frac{1}{n^k}\theta\left(n^k x_1, n^k x_2, \cdots, n^k x_n\right)$$

for all $x_1, x_2, \cdots, x_n \in X$. Letting $k \to \infty$ in the above inequality and using the definition of $A(x)$, we obtain that $DA(x_1, x_2, \cdots, x_n) = 0$. By Lemma 2.1, A is additive.

To show that A is unique, let $B(x)$ be another additive mapping satisfying (1.2) and (3.8). Then

$$\begin{aligned}
&\|A(x)-B(x)\| = \frac{1}{n^k}\left\|A(n^kx)-B(n^kx)\right\| \\
&= \frac{1}{n^k}\left\|A(n^kx)-\Im(n^kx)+\Im(n^kx)-B(n^kx)\right\| \\
&\le \frac{1}{n^k}\left\|A(n^kx)-\Im(n^kx)\right\| + \frac{1}{n^k}\left\|\Im(n^kx)-B(n^kx)\right\| \\
&\le \frac{2}{n^{k+1}}\Phi(n^kx,-n^kx,n^kx,-n^kx,\cdots,n^kx,-n^kx) \\
&\to 0 \quad \text{as} \quad n\to\infty
\end{aligned}$$

for all $x \in X$. Hence A is unique. □

THEOREM 3.2. *Let $\theta : X^n \to [0,\infty)$ be a function such that*

$$\Psi(x_1,\cdots,x_n) := \sum_{k=1}^{\infty} n^k\theta\left(\frac{x_1}{n^k},\frac{x_2}{n^k},\cdots,\frac{x_n}{n^k}\right) < \infty$$

for all $x_1,x_2,\cdots,x_n \in X$. Let $\Im : X \to Y$ be a mapping satisfying (3.7). There exists a unique additive mapping $A : X \to Y$ which satisfies

$$\|\Im(x)-A(x)\| \le \frac{1}{n}\Psi(x,-x,x,-x,\cdots,\cdots,x,-x)$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{k\to\infty} n^k\Im\left(\frac{x}{n^k}\right)$$

for all $x \in X$.

Proof. It follows from (3.9) that

$$\left\|\Im(x)-n\Im\left(\frac{x}{n}\right)\right\| \le \theta\left(\frac{x}{n},-\frac{x}{n},\cdots,\frac{x}{n},-\frac{x}{n}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.1. □

The following corollary is an immediate consequence of Theorems 3.1 and 3.2.

Corollary 3.3. *Let λ and γ be positive real numbers with $\gamma \neq 1$. Let $\Im : X \to Y$ be a mapping satisfying the inequality*

$$\|D\Im(x_1, x_2, \cdots, x_n)\| \leq \lambda \sum_{i=1}^{n} \|x_i\|^\gamma \tag{3.13}$$

for all $x_1, x_2, \cdots, x_n \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|\Im(x) - A(x)\| \leq \frac{n\lambda\|x\|^\gamma}{|n - n^\gamma|}$$

for all $x \in X$.

4. Stability results for odd positive integers in (1.2)

In this section, we obtain the Hyers-Hyers stability of the functional equation (1.2) for odd positive integers. Assume that n is odd.

Theorem 4.1. *Let $\theta : X^n \to [0, \infty)$ be a function such that*

$$\Phi(x_1, \cdots, x_n) := \sum_{k=0}^{\infty} \frac{\theta\left((n-1)^k x_1, (n-1)^k x_2, \cdots, (n-1)^k x_n\right)}{(n-1)^k} < \infty$$

for all $x_1, x_2, \cdots, x_n \in X$. Let $\Im : X^n \to Y$ be an odd mapping satisfying (3.7). There exists a unique additive mapping $A : X \to Y$ which satisfies

$$\|\Im(x) - A(x)\| \leq \frac{1}{n-1}\Phi(x, -x, x, -x, \cdots, x, -x, 0) \tag{4.14}$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{k\to\infty} \frac{\left((n-1)^k x\right)}{(n-1)^k}$$

for all $x \in X$.

Proof. Since $\Im$ is odd, $\Im(0) = 0$.

Letting $(x_1, x_2, \ldots, x_{n-1}, x_n) = (x, -x, x, -x, \cdots, x, -x, 0)$ in (3.7), we have

$$\|\Im((n-1)x) - (n-1)\Im(x)\| \leq \theta(x, -x, x, -x, \ldots, x, -x, 0) \tag{4.15}$$

for all $x \in X$. It follows from (4.15) that

$$\left\|\frac{\Im((n-1)x)}{n-1} - \Im(x)\right\| \leq \frac{1}{n-1}\theta(x, -x, x, -x, \cdots, x, -x, 0) \tag{4.16}$$

for all $x \in X$. Replacing x by $(n-1)^{l-1}x$ in (4.16) and dividing by $(n-1)^{l-1}$, we obtain

$$\left\| \frac{\Im((n-1)^l x)}{(n-1)^l} - \frac{\Im((n-1)^{l-1} x)}{(n-1)^{l-1}} \right\| \tag{4.17}$$
$$\leq \frac{1}{(n-1)^l}\theta((n-1)^{l-1}x, -(n-1)^{l-1}x, \cdots, (n-1)^{l-1}x, -(n-1)^{l-1}x, 0)$$

for all $x \in X$. It follows from (4.17) and the triangle inequality that

$$\left\| \frac{\Im((n-1)^k x)}{(n-1)^k} - \Im(x) \right\| \tag{4.18}$$
$$\leq \frac{1}{n-1}\sum_{l=0}^{k-1} \frac{1}{(n-1)^l}\theta\left((n-1)^l x, -(n-1)^l x, \cdots, (n-1)^l x, -(n-1)^l x, 0\right)$$

for all $x \in X$.

Replacing x by $(n-1)^m x$ and dividing $(n-1)^m$ in (4.18), we obtain that

$$\left\| \frac{\Im((n-1)^{k+m} x)}{(n-1)^{k+m}} - \frac{\Im((n-1)^m x)}{(n-1)^m} \right\|$$
$$\leq \frac{1}{n-1}\sum_{l=0}^{k-1} \frac{1}{(n-1)^{l+m}}\theta$$
$$\times \left((n-1)^{l+m} x, -(n-1)^{l+m} x, \cdots, (n-1)^{l+m} x, -(n-1)^{l+m} x, 0\right)$$

for all $x \in X$. Hence the sequence $\left\{ \frac{\Im((n-1)^m x)}{(n-1)^m} \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A : X \to Y$ defined by $A(x) = \lim_{m\to\infty} \frac{\Im((n-1)^m x)}{(n-1)^m}$ for all $x \in X$. Letting $k \to \infty$ in (4.18), we get that (4.14) holds for $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.1. □

Theorem 4.2. *Let $\theta : X^n \to [0, \infty)$ be a function such that*

$$\Psi(x_1, \cdots, x_n) := \sum_{k=1}^{\infty} (n-1)^k \theta\left(\frac{x_1}{n^k}, \frac{x_2}{(n-1)^k}, \cdots, \frac{x_n}{(n-1)^k} \right) < \infty$$

for all $x_1, x_2, \cdots, x_n \in X$. Let $\Im : X \to Y$ be an odd mapping satisfying (3.7). There exists a unique additive mapping $A : X \to Y$ which satisfies

$$\|\Im(x) - A(x)\| \leq \frac{1}{n-1}\Psi(x, -x, x, -x, \cdots, \cdots, x, -x, 0)$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{k\to\infty} (n-1)^k \Im\left(\frac{x}{(n-1)^k}\right)$$

for all $x \in X$.

Proof. It follows from (4.15) that

$$\left\|\Im(x) - (n-1)\Im\left(\frac{x}{n-1}\right)\right\| \le \theta\left(\frac{x}{n-1}, -\frac{x}{n-1}, \cdots, \frac{x}{n-1}, -\frac{x}{n-1}, 0\right)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 3.1 and 4.1. □

The following corollary is an immediate consequence of Theorems 4.1 and 4.2.

COROLLARY 4.3. *Let λ and γ be positive real numbers with $\gamma \neq 1$. Let $\Im : X \to Y$ be an odd mapping satisfying* (3.13) *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|\Im(x) - A(x)\| \le \frac{(n-1)\lambda\|x\|^\gamma}{|(n-1) - (n-1)^\gamma|}$$

for all $x \in X$.

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