Korean J. Math. **29** (2021), No. 1, pp. 1–12
http://dx.doi.org/10.11568/kjm.2021.29.1.1

THE FORCING NONSPLIT DOMINATION NUMBER OF A GRAPH

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Abstract. A dominating set S of a graph G is said to be nonsplit dominating set if the subgraph $\langle V - S\rangle$ is connected. The minimum cardinality of a nonsplit dominating set is called the nonsplit domination number and is denoted by $\gamma_{ns}(G)$. For a minimum nonsplit dominating set S of G, a set $T \subseteq S$ is called a forcing subset for S if S is the unique γ_{ns}-set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing nonsplit domination number of S, denoted by $f_{\gamma ns}(S)$, is the cardinality of a minimum forcing subset of S. The forcing nonsplit domination number of G, denoted by $f_{\gamma ns}(G)$ is defined by $f_{\gamma ns}(G) = min\{f_{\gamma ns}(S)\}$, where the minimum is taken over all γ_{ns}-sets S in G. The forcing nonsplit domination number of certain standard graphs are determined. It is shown that, for every pair of positive integers a and b with $0 \leq a \leq b$ and $b \geq 1$, there exists a connected graph G such that $f_{\gamma ns}(G) = a$ and $\gamma_{ns}(G) = b$. It is shown that, for every integer $a \geq 0$, there exists a connected graph G with $f_{\gamma}(G) = f_{\gamma ns}(G) = a$, where $f_{\gamma}(G)$ is the forcing domination number of the graph. Also, it is shown that, for every pair a, b of integers with $a \geq 0$ and $b \geq 0$ there exists a connected graph G such that $f_{\gamma}(G) = a$ and $f_{\gamma ns}(G) = b$.

1. Introduction

By a graph $G = (V, E)$, we mean finite, undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For graph theoretic terminology we refer to [8, 9]. Two vertices u and v are said to be adjacent if uv is an edge of G. If $uv \in E(G)$, we say that u is a neighbor of v and denote by $N(v)$, the set of neighbors of v. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. The minimum and maximum degrees of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A vertex of degree 1 is called an *end vertex.* A vertex v is an *universal vertex* of a graph G, if it is a full degree vertex of G. The *distance* $d(u, v)$ between u and v in a connected graph G is the length of a shortest u-v path in G. A u-v path of length $d(u, v)$ is called a *u-v geodesic.* The *diameter* of a graph G is the maximum distance between the pair of vertices of G. For any set S of vertices of G, the induced subgraph $\langle S\rangle$ is the maximal subgraph of G with vertex set S. A vertex v of a graph G is a *simplicial vertex* if $\langle N(v)\rangle$ is complete. The *join* $G + H$

Received April 25, 2020. Accepted December 28, 2020. Published online March 30, 2021.
2010 Mathematics Subject Classification: 05C69.
Key words and phrases: dominating set, domination number, forcing domination number, nonsplit domination number, forcing nonsplit domination number.

of graphs G and H is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G)$ and $v \in V(H)\}$. F_n is the graph obtained from K_1 and P_{n-1}, where $F_n = K_1 + P_{n-1}$.

A set $S \subset V$ of a graph G is a *dominating set* if for every vertex $v \in V - S$, there exists a vertex $u \in S$ such that v is adjacent to u. The minimum cardinality of a dominating set is the *domination number* and is denoted by $\gamma(G)$. This was studied in [9]. A minimum dominating set of a graph G is hence often called as a γ*-set* of G. A vertex v of a connected graph G is said to be a *dominating vertex* of G if v belongs to every γ-set of G. Let S be a γ-set of G. A set $T \subseteq S$ is called a *forcing subset* for S if S is the unique γ-set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The *forcing domination number* of S, denoted by $f_\gamma(S)$, is the cardinality of a minimum forcing subset of S. The forcing domination number of G, denoted by $f_\gamma(G)$, is $f_\gamma(G) = min\{f_\gamma(S)\}$, where the minimum is taken over all γ-sets in G. The forcing concept in domination was first introduced and studied in [4] and further studied in [1–3,5–7,12]. The forcing concept for various parameters were further studied in [10,11,15–17]. The forcing sets in a graph is a very interesting concept. In the management of an institution, the executive committee consists of senior members who have adequate rapport with other members of the institution. Some members of the executive committee may sit in other important committees also. Some times, restrictions are imposed on members that they can be part of exactly one committee. This precisely leads to the concept of the forcing dominating set. A dominating set S of a graph G is said to be a *nonsplit dominating set* if the subgraph $\langle V - S\rangle$ is connected. The minimum cardinality of a nonsplit dominating set is called the *nonsplit domination number* and is denoted by $\gamma_{ns}(G)$. A minimum nonsplit dominating set of a graph G is often called as a γ_{ns}*-set* of G. This concept was introduced in [13] and further studied in [14, 18]. The nonsplit domination number is also known as *complementary connected domination number*. A communication network can be represented by a connected graph G, where the vertices of G represent processors and edges represent bi-directional communication channels. A dominating set in a graph can be interpreted as a set of processors from which information can be passed on to all the other processors. Hence determination of non split domination parameter of a graph is an important problem. Nonsplit domination is very effective in modeling problems in social network analysis. It can be used to analyze the social relations among individuals and to select representatives of a group subject to some constrains. Members of a group usually have different opinions and they divide among themselves based on their opinion. Good relations among the rest of the members can be represented by the presence of an edge between them. The constraints imposed on the individuals so that they can be representatives of exactly one group lead to the concept of forcing nonsplit dominating set.

The following theorem is used in the sequel.

THEOREM 1.1. [9] *Let G be a connected graph and W be the set of all dominating vertices of G. Then $f_\gamma(G) \leq \gamma(G) - |W|$.*

Throughout the following G denotes a connected graph with at least two vertices.

2. The forcing nonsplit domination number of a graph

Definition 2.1. Let G be a connected graph and S a γ_{ns}-set of G. A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique γ_{ns}-set containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset* of S. The *forcing nonsplit domination number* of S, denoted by $f_{\gamma ns}(S)$, is the cardinality of a minimum forcing subset of S. The *forcing nonsplit domination number* of G, denoted by $f_{\gamma ns}(G)$ is defined by $f_{\gamma ns}(G) = min\{f_{\gamma ns}(S)\}$, where the minimum is taken over all γ_{ns}-sets S in G.

Example 2.2. For the graph G given in Figure 2.1, $S_1 = \{v_1, v_5\}$, $S_2 = \{v_2, v_5\}$ and $S_3 = \{v_3, v_5\}$ are the only three γ_{ns}-sets of G such that $f_{\gamma ns}(S_1) = f_{\gamma ns}(S_2) = f_{\gamma ns}(S_3) = 1$ so that $f_{\gamma ns}(G) = 1$.

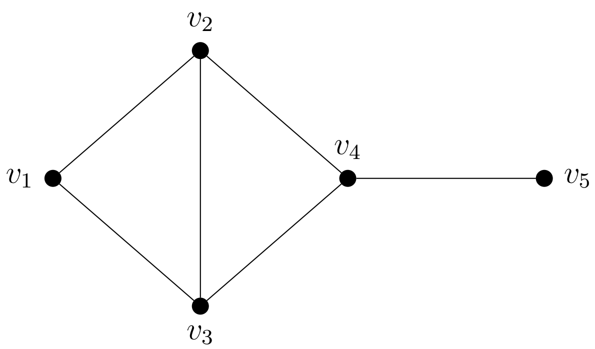

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Figure 2.1

The next theorem follows immediately from the definition of the nonsplit domination number and the forcing nonsplit domination number of a connected graph G.

Theorem 2.3. *For every connected graph G, $0 \leq f_{\gamma ns}(G) \leq \gamma_{ns}(G)$.*

Theorem 2.4. *Let G be a connected graph with diameter at least 3. Then $\gamma_{ns}(G) \leq n - 2$.*

Proof. Let $P : x_0, x_1, x_2, ..., x_p$ be a diametral path in G. Since diameter of G is at least 3, P contains at least two internal vertices. Let $S = V - \{x_1, x_2\}$. Then S is a γ_{ns}-set of G so that $\gamma_{ns}(G) \leq n - 2$. □

Remark 2.5. The bound in Theorem 2.4 is sharp. For the graph $G = P_n$ $(n \geq 4)$, $\gamma_{ns}(G) = n - 2$.

Theorem 2.6. *Let G be a connected graph with diameter at least 3. Then each end vertex of G belongs to every γ_{ns}-set of G.*

Proof. Since diameter of G is at least 3, by Theorem 2.4, $\gamma_{ns}(G) \leq n - 2$. Let S be a γ_{ns}-set of G. Let v be a cut vertex of G and vx be an end edge of G. We prove that $x \in S$. Suppose that $x \notin S$. Then it follows that $v \in S$. This implies that S contains all the vertices of G except v. Therefore $\gamma_{ns}(G) \geq n - 1$, which is a contradiction. Hence $x \in S$. □

REMARK 2.7. Theorem 2.6 need not be true if diameter is at most 2. For the star $G = K_{1,n-1}$ with $V(G) = \{v, v_1, v_2, ..., v_{n-1}\}$, there exists a γ_{ns}-set S such that $v_{n-1} \notin S$.

DEFINITION 2.8. A vertex v of a connected graph G is a *nonsplit dominating vertex* of G if v belongs to every nonsplit dominating set of G. If G has a unique nonsplit dominating set S, then every vertex of S is a nonsplit dominating vertex of G.

The proof of the following theorems are straightforward, so we omit the proofs.

THEOREM 2.9. *Let G be a connected graph.*

(a) $f_{\gamma ns}(G) = 0$ if and only if G has a unique minimum nonsplit dominating set.
(b) $f_{\gamma ns}(G) = 1$ if and only if G has at least two minimum nonsplit dominating sets, one of which is a unique minimum nonsplit dominating set containing one of its elements, and
(c) $f_{\gamma ns}(G) = \gamma_{ns}(G)$ if and only if no minimum nonsplit dominating set of G is the unique minimum nonsplit dominating set containing any of its proper subsets.

THEOREM 2.10. *Let G be a connected graph and W be the set of all nonsplit dominating vertices of G. Then $f_{\gamma ns}(G) \leq \gamma_{ns}(G) - |W|$.*

COROLLARY 2.11. *Let G be a connected graph with $d \geq 3$ and l be the number of end vertices of G. Then $f_{\gamma ns}(G) \leq \gamma_{ns}(G) - l$.*

Proof. This follows from Theorems 2.6 and 2.10. □

REMARK 2.12. The bound in Theorem 2.10 is sharp. For the graph G given in Figure 2.1, $S_1 = \{v_1, v_5\}$, $S_2 = \{v_2, v_5\}$ and $S_3 = \{v_3, v_5\}$ are the only three γ_{ns}-sets of G such that $f_{\gamma ns}(S_1) = f_{\gamma ns}(S_2) = f_{\gamma ns}(S_3) = 1$ so that $f_{\gamma ns}(G) = 1$ and $\gamma_{ns}(G) = 2$. Also, $W = \{v_5\}$ is the only nonsplit dominating vertex of G and so $f_{\gamma ns}(G) = \gamma_{ns}(G) - |W|$. Also, the inequality in Theorem 2.10, can be strict. For the graph G given in Figure 2.2, $S_1 = \{v_1, v_4, v_5, v_7\}, S_2 = \{v_1, v_4, v_5, v_8\}, S_3 = \{v_2, v_3, v_5, v_7\}$, $S_4 = \{v_2, v_3, v_5, v_8\}$, $S_5 = \{v_2, v_4, v_5, v_7\}$ and $S_6 = \{v_2, v_4, v_5, v_8\}$ are the γ_{ns}-sets of G such that $f_{\gamma ns}(S_i) = 2$ for $i = 1$ to 4 and $f_{\gamma ns}(S_i) = 3$ for $i = 5, 6$ so that $\gamma_{ns}(G) = 4, f_{\gamma ns}(G) = 2$ and $|W| = 1$. Thus $f_{\gamma ns}(G) < \gamma_{ns}(G) - |W|$.

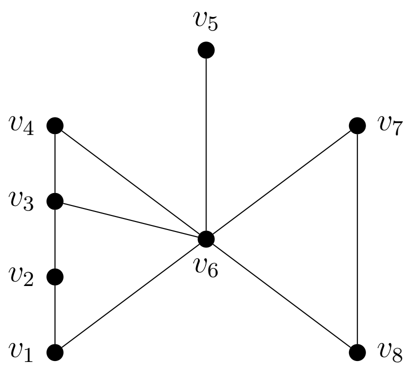

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Figure 2.2

In the following we determine the forcing nonsplit domination number of some standard graphs.

THEOREM 2.13. *For the non trivial path $G = P_n (n \geq 2)$,*

$$f_{\gamma ns}(G) = \begin{cases} 1, & \text{for } n = 2; \\ 2, & \text{for } n = 3; \\ 0, & \text{for } n = 4; \\ 1, & \text{for } n = 5, 6; \\ \frac{n-3}{2}, & \text{for odd } n \geq 7; \\ \frac{n-2}{2}, & \text{for even } n \geq 8. \end{cases}$$

Proof. Let $P_n : v_1, v_2, ..., v_{n-1}, v_n$. Let $n = 2$. Then $S_1 = \{v_1\}$ and $S_2 = \{v_2\}$ are the only two γ_{ns}-sets of G such that $f_{\gamma ns}(S_1) = f_{\gamma ns}(S_2) = 1$ so that $f_{\gamma ns}(G) = 1$. Let $n = 3$. Then $S_1 = \{v_1, v_2\}, S_2 = \{v_1, v_3\}$ and $S_3 = \{v_2, v_3\}$ are the three γ_{ns}-sets of G such that $f_{\gamma ns}(S_1) = f_{\gamma ns}(S_2) = f_{\gamma ns}(S_3) = 2$ so that $f_{\gamma ns}(G) = 2$. Let $n = 4$. Then $S = \{v_1, v_4\}$ is the unique γ_{ns}-set of G so that $f_{\gamma ns}(G) = 0$. Let $n = 5$. Then $S_1 = \{v_1, v_4, v_5\}$ and $S_2 = \{v_1, v_2, v_5\}$ are the only two γ_{ns}-sets of G such that $f_{\gamma ns}(S_1) = f_{\gamma ns}(S_2) = 1$. Let $n = 6$. Then $S_1 = \{v_1, v_4, v_5, v_6\}$, $S_2 = \{v_1, v_2, v_5, v_6\}$ and $S_3 = \{v_1, v_2, v_3, v_6\}$ are the three γ_{ns}-sets of G such that $f_{\gamma ns}(S_1) = f_{\gamma ns}(S_3) = 1$ and $f_{\gamma ns}(S_2) = 2$ so that $f_{\gamma ns}(G) = 1$.

For odd $n \geq 7$, there are $n-3$ γ_{ns}-sets viz., $S_1 = \{v_1, v_4, v_5, ..., v_n\}, S_2 = \{v_1, v_2, v_5, ..., v_n\}, S_3 = \{v_1, v_2, v_3, v_6, ..., v_n\}, ..., S_{n-5} = \{v_1, v_2, v_3, ..., v_{n-5}, v_{n-2}, v_{n-1}, v_n\}, S_{n-4} = \{v_1, v_2, v_3, ..., v_{n-4}, v_{n-1}, v_n\}, S_{n-3} = \{v_1, v_2, v_3, ..., v_{n-3}, v_n\}$. We observe that $T_1 = \{v_4, v_6, ..., v_{n-1}\}$ is a minimum forcing subset of S_1 and so $f_{\gamma ns}(S_1) = \frac{n-3}{2}$, $T_{n-5} = \{v_2, v_4, ..., v_{\frac{n-1}{2}}, v_6, v_9, ..., v_{n-2}\}$ is a minimum forcing subset of S_{n-5} and so $f_{\gamma ns}(S_{n-5}) = \frac{n-3}{2}$, $T_{n-4} = \{v_3, v_6, ..., v_{\frac{n+1}{2}}, v_7, v_{10}, ..., v_{n-1}\}$ is a minimum forcing subset of S_{n-4} and so $f_{\gamma ns}(S_{n-4}) = \frac{n-3}{2}$ and $T_{n-3} = \{v_2, v_4, v_6, ..., v_{n-3}\}$ is a minimum forcing subset of S_{n-3} and so $f_{\gamma ns}(S_{n-3}) = \frac{n-3}{2}$. For $i = 2, 4, 6, ..., n-7$, $S_i = \{v_1, v_2, ..., v_i, v_{i+3}, v_{i+4}, ..., v_n\}$. Then $T_i = \{v_2, v_4, ..., v_i, v_{i+3}, v_{i+5}, ..., v_{n-2}\}$ is a minimum forcing subset of S_i and so $f_{\gamma ns}(S_i) = \frac{n-3}{2}$. For $j = 3, 5, 7, ..., n-6$, $S_j = \{v_1, v_2, ..., v_i, v_{i+3}, v_{i+4}, ..., v_n\}$. Then $T_j = \{v_3, v_5, ..., v_{j+2}, v_{j+5}, ..., v_{n-1}\}$ is a minimum forcing subset of S_j and so $f_{\gamma ns}(S_j) = \frac{n-3}{2}$. Therefore $f_{\gamma ns}(G) = \frac{n-3}{2}$ for odd $n \geq 7$.

For even $n \geq 8$, there are $n-3$ γ_{ns}-sets viz., $S_1 = \{v_1, v_4, v_5, ..., v_n\}, S_2 = \{v_1, v_2, v_5, ..., v_n\}, S_3 = \{v_1, v_2, v_3, v_6, ..., v_n\}, ..., S_{n-5} = \{v_1, v_2, v_3, ..., v_{n-5}, v_{n-2}, v_{n-1}, v_n\}, S_{n-4} = \{v_1, v_2, v_3, ..., v_{n-4}, v_{n-1}, v_n\}, S_{n-3} = \{v_1, v_2, v_3, ..., v_{n-3}, v_n\}$. Then $T_1 = \{v_4, v_6, ..., v_n\}$ is a minimum forcing subset of S_1 and so $f_{\gamma ns}(S_1) = \frac{n-2}{2}$, $T_{n-5} = \{v_3, v_5, ..., v_{\frac{n}{2}+1}, v_{\frac{n-2}{2}}, ..., v_{n-1}\}$ is a minimum forcing subset of S_{n-5} and so $f_{\gamma ns}(S_{n-5}) = \frac{n-2}{2}$, $T_{n-4} = \{v_2, v_4, ..., v_{\frac{n}{2}}, v_{\frac{n}{2}+2}, ..., v_{n-1}\}$ is a minimum forcing subset of S_{n-4} and so $f_{\gamma ns}(S_{n-4}) = \frac{n-2}{2}$ and $T_{n-3} = \{v_2, v_4, ..., v_{\frac{n}{2}}, v_{\frac{n}{2}+2}, ..., v_{n-3}\}$ is a minimum forcing subset of S_{n-3} and so $f_{\gamma ns}(S_{n-3}) = \frac{n-2}{2}$. For $i = 2, 4, 6, ..., n-6$, $S_i = \{v_1, v_2, ..., v_i, v_{i+3}, v_{i+4}, ..., v_n\}$. Then $T_i = \{v_2, v_4, ..., v_i, v_{i+3}, v_{i+5}, ..., v_{n-1}\}$ is a minimum forcing subset of S_i and so $f_{\gamma ns}(S_i) = \frac{n-2}{2}$. For $j = 3, 5, 7, ..., n-7$, $S_j = \{v_1, v_2, ..., v_i, v_{i+3}, v_{i+4}, ..., v_n\}$. Then $T_j = \{v_2, v_4, ..., v_{j-2}, v_j, v_{j+1}, v_{j+3}, ..., v_{n-3}\}$ is a minimum forcing subset of S_j and so $f_{\gamma ns}(S_j) = \frac{n-2}{2}$. Therefore $f_{\gamma ns}(G) = \frac{n-2}{2}$ for even $n \geq 8$. □

THEOREM 2.14. *For the complete graph $G = K_n (n \geq 2)$, $f_{\gamma ns}(G) = 1$.*

Proof. For $G = K_n$, any singleton subset of G is a γ_{ns}-set of G so that $\gamma_{ns}(G) = 1$. Since $n \geq 2$, G contains more than one γ_{ns}-set and so $f_{\gamma ns}(G) \geq 1$. By Theorem 2.3, $f_{\gamma ns}(G) = 1$. □

THEOREM 2.15. *For the complete bipartite graph* $G = K_{r,s} (2 \leq r \leq s)$, $f_{\gamma ns}(G) = 2$.

Proof. Let $X = \{x_1, x_2, ..., x_r\}$ and $Y = \{y_1, y_2, ..., y_s\}$ be the bipartite sets of G. Let $S_{ij} = \{x_i, y_j\}$ $(1 \leq i \leq r)$ and $(1 \leq j \leq s)$. Then S_{ij} is a γ_{ns}-set of G so that $\gamma_{ns}(G) = 2$. Since any singleton subset of S_{ij} is a subset of more than one γ_{ns}-set of G for some i and j. Therefore $f_{\gamma ns}(G) \geq 2$. By Theorem 2.3, $f_{\gamma ns}(G) = 2$. $\square$

THEOREM 2.16. *For the star* $G = K_{1,n-1}$, $f_{\gamma ns}(G) = n - 1$.

Proof. Let v be the center and $v_1, v_2, ..., v_{n-1}$ be the set of all end vertices of G. Then $S = \{v_1, v_2, ..., v_{n-1}\}, S_1 = \{v, v_2, v_3, ..., v_{n-1}\}, S_2 = \{v, v_1, v_3, ..., v_{n-1}\}, ..., S_{n-1} = \{v, v_1, v_2, ..., v_{n-2}\}$ are the γ_{ns}-sets of G with cardinality $n - 1$. We notice that no γ_{ns}-set of G is the unique γ_{ns}-set containing any of its proper subsets. Therefore $f_{\gamma ns}(G) = \gamma_{ns}(G) = n - 1$. $\square$

THEOREM 2.17. *For the cycle* $G = C_n$ $(n \geq 3)$, $f_{\gamma ns}(G) = \gamma_{ns}(G) = n - 2$.

Proof. Let $C_n : v_1, v_2, ..., v_{n-1}, v_n$ be the cycle of order n. Then $S_1 = \{v_1, v_2, v_3, ..., v_{n-2}\}$, $S_2 = \{v_2, v_3, v_4, ..., v_{n-2}, v_{n-1}\}$, $S_3 = \{v_3, v_4, ..., v_{n-1}, v_n\}$, ..., $S_n = \{v_n, v_1, v_2, v_3, ..., v_{n-3}\}$ are the γ_{ns} - sets of G with cardinality $n - 2$. We notice that no γ_{ns}-set of G is the unique γ_{ns}-set containing any of its proper subsets. Therefore $f_{\gamma ns}(G) = \gamma_{ns}(G) = n - 2$. $\square$

THEOREM 2.18. *Let G be a connected graph without cut vertices. If* $\Delta(G) = n-1$, *then* $0 \leq f_{\gamma ns}(G) \leq 1$.

Proof. By the definition of forcing nonsplit domination number, we have $f_{\gamma ns}(G) \geq 0$. Since G contains no cut vertices, $\gamma_{ns}(G) = 1$. By Theorem 2.3, $f_{\gamma ns}(G) \leq 1$. Thus $0 \leq f_{\gamma ns}(G) \leq 1$. $\square$

The following theorem shows the sharpness of the lower and the upper bounds.

THEOREM 2.19. *Let G be a connected graph of order $n \geq 4$ without cut vertices and* $\Delta(G) = n - 1$.

(i) If G contains only one universal vertex, then $f_{\gamma ns}(G) = 0$.
(ii) If G contains more than one universal vertex, then $f_{\gamma ns}(G) = 1$.

Proof. (i) Let u be the universal vertex of G which is not a cut vertex. Then $S = \{u\}$ is the unique γ_{ns}-set of G so that $f_{\gamma ns}(G) = 0$.

(ii) Suppose that G contains more than one universal vertex. Let $x_1, x_2, ..., x_r$, $(2 \leq r \leq n)$ be the universal vertices of G. Since G contains no cut vertices, $S_i = \{x_i\}$ is a γ_{ns}-set of G for $1 \leq i \leq r$ such that $f_{\gamma ns}(S_i) = 1$ for all $1 \leq i \leq r$. Therefore $f_{\gamma ns}(G) = 1$. $\square$

LEMMA 2.20. *Let* $G = K_1 + (m_1K_1 \cup m_2K_2 \cup m_3K_3 \cup ... \cup m_rK_r)$, *where* $m_1 + m_2 + m_3 + ...m_r \geq 2$ *and S be a γ_{ns}-set of G. If $\delta(G) \geq 2$, then S contains at least one element from each component of $G - v$, where* $V(K_1) = \{v\}$.

Proof. Let v be the cut vertex of G and S be a γ_{ns}-set of G. Let $G_1, G_2, ..., G_r (r \geq 2)$ be the components of $G - v$. Since $\delta(G) \geq 2$, $|V(G_i)| \geq 2$ for all i $(1 \leq i \leq r)$. Let x_i be a vertex of $V(G_i)$ for all $i, 1 \leq i \leq r$. We prove that S contains at least one element from each G_i $(1 \leq i \leq r)$. On the contrary, suppose that there exists a component say G_1, such that S contains no elements of G_1. Then it follows that

$v \in S$ and S contains all the elements from each G_i for all i $(2 \le i \le r)$ and so $\gamma_{ns}(G) \ge 1 + 2(r-1) = 2r-1$. Let $S_1 = \{x_1, x_2, x_3, ..., x_r\}$. Then S_1 is a γ_{ns}-set of G so that $\gamma_{ns}(G) = r < 2r-1$, which is a contradiction. Therefore S contains at least one element from each component of $G - v$. □

Theorem 2.21. *Let*$G = K_1 + (m_1K_1 \cup m_2K_2 \cup m_3K_3 \cup ... \cup m_rK_r)$, *where* $m_1 + m_2 + m_3 + ...m_r \ge 2$. *If* $\delta(G) \ge 2$, *then* $f_{\gamma ns}(G) = \gamma_{ns}(G)$.

Proof. Let v be the cut vertex of G and S be a γ_{ns}-set of G. Let $G_1, G_2, ..., G_r (r \ge 2)$ be the components of $G - v$. Since $\delta(G) \ge 2$, $|V(G_i)| \ge 2$ for $1 \le i \le r$. Let $x_i, y_i \in V(G_i)$ for $1 \le i \le r$ and let $H_i = \{x_i, y_i\}$ $(1 \le i \le r)$. By Lemma 2.20, S contains at least one element from each H_i $(1 \le i \le r)$ and so $\gamma_{ns}(G) \ge r$. Let $S = \{x_1, x_2, ..., x_r\}$. Then S is a minimum nonsplit dominating set of G so that $\gamma_{ns}(G) = r$.

Next we show that $f_{\gamma ns}(G) = r$. Since $\gamma_{ns}(G) = r$ and every γ_{ns}-set of G contains at least one element from each H_i $(1 \le i \le r)$, it is easily seen that every γ_{ns}-set is of the form $S = \{c_1, c_2, ..., c_r\}$, where $c_i \in H_i$ $(1 \le i \le r)$. Let T be a proper subset of S with $|T| < r$. Then there exists some i such that $H_i \cap T = \phi$ which shows that $f_{\gamma ns}(G) = r = \gamma_{ns}(G)$. □

3. Realization results

In this section, we present some graphs from which various graphs arising in theorems are generated using identification.

Definition 3.1. Let $P_i : u_i, v_i$ $(1 \le i \le a)$ be a copy of path on two vertices and let $P_i^{'} : x_i, y_i$ $(1 \le i \le b)$ be another copy of path on two vertices. Let $J_{a,\,b}$ be the graph obtained from P_i $(1 \le i \le a)$ and $P_i^{'}$ $(1 \le i \le b)$ by adding a new vertex x and introducing the edges xx_i $(1 \le i \le b)$, xu_i $(1 \le i \le a)$ and xv_i $(1 \le i \le a)$.

Definition 3.2. Let $K_3^i : x_i, y_i, z_i$ $(1 \le i \le a)$ be a copy of the complete graph K_3. Let G_a be the graph obtained from K_3^i by adding a new vertex x and introducing the edges xx_i $(1 \le i \le a)$ and xz_i $(1 \le i \le a)$.

Definition 3.3. Let $K_4^i : p_i, q_i, r_i, s_i$ $(1 \le i \le a)$ be a copy of the complete graph K_4. Let H_a be the graph obtained from K_4^i $(1 \le i \le a)$ by adding new vertices $y, t_1, t_2, ..., t_a$ and introducing the edges yp_i, yq_i and s_it_i $(1 \le i \le a)$.

Definition 3.4. Let $P_i : u_i, v_i$ $(1 \le i \le a)$ be a copy of path on two vertices. Let R_a be the graph obtained from P_i $(1 \le i \le a)$ by adding the vertex z and introducing the edges zu_i $(1 \le i \le a)$.

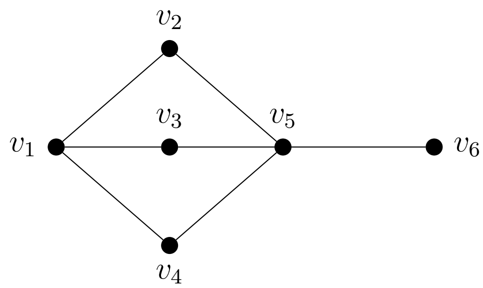
In view of Theorem 2.3, we have the following realization result.

Theorem 3.5. *For every pair of positive integers a and b with $0 \le a \le b$ and $b \ge 1$, there exists a connected graph G such that $f_{\gamma ns}(G) = a$ and $\gamma_{ns}(G) = b$.*

Proof. **Case 1.** $a = 0, b \ge 1$.

Subcase 1.1. $a = 0, b = 1$. Consider the graph $G = F_n$ $(n \ge 5)$. Since G contains only one universal vertex, $\gamma_{ns}(G) = 1$ and by Theorem 2.19 (i) $f_{\gamma ns}(G) = 0$.

Subcase 1.2. $a = 0, b = 2$. Consider the graph G given in Figure 3.1. Then $S = \{v_1, v_6\}$ is the unique γ_{ns}-set of G so that $\gamma_{ns}(G) = 2$ and $f_{\gamma ns}(G) = 0$.



G
Figure 3.1

Subcase 1.3. $a = 0, b \geq 3$. Let G be the graph obtained from a path P_b $(b \geq 3)$ by adding an end edge to each vertex of P_b. Since $b \geq 3$, the diameter is at least 3. Let S be the set of end vertices of G. Then by Theorem 2.6, S is a subset of every γ_{ns}-set of G and so $\gamma_{ns}(G) \geq b$. Since S is the unique γ_{ns}-set of G, it follows from Theorem 2.10 (a) that $f_{\gamma ns}(G) = 0$ and $\gamma_{ns}(G) = b$.

Case 2. $0 < a = b$. Let $G = K_1 + aK_2$ $(a \geq 1)$. Then by Theorem 2.21, we have $\gamma_{ns}(G) = f_{\gamma ns}(G) = a$.

Case 3. $0 < a < b$. Consider the graph $G = J_{a,\, b-a}$. First we show that $\gamma_{ns}(G) = b$. Let $Y = \{y_1, y_2, ..., y_{b-a}\}$. By Theorem 2.6, Y is a subset of every minimum nonsplit dominating set of G. Let $A_i = \{u_i, v_i\}(1 \leq i \leq a)$. Then it is easily observed that every nonsplit dominating set of G contains at least one vertex from each $A_i(1 \leq i \leq a)$ and so $\gamma_{ns}(G) \geq b - a + a = b$. Let $S = Y \cup \{u_1, u_2, ..., u_a\}$. Then S is a minimum nonsplit dominating set of G so that $\gamma_{ns}(G) = b$. Next we show that $f_{\gamma ns}(G) = a$. By Corollary 2.11, $f_{\gamma ns}(G) \leq \gamma_{ns}(G) - |Y| = b - (b - a) = a$. Now, since $\gamma_{ns}(G) = b$ and Y is a subset of every minimum nonsplit dominating set of G, it is easily seen that every γ_{ns}-set of G is of the form $S' = Y \cup \{c_1, c_2, ..., c_a\}$, where $c_i \in A_i(1 \leq i \leq a)$. Let T be any proper subset of S' with $|T| < a$. Then it is clear that there exists some i such that $T \cap A_i = \phi$, which shows that $f_{\gamma ns}(G) = a$. □

Proposition 3.6. The difference $|f_\gamma(G) - f_{\gamma ns}(G)|$ can be arbitrarily large.

Proof. Consider the graph $J_{a,\, 2a}$ $(a \geq 2)$. We notice that $\gamma(G) = 2a + 1$ and $f_\gamma(G) = 2a$. Also, by Theorem 3.5, $\gamma_{ns}(G) = 3a$ and $f_{\gamma ns}(G) = a$. Therefore $|f_\gamma(G) - f_{\gamma ns}(G)| = |2a - a| = a$. □

We know that $\gamma(G) \leq \gamma_{ns}(G)$. However, there was no known relationship between $f_\gamma(G)$ and $f_{\gamma ns}(G)$. So we have the following realization results.

Theorem 3.7. *For every integer $a \geq 0$, there exists a connected graph G with $f_\gamma(G) = f_{\gamma ns}(G) = a$.*

Proof. **Case 1.** $a = 0$. Let $G = F_n$ $(n \geq 5)$. Since G contains only one universal vertex, $f_\gamma(G) = 0$. By Theorem 2.19 (i), $f_{\gamma ns}(G) = 0$.

Case 2. $a \geq 1$. Consider the graph $G = G_a$. First we prove that $\gamma(G) = a$. For $1 \leq i \leq a$, let $A_i = \{x_i, z_i\}$. It is easily observed that every minimum dominating set of G contains at least one vertex from each A_i $(1 \leq i \leq a)$ and so $\gamma(G) \geq a$. Let $D = \{x_1, x_2, ..., x_a\}$. Then D is a minimum dominating set of G so that $\gamma(G) = a$. Next we prove that $f_\gamma(G) = a$. Since $\gamma(G) = a$ and every minimum dominating set of G contains at least one vertex from each A_i $(1 \leq i \leq a)$, it is easily seen that every γ-set is of the form $D_1 = \{c_1, c_2, ..., c_a\}$ where $c_i \in A_i$ $(1 \leq i \leq a)$. Let T be any proper subset of D_1 with $|T| < a$. Then it is clear that there exists some i such that $T \cap A_i = \phi$, which shows that $f_\gamma(G) = a$.

Next we show that $\gamma_{ns}(G) = a$. It is easily observed that every nonsplit dominating set of G contains at least one vertex from each A_i $(1 \leq i \leq a)$ and so $\gamma_{ns}(G) \geq a$. Let $D = \{x_1, x_2, ..., x_a\}$. Then D is a minimum dominating set of G and $\langle V - D \rangle$ is connected. Hence D is a minimum nonsplit dominating set of G and so $\gamma_{ns}(G) = a$. By the similar argument as in the proof of $f_\gamma(G) = a$, we can prove that $f_{\gamma ns}(G) = a$. □

THEOREM 3.8. *For every pair a, b of integers with $a \geq 0$ and $b \geq 0$ there exists a connected graph G such that $f_\gamma(G) = a$ and $f_{\gamma ns}(G) = b$.*

Proof. **Case 1.** $0 \leq a \leq b$.

Subcase 1.1. $0 \leq a = b$. Then the graph constructed in Theorem 3.7 satisfies the requirements of this case.

Subcase 1.2. $a = 0$, $b = 1$. Let $G = K_{1,3} + e$. Then $f_\gamma(G) = 0$ and $f_{\gamma ns}(G) = 1$.

Subcase 1.3. $a = 0$, $b \geq 2$. Consider the graph $G = H_b$. First we prove that $\gamma(G) = b + 1$ and $f_\gamma(G) = 0$. For $1 \leq i \leq b$, it is easily observed that every minimum dominating set contains the vertex y and each s_i $(1 \leq i \leq b)$ and so $\gamma(G) = b + 1$. Let $D = \{y, s_1, s_2, ..., s_b\}$. Then D is the unique γ-set of G so that $\gamma(G) = b + 1$ and $f_\gamma(G) = 0$.

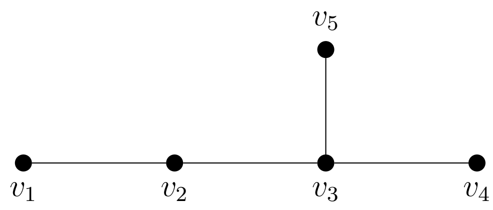
Next we prove that $\gamma_{ns}(G) = 2b$ and $f_{\gamma ns}(G) = b$. Let $Z = \{t_1, t_2, ..., t_b\}$. By Theorem 2.6, Z is a subset of every minimum nonsplit dominating set of G. For $1 \leq i \leq b$, let $B_i = \{p_i, q_i\}$. It is easily observed that every nonsplit dominating set of G contains at least one vertex from each B_i $(1 \leq i \leq b)$ and so $\gamma_{ns}(G) \geq 2b$. Let $D_1 = Z \cup \{p_1, p_2, ..., p_b\}$. Then D_1 is a minimum nonsplit dominating set of G so that $\gamma_{ns}(G) = 2b$. By Corollary 2.11, $f_{\gamma ns}(G) \leq \gamma_{ns}(G) - |Z| = 2b - b = b$. Now, since $\gamma_{ns}(G) = 2b$ and Z is a subset of every minimum nonsplit dominating set, it is easily seen that every γ_{ns}-set D_2 is of the form $D_2 = Z \cup \{c_1, c_2, ..., c_b\}$ where $c_i \in B_i$ $(1 \leq i \leq b)$. Let T be any proper subset of D_2 with $|T| < b$. Then it is clear that there exists some i such that $T \cap B_i = \phi$, which shows that $f_{\gamma ns}(G) = b$.

Subcase 1.4. $0 < a < b$. Let G be the graph obtained from G_a and H_{b-a} by identifying the vertex x of G_a and y of H_{b-a}. First we prove that $\gamma(G) = b$. For $1 \leq i \leq a$, let $A_i = \{x_i, z_i\}$. It is easily observed that every minimum dominating set of G contains the vertex $s_i (1 \leq i \leq b-a)$ and at least one vertex from each $A_i (1 \leq i \leq a)$ and so $\gamma(G) \geq b - a + a = b$. Let $Z = \{s_1, s_2, ..., s_{b-a}\}$ and $D_3 = Z \cup \{x_1, x_2, ..., x_a\}$. Then D_3 is a minimum dominating set of G and so $\gamma(G) = b$. Next we prove that $f_\gamma(G) = a$. Now, since $\gamma(G) = b$ and Z is a subset of every γ-set of G, every γ-set is of the form $D_4 = Z \cup \{c_1, c_2, ..., c_a\}$ where $c_i \in A_i$ $(1 \leq i \leq a)$. Let T be any proper subset of D_4 with $|T| < a$. It is clear that there exists some i such that $T \cap A_i = \phi$, which shows that $f_\gamma(G) = a$.

Next we prove that $\gamma_{ns}(G) = 2b - a$. Let $Z_1 = \{t_1, t_2, ..., t_{b-a}\}$. Then by Theorem 2.6, Z_1 is a subset of every minimum nonsplit dominating set of G. For $1 \leq i \leq b - a$, let $B_i = \{p_i, q_i\}$. Then every minimum nonsplit dominating set of G contains at least one vertex from A_i $(1 \leq i \leq a)$ and at least one vertex from B_i $(1 \leq i \leq b - a)$ and so $\gamma_{ns}(G) \geq b - a + a + b - a = 2b - a$. Let $D_5 = Z \cup \{x_1, x_2, ..., x_a, p_1, p_2, ..., p_{b-a}\}$. Then D_5 is a nonsplit dominating set of G so that $\gamma_{ns}(G) = 2b - a$. Next we prove that $f_{\gamma ns}(G) = b$. Since $\gamma_{ns}(G) = 2b - a$ and every γ_{ns}-set of G contains Z_1, it is easily seen that every γ_{ns}-set is of the form $D_6 = Z_1 \cup \{c_1, c_2, ..., c_a\} \cup \{d_1, d_2, ..., d_{b-a}\}$ where $c_i \in A_i$ $(1 \leq i \leq a)$ and $d_i \in B_i$ $(1 \leq i \leq b-a)$. Let T be any proper subset of D_6 with $|T| < b$. Then it is clear that there exists some i and j such that $T \cap A_i \cap B_j = \phi$ which shows $f_{\gamma ns}(G) = b$.

Case 2. $0 \leq b < a$

Subcase 2.1. $b = 0$, $a = 1$. Consider the graph G given in Figure 3.2. Then $S_1 = \{v_1, v_3\}$ and $S_2 = \{v_2, v_3\}$ are the two γ-sets of G so that $\gamma(G) = 2$ and $f_\gamma(G) = 1$. Also, $S_3 = \{v_1, v_4, v_5\}$ is the unique γ_{ns}-set of G and so $\gamma_{ns}(G) = 3$ and $f_{\gamma ns}(G) = 0$.



G

Figure 3.2

Subcase 2.2. $b = 0$, $a \geq 2$. Consider the graph R_a. Let G be the graph obtained from R_a by adding two new vertices u and v and introducing the edges zu and zv. First we show that $\gamma(G) = a + 1$. For $1 \leq i \leq a$, let $T_i = \{u_i, v_i\}$. Then it is easily observed that every minimum dominating set of G contains the vertex z and at least one vertex from each $T_i (1 \leq i \leq a)$ and so $\gamma(G) \geq a+1$. Let $S_4 = \{z\} \cup \{u_1, u_2, ..., u_a\}$. Then S_4 is a minimum dominating set of G so that $\gamma(G) = a + 1$. Next we prove that $f_\gamma(G) = a$. By Theorem 1.1, $f_\gamma(G) \leq \gamma(G) - |Z|$. Since $\gamma(G) = a + 1$ and every γ-set of G contains z, it is easily seen that every γ-set is of the form $S_5 = \{z\} \cup \{c_1, c_2, ..., c_a\}$, where $c_i \in T_i$ $(1 \leq i \leq a)$. Let T be a proper subset of S_5 with $|T| < a$. Then it is clear that there exists some i such that $T \cap T_i = \phi$, which shows that $f_\gamma(G) = a$. Next we prove that $\gamma_{ns}(G) = a + 2$ and $f_{\gamma ns}(G) = 0$. Let $W = \{u, v, v_1, v_2, ..., v_a\}$. Then by Theorem 2.6, W is a subset of every minimum nonsplit dominating set of G and so $\gamma_{ns}(G) \geq a + 2$. It is clear that W is the unique γ_{ns}-set of G so that $\gamma_{ns}(G) = a + 2$ and $f_{\gamma ns}(G) = 0$.

Subcase 2.3. $0 < b < a$. Let G be the graph obtained from G_b and R_{a-b} by identifying the vertex x of G_b and the vertex z of R_{a-b} and also adding two new vertices u and v and introducing the edges zu and zv. First we prove that $\gamma(G) = a+1$. For $1 \leq i \leq b$, let $A_i = \{x_i, y_i, z_i\}$ and for $1 \leq i \leq a-b$, let $B_i = \{u_i, v_i\}$. It is easily observed that every dominating set of G contains the vertex z and at least one vertex from each A_i $(1 \leq i \leq b)$ and at least one vertex from each B_i $(1 \leq i \leq a-b)$ and so $\gamma(G) \geq 1 + b + a - b = a + 1$. Now as in earlier cases, every γ-set of G is of the form $S_6 = \{z\} \cup \{u_1, u_2, ..., u_{a-b}\} \cup \{x_1, x_2, ..., x_b\}$. Then S_6 is a minimum dominating set of G which shows that $\gamma(G) = 1 + a - b + b = a + 1$. Next we prove that $f_\gamma(G) = a$. Since $\gamma(G) = a + 1$ and every γ-set of G contains z, it is easily seen that every γ-set is of the form $S_7 = \{z\} \cup \{c_1, c_2, ..., c_b\} \cup \{d_1, d_2, ..., d_{a-b}\}$ where $c_i \in A_i$ $(1 \leq i \leq b)$ and $d_i \in B_i$ $(1 \leq i \leq a-b)$. Let T be any proper subset of S_7 with $|T| < a$. It is clear that there exists some i and j such that $T \cap A_i \cap B_j = \phi$, which shows that $f_\gamma(G) = a$.

Next we prove that $\gamma_{ns}(G) = a+2$ and $f_{\gamma ns}(G) = b$. Let $W_1 = \{u, v, v_1, v_2, ..., v_{a-b}\}$. Then by Theorem 2.6, W_1 is a subset of every minimum nonsplit dominating set of

G. It is easily observed that every nonsplit dominating set of G contains at least one vertex from A_i $(1 \leq i \leq b)$ and so $\gamma_{ns}(G) \geq a - b + 2 + b = a + 2$. Let $S_8 = W_1 \cup \{x_1, x_2, ..., x_b\}$. Then S_8 is a minimum nonsplit dominating set of G so that $\gamma_{ns}(G) = a - b + 2 + b = a + 2$. Next we prove that $f_{\gamma ns}(G) = b$. Since $\gamma_{ns}(G) = a + 2$ and every γ_{ns}-set of G contains W_1, it is easily seen that every γ_{ns}-set is of the form $S_9 = W_1 \cup \{c_1, c_2, ..., c_b\}$ where $c_i \in A_i$ $(1 \leq i \leq b)$. Let T be any proper subset of S_9 with $|T| < b$. Then it is clear that there exists some i such that $T \cap A_i = \phi$, which shows $f_{\gamma ns}(G) = b$. □

Remark 3.9. (i) Let $C_6 : v_1, v_2, v_3, v_4, v_5, v_6, v_1$. Let G be the graph obtained from C_6 by introducing the edge v_1v_4. Then it is easily verified that $\gamma(G) = 2$, $f_\gamma(G) = 0$, $\gamma_{ns}(G) = 2$ and $f_{\gamma ns}(G) = 1$. Thus $f_\gamma(G) < f_{\gamma ns}(G) < \gamma(G) = \gamma_{ns}(G)$.

(ii) For the graph G given in Figure 3.3, $\gamma(G) = 2$, $f_\gamma(G) = 1$, $\gamma_{ns}(G) = 2$ and $f_{\gamma ns}(G) = 0$. Thus $f_{\gamma ns}(G) < f_\gamma(G) < \gamma(G) = \gamma_{ns}(G)$.

(iii) For $G = C_6$, $\gamma(G) = 2$ and $\gamma_{ns}(G) = 4$. Also, $f_\gamma(G) = 1$ and $f_{\gamma ns}(G) = 4$. Thus $f_\gamma(G) < \gamma(G) < f_{\gamma ns}(G) = \gamma_{ns}(G)$.

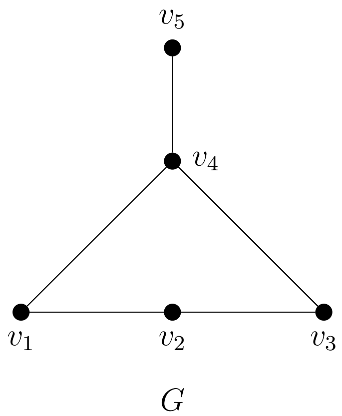


Figure 3.3

So we leave the following problem as open question.

Problem 1. For any four positive integers with $a \geq 0$, $b \geq 0$, $c \leq b \leq d$ and $d \geq 1$, does there exist a connected graph G such that $f_\gamma(G) = a$, $f_{\gamma ns}(G) = b$, $\gamma(G) = c$ and $\gamma_{ns}(G) = d$?

Acknowledments

We are thankful to the referees for their constructive and detailed comments and suggestions which improved the paper overall.

References

[1] H. Abdollahzadeh Ahangar and L. Pushpalatha, *The forcing domination number of Hamiltonian Cubic graphs*, International Journal of Mathematical Combinatorics **2** (2009), 53–57.

[2] D. Ali Mojdeh and N. Jafari Rad, *On domination and its forcing in Mycielski's graphs*, Scientia Iranica **15** (2) (2008), 218–222.

[3] C. L. Armada, S. R. Canoy, Jr. and C. E. Go, *Forcing subsets for γ_c-sets and γ_t-sets in the Lexicographic Product of graphs*, European Journal of Pure And Applied Mathematics **12** (4) (2019), 1779–1786.
[4] G. Chartrand, H. Gavlas, R. C. Vandell and F. Harary, *The forcing domination number of a graph*, Journal of Combinatorial Mathematics and Combinatorial Computing **25** (1997), 161–174.
[5] R. Davila and M. A. Henning, *Total forcing versus total domination in Cubic graphs*, Applied Mathematics and Computation, **354** (2019), 385–395.
[6] D. Ferrero, L. Hogben, F. H. J. Kenter and M. Young, *The relationship between k-forcing and k-power domination*, Discrete Mathematics **341** (6) (2018), 1789–1797.
[7] M. Hajian, M. A. Henning and N. Jafari Rad, *A new lower bound on the domination number of a graph*, Journal of combinatorial Optimization, doi.org/10.1007/s10878-019-00409-x.
[8] F. Harary, *Graph Theory*, Addison-Wesley, (1969).
[9] T. W. Haynes, S. T. Hedetnimi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Incorporated, New York, (1998).
[10] J. John, V. Mary Gleeta, *The forcing monophonic hull number of a graph*, International Journal of Mathematics Trends and Technology **3** (2) (2012), 43–46.
[11] J. John, S. Panchali, *The forcing monophonic number of a graph*, International Journal of Mathematical Archive **3** (3) (2012), 935–938.
[12] S. Kavitha, S. Robinson Chellathurai and J. John, *On the forcing connected domination number of a graph*, Journal of Discrete Mathematical Sciences and Cryptography **23** (3) (2017), 611–624.
[13] V. R. Kulli and B. Janakiram, *The nonsplit domination number of a graph*, Indian Journal of Pure and Applied Mathematics **31** (5) (2000), 545–550.
[14] C. Sivagnanam and M. P. Kulandaivel, *Complementary connected domination number and connectivity of a graph*, General Mathematics Notes **29** (2), (2015), 27–35.
[15] A. P. Santhakumaran and J. John, *The upper edge geodetic number and the forcing edge geodetic number of a graph*, Opuscula Mathematica **29** (4) (2009), 427–441.
[16] A. P. Santhakumaran and J. John, *The forcing Steiner number of a graph*, Discussiones Mathematicae Graph Theory, **31** (2011), 171–181.
[17] A. P. Santhakumaran and J. John, *On the forcing geodetic and the forcing Steiner numbers of a graph*, Discussiones Mathematicae Graph Theory **31** (2011), 611–624.
[18] T. Tamizh Chelvam and B. J. Prasad, *Complementary connected domination number*, International Journal of Management Systems **18** (2) (2002), 147–154.

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