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FEW RESULTS ON RELATIVE (k,n) VALIRON DEFECTS FROM THE VIEW POINTS OF INTEGRATED MODULI OF LOGARITHMIC DERIVATIVE OF ENTIRE AND MEROMORPHIC FUNCTIONS

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Abstract. The prime target of this paper is to compare some relative (k,n) Nevanlinna defects with relative (k,n) Valiron defects from the view point of integrated moduli of logarithmic derivative of entire and meromorphic functions where k and n are any two non-negative integers.

1. Introduction

Let f be a non constant meromorphic function defined in the open complex plane $\mathbb{C}$. For $\alpha \in \mathbb{C} \cup \{\infty\}$, let $n(t, \alpha; f)$ denote the number of roots of $f = \alpha$ in $|z| \leq t$, the multiple roots being counted according to their multiplicities and $N(t, \alpha; f)$ is defined in the usual way in terms of $n(t, \alpha; f)$. Similarly, $\bar{n}(t, \alpha; f)$ denotes the number of distinct roots of $f = \alpha$ in $|z| \leq t$ and $\bar{N}(t, \alpha; f)$ is also defined in the usual way in terms of $\bar{n}(t, \alpha; f)$.

The Nevanlinna defect $\delta(\alpha; f)$ and the Valiron defect $\Delta(\alpha; f)$ of α are respectively defined in the following manner:

$$\delta(\alpha; f) = 1 - \limsup_{r\to\infty} \frac{N(r, \alpha; f)}{T(r, f)} = \liminf_{r\to\infty} \frac{m(r, \alpha; f)}{T(r, f)}$$

and

$$\Delta(\alpha, f) = 1 - \liminf_{r\to\infty} \frac{N(r, \alpha; f)}{T(r, f)} = \limsup_{r\to\infty} \frac{m(r, \alpha; f)}{T(r, f)}.$$

Milloux [6] introduced the concept of absolute defect of 'α' with respect to the derivative f'. Later Xiong [10] extended this definition. He introduced the term

$$\delta_R^{(k)}(\alpha; f) = 1 - \limsup_{r\to\infty} \frac{N(r, \alpha; f^{(k)})}{T(r, f)} \quad \text{for} \quad k = 1, 2, 3, ...$$

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and called it the relative Nevanlinna defect of 'α' with respect to $f^{(k)}$.Xiong [10] showed various relations between the usual defects and the relative defects. Singh [8] introduced the term relative defect for distinct zeros and poles and established various relations between the relative defects and the usual defects. In the paper we call the following two terms

$$_R\delta^{(k)}_{(n)}(\alpha; f) = 1 - \limsup_{r\to\infty} \frac{N\left(r, \alpha; f^{(k)}\right)}{T\left(r, f^{(n)}\right)}$$

and

$$_R\Delta^{(k)}_{(n)}(\alpha; f) = 1 - \liminf_{r\to\infty} \frac{N(r, \alpha; f^{(k)})}{T(r, f^{(n)})}$$

respectively the relative (k, n) Nevanlinna defect and the relative (k, n) Valiron defect of 'α' with respect to $f^{(k)}$ for $k = 1, 2, 3, ...$ and $n = 0, 1, 2, 3, ...$ and prove various relations between them. For $n = 0$, the above definitions coincide with the relative Nevanlinna defect and the relative Valiron defect respectively.

The term $S(r, f)$ denotes any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ through all values of r if f is of finite order and except possibly for a set of r of finite linear measure otherwise. We do not explain the standard definitions and notations of the value distribution theory and the Nevanlinna theory as those are available in [4].

The following definitions are well known.

Definition 1.1. The order ρ_f of a meromorphic function f is defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

If f is entire, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r}.$$

If $\rho_f < \infty$ then f is of finite order.

Definition 1.2. The lower order λ_f of a meromorphic function f is defined as

$$\lambda_f = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r} .$$

If f is entire, it can easily be verified that

$$\lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r} .$$

We may now recall the following.

If f is a meromorphic function in the complex plane. Then the integrated moduli of the logarithmic derivative $I(r, f)$ is defined by

$$I(r, f) = \frac{r}{2\pi} \int_0^{2\pi} \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta$$

for $0 < r < +\infty$ $\{cf.\ [9]\}$.

We now define the following two terms by using the concept of $I\,(r,f)$

$$_{I}\delta_{n}^{(k)}(a;f)=1-\lim\sup_{r\rightarrow\infty}\frac{N(r,a;f^{k})}{I(r,f^{n})},$$

and

$$_{I}\Delta_{n}^{(k)}(a;f)=1-\lim\inf_{r\rightarrow\infty}\frac{N(r,a;f^{k})}{I(r,f^{n})}.$$

These are respectively known as relative (k,n) Nevanlinna defect and relative (k,n) Valiron defect with respect to $I\,(r,f)$. In this paper we obtain different kind of relative (k,n) defects of entire and meromorphic functions under the flavour of their integrated moduli of logarithmic derivative. Further, the estimations are sharper as ensured by suitable examples.

2. Preliminaries

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [8] *Let f is a meromorphic function of finite order such that $\sum\limits_{a\neq\infty}\delta(a;f)=1$ and $\delta(\infty;f)=1$. Then for any non negative integer k,*

$$\lim_{r\rightarrow\infty}\frac{T(r,f^{(k)})}{T(r,f)}=1.$$

Lemma 2.2. [8] *If f is any meromorphic function of finite order such that $\sum\limits_{a\neq\infty}\delta(a;f)=1$ and $\delta(\infty;f)=1$, then*

$$\lim_{r\rightarrow\infty}\frac{T(r,f^{(k)})}{T(r,f^{(n)})}=1$$

where k and n are any two non negative integers.

Lemma 2.3. [1] *Let f is a meromorphic function of finite order with $\sum\limits_{a\neq\infty}\delta(a;f)=\delta(\infty;f)=1$. Then for any $'\alpha'$,*

$$_{R}\delta_{(n)}^{(k)}(\alpha;f)=\lim_{r\rightarrow\infty}\inf\frac{m\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(n)}\right)}.$$

Lemma 2.4. [1] *If f is any meromorphic function of finite order with $\sum\limits_{a\neq\infty}\delta(a;f)=\delta(\infty;f)=1$, then for any ‘$\alpha$’*

$$_{R}\Delta_{(n)}^{(k)}(\alpha;f)=\lim_{r\rightarrow\infty}\sup\frac{m\left(r,\alpha;f^{(k)}\right)}{T(r,f^{(n)})}.$$

Lemma 2.5. [9] *Let f is an entire function of finite order $'\rho'$ having no zeros in $\mathbb{C}$. Then*

$$\lim_{r\rightarrow\infty}\frac{I(r,f)}{T(r,f)}=\pi\rho$$

and

$$\lim_{r\rightarrow\infty}\frac{I(r,f^{n})}{T(r,f^{n})}=\pi\rho.$$

Lemma 2.6. *If f is any entire function of non zero finite order $'\rho'$ with no zeros in $\mathbb{C}$ such that $\sum\limits_{a\neq\infty}\delta(a;f)=1$ and $\delta(\infty;f)=1$, then for any non negative integer k,*

$$\lim_{r\to\infty}\frac{I(r,f^{(k)})}{I(r,f)}=1.$$

Proof. In view of Lemma 2.1, Lemma 2.2, and Lemma 2.5 we get that

$$\begin{aligned}\lim_{r\to\infty}\frac{I(r,f^{(k)})}{I(r,f)} &= \lim_{r\to\infty}\left[\frac{I(r,f^{(k)})}{T(r,f^{k})}\cdot\frac{T(r,f^{(k)})}{I(r,f)}\right]\\ &= \lim_{r\to\infty}\frac{I(r,f^{(k)})}{T(r,f^{k})}\cdot\lim_{r\to\infty}\frac{T(r,f^{(k)})}{I(r,f)}\\ &= \pi\rho\cdot\lim_{r\to\infty}\left[\frac{T(r,f^{(k)})}{T(r,f)}\cdot\frac{T(r,f)}{I\,(r,f)}\right]\\ &= \pi\rho\cdot\lim_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f)}\cdot\lim_{r\to\infty}\frac{T(r,f)}{I\,(r,f)}\\ &= \pi\rho\cdot 1\cdot\frac{1}{\pi\rho}=1.\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.7. *If f is any entire function of non zero finite order $'\rho'$ with no zeros in $\mathbb{C}$ and $\sum\limits_{a\neq\infty}\delta(a;f)=1$ and $\delta(\infty;f)=1$, then for any two non negative integer k and n,*

$$\lim_{r\to\infty}\frac{I(r,f^{(k)})}{I(r,f^{(n)})}=1.$$

We omit the proof of Lemma 2.7 because it can be carried out in the line of Lemma 2.6.

Lemma 2.8. *Let f is an entire function of non zero finite order $'\rho'$ having no zeros in $\mathbb{C}$ with $\sum\limits_{a\neq\infty}\delta(a;f)=\delta(\infty;f)=1$. Then for any $'\alpha'$,*

$${}_{I}\delta^{(k)}_{(n)}(\alpha;f)=\left(1-\frac{1}{\pi\rho}\right)+\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}.$$

Proof. In view of Lemma 2.1, Lemma 2.2, Lemma 2.3, Lemma 2.5, Lemma 2.6 and Lemma 2.7 we get that

$${}_{I}\delta^{(k)}_{(n)}(\alpha;f)=1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}$$

$$
\begin{aligned}
&= 1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}\cdot\lim_{r\to\infty}\frac{T(r,f^{(k)})}{I(r,f^{(n)})}\\
&= 1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}\cdot\lim_{r\to\infty}\left(\frac{T(r,f^{(k)})}{T(r,f)}\cdot\frac{T(r,f)}{I(r,f^{(n)})}\right)\\
&= 1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}\cdot\lim_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f)}\cdot\lim_{r\to\infty}\frac{T(r,f)}{I(r,f^{(n)})}\\
&= 1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}\cdot\lim_{r\to\infty}\left(\frac{T(r,f)}{I\left(r,f\right)}\cdot\frac{I(r,f)}{I(r,f^{(n)})}\right)\\
&= 1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}\cdot\lim_{r\to\infty}\frac{T(r,f)}{I\left(r,f\right)}\cdot\lim_{r\to\infty}\frac{I(r,f)}{I(r,f^{(n)})}\\
&= 1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}\cdot\frac{1}{\pi\rho}\cdot 1\\
&= \left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\left(1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}\right)\\
&= \left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}\cdot\lim_{r\to\infty}\frac{I(r,f^{(n)})}{T(r,f^{(k)})}\\
&= \left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}\cdot\lim_{r\to\infty}\left(\frac{I(r,f^{(n)})}{I(r,f)}\cdot\frac{I(r,f)}{T(r,f^{(k)})}\right)\\
&= \left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}\cdot\lim_{r\to\infty}\frac{I(r,f^{(n)})}{I(r,f)}\cdot\lim_{r\to\infty}\frac{I(r,f)}{T(r,f^{(k)})}\\
&= \left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}\cdot\lim_{r\to\infty}\frac{I(r,f^{(n)})}{I(r,f)}\cdot\lim_{r\to\infty}\frac{I(r,f)}{T(r,f^{(k)})}\\
&= \left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}\cdot 1\cdot\lim_{r\to\infty}\left(\frac{I(r,f)}{T\left(r,f\right)}\cdot\frac{T\left(r,f\right)}{T(r,f^{(k)})}\right)\\
&= \left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}\cdot\lim_{r\to\infty}\frac{I(r,f)}{T\left(r,f\right)}\cdot\lim_{r\to\infty}\frac{T\left(r,f\right)}{T(r,f^{(k)})}\\
&= \left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}\cdot\pi\rho\cdot 1\\
&= \left(1-\frac{1}{\pi\rho}\right)+\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}.
\end{aligned}
$$

This proves the lemma. □

Lemma 2.9. *Let f is an entire function of non zero finite order $'\rho'$ such that f has no zeros in $\mathbb{C}$ with $\sum\limits_{a\neq\infty}\delta(a;f)=\delta(\infty;f)=1$ .Then for any 'α',*

$$
{}_{I}\Delta_{(n)}^{(k)}(\alpha;f)=\left(1-\frac{1}{\pi\rho}\right)+\limsup_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I(r,f^{(n)})}.
$$

The proof is omitted.

3. Main results.

In this section we present the main results of the paper.

THEOREM 3.1. *Let f be an entire function of non-zero finite order $'\rho'$ having no zeros in $\mathbb{C}$ such that $m(r,f) = S(r,f)$. If $'a'$, $'b'$ and $'c'$ are three distinct complex numbers then for any two positive integers k and n,*

$$3{}_I\delta_{(n)}^{(0)}(a;f)+2{}_I\delta_{(n)}^{(0)}(b;f)+{}_I\delta_{n}^{(0)}(c;f)+5{}_I\delta_{(n)}^{(k)}(\infty;f)+\frac{1}{\pi\rho} \le 5{}_I\Delta_{(n)}^{(0)}(\infty;f)+5{}_I\Delta_{(n)}^{(k)}(0;f)+1.$$

Proof. Let us consider the following identity

$$\frac{b-a}{f-a} = \left[\frac{f^{(k)}}{f-a}\left\{\frac{f-a}{f^{(k)}} - \frac{f-b}{f^{(k)}}\right\} - \frac{f-c}{f^{(k)}}\cdot\frac{f^{(k)}}{f}\cdot\frac{f^{(k)}}{f-a}\cdot\left\{\frac{f-a}{f^{(k)}} - \frac{f-b}{f^{(k)}}\right\}\right]\cdot\frac{f}{c}.$$

Since $m\left(r,\frac{1}{f-a}\right) \le m\left(r,\frac{b-a}{f-a}\right) + O(1)$ and $m\left(r,\frac{f}{c}\right) \le m(r,f) + O(1)$, we get from the above identity that

$$\begin{aligned} m\left(r,\frac{b-a}{f-a}\right) &\le m\left(r,\frac{f-a}{f^{(k)}}\right) + m\left(r,\frac{f-b}{f^{(k)}}\right) + m\left(r,\frac{f-c}{f^{(k)}}\right) \\ &\quad + m\left(r,\frac{f-a}{f^{(k)}}\right) + m\left(r,\frac{f-b}{f^{(k)}}\right) + m\left(r,\frac{f}{c}\right) \\ &\quad + S(r,f) \end{aligned}$$

$$\begin{aligned} i.e.,\ m\left(r,\frac{1}{f-a}\right) &\le 2m\left(r,\frac{f-a}{f^{(k)}}\right) + 2m\left(r,\frac{f-b}{f^{(k)}}\right) + m\left(r,\frac{f-c}{f^{(k)}}\right) \\ &\quad + m(r,f) + S(r,f) + O(1) \end{aligned}$$

$$\begin{aligned} i.e.,\ m\left(r,\frac{1}{f-a}\right) &\le 2T\left(r,\frac{f-a}{f^{(k)}}\right) - 2N\left(r,\frac{f-a}{f^{(k)}}\right) + 2T\left(r,\frac{f-b}{f^{(k)}}\right) \\ &\quad - 2N\left(r,\frac{f-b}{f^{(k)}}\right) + T\left(r,\frac{f-c}{f^{(k)}}\right) - N\left(r,\frac{f-c}{f^{(k)}}\right) \\ &\quad + m(r,f) + S(r,f) + O(1). \end{aligned} \tag{1}$$

Now by the relation $T\left(r,\frac{1}{f}\right) = T(r,f) + O(1)$ and in view of Lemma 2.1, we get from Equation (1) that

$$\begin{aligned} i.e.,\ m\left(r,\frac{1}{f-a}\right) &\le 2T\left(r,\frac{f^{(k)}}{f-a}\right) - 2N\left(r,\frac{f-a}{f^{(k)}}\right) + 2T\left(r,\frac{f^{(k)}}{f-b}\right) \\ &\quad - 2N\left(r,\frac{f-b}{f^{(k)}}\right) + T\left(r,\frac{f^{(k)}}{f-c}\right) - N\left(r,\frac{f-c}{f^{(k)}}\right) \\ &\quad + m(r,f) + S(r,f) + O(1). \end{aligned}$$

$$\begin{aligned} i.e., \ m\left(r, \frac{1}{f-a}\right) &\leq 2\left\{N\left(r, \frac{f^{(k)}}{f-a}\right) - N\left(r, \frac{f-a}{f^{(k)}}\right)\right\} \\ &+2\left\{N\left(r, \frac{f^{(k)}}{f-b}\right) - N\left(r, \frac{f-b}{f^{(k)}}\right)\right\} \\ &+N\left(r, \frac{f^{(k)}}{f-c}\right) - N\left(r, \frac{f-c}{f^{(k)}}\right) \\ &+m(r, f) + S(r, f) + O(1). \end{aligned} \tag{2}$$

In view of [5, p. 34], it follows from Equation (2) that

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) &\leq 2N\left(r, f^{(k)}\right) + 2N\left(r, \frac{1}{f-a}\right) - 2N(r, f-a) - 2N\left(r, \frac{1}{f^{(k)}}\right) \\ &+2N\left(r, f^{(k)}\right) + 2N\left(r, \frac{1}{f-b}\right) - 2N(r, f-b) - 2N\left(r, \frac{1}{f^{(k)}}\right) \\ &+N\left(r, f^{(k)}\right) + N\left(r, \frac{1}{f-c}\right) - N(r, f-c) - N\left(r, \frac{1}{f^{(k)}}\right) \\ &+m(r, f) + S(r, f) + O(1) \end{aligned}$$

$$\begin{aligned} i.e., \ m\left(r, \frac{1}{f-a}\right) &\leq 5N\left(r, f^{(k)}\right) - 5N(r, f) - 5N\left(r, \frac{1}{f^{(k)}}\right) \\ &+2N\left(r, \frac{1}{f-a}\right) + 2N\left(r, \frac{1}{f-b}\right) + 2N\left(r, \frac{1}{f-c}\right) \\ &+m(r, f) + S(r, f) + O(1). \end{aligned} \tag{3}$$

In view of Lemma 2.5 and $m(r, f) = S(r, f)$, we obtain from Equation (3) that

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) &\leq 5N\left(r, f^{(k)}\right) - 5N(r, f) - 5N\left(r, \frac{1}{f^{(k)}}\right) \\ &+2N\left(r, \frac{1}{f-a}\right) + 2N\left(r, \frac{1}{f-b}\right) + 2N\left(r, \frac{1}{f-c}\right) \\ &+S(r, f) \end{aligned}$$

$$\begin{aligned} i.e., \ \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{I\left(r, f^{(n)}\right)} &\leq 5\liminf_{r\to\infty}\left\{\frac{N\left(r, f^{(k)}\right)}{I\left(r, f^{(n)}\right)} - \frac{N(r, f)}{I\left(r, f^{(n)}\right)} - \frac{N\left(r, \frac{1}{f^{(k)}}\right)}{I\left(r, f^{(n)}\right)}\right\} \\ &+2\limsup_{r\to\infty}\frac{N\left(r, \frac{1}{f-a}\right)}{I\left(r, f^{(n)}\right)} + 2\limsup_{r\to\infty}\frac{N\left(r, \frac{1}{f-b}\right)}{I\left(r, f^{(n)}\right)} \\ &+2\limsup_{r\to\infty}\frac{N\left(r, \frac{1}{f-c}\right)}{I\left(r, f^{(n)}\right)} \end{aligned}$$

$$i.e.,\ {}_I\delta_{(n)}^{(0)}(a;f)-\left(1-\frac{1}{\pi\rho}\right) \leq 5\left\{1-\ {}_I\Delta_{(n)}^{(k)}(\infty;f)\right\}-5\left\{1-\ {}_I\Delta_{(n)}^{(0)}(\infty;f)\right\}$$
$$-5\left\{1-\ {}_I\Delta_{(n)}^{(k)}(0;f)\right\}+2\left\{1-\ {}_I\delta_{(n)}^{(0)}(a;f)\right\}$$
$$+2\left\{1-\ {}_I\delta_{(n)}^{(0)}(b;f)\right\}+\left\{1-\ {}_I\delta_{(n)}^{(0)}(c;f)\right\}$$

$$i.e.,\ 3{}_I\delta_{(n)}^{(0)}(a;f)+2{}_I\delta_{(n)}^{(0)}(b;f)+\ {}_I\delta_{n}^{(0)}(c;f)+5{}_I\delta_{(n)}^{(k)}(\infty;f)+\frac{1}{\pi\rho}$$
$$\leq 5{}_I\Delta_{(n)}^{(0)}(\infty;f)+5{}_I\Delta_{(n)}^{(k)}(0;f)+1.$$

This proves the theorem. □

Remark 3.1. *The condition $\rho > 0$ in Theorem 3.1 is necessary as we see from the following example.*

Example 1. Let $f(z) = z$. Then $N(r,f) = 0$ and

$$\begin{aligned} T(r,f) &= m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|re^{i\theta}\right|d\theta=\frac{1}{2\pi}\int_0^{2\pi}\log^+(r\cos\theta)\,d\theta \\ &= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\log(r\cos\theta)\,d\theta-\frac{1}{2\pi}\int_0^{-\frac{\pi}{2}}\log(r\cos\theta)\,d\theta \\ &= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\log(r\cos\theta)\,d\theta+\frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\log(r\cos\theta)\,d\theta=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\log(r\cos\theta)\,d\theta \\ &= \frac{1}{\pi}\cdot 2\pi\log\left(\frac{r^2}{2}\right)=2\log\left(\frac{r^2}{2}\right)\neq 0. \end{aligned}$$

Now,

$$\rho=\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r}=\limsup_{r\to\infty}\frac{\log^{[2]}r}{\log r}=\limsup_{r\to\infty}\frac{1}{\log r}=0$$

and

$$I(r,f)=\frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right|d\theta=\frac{r}{2\pi}\int_0^{2\pi}\left|\frac{re^{i\theta}\cdot i}{re^{i\theta}}\right|d\theta=\frac{r}{2\pi}\cdot 2\pi=r\neq 0.$$

Therefore,

$${}_I\delta_{(n)}^{(0)}(a;f)=\ {}_I\delta_{(n)}^{(0)}(b;f)=\ {}_I\delta_{n}^{(0)}(c;f)=\ {}_I\delta_{(n)}^{(k)}(\infty;f)=1$$

and

$${}_I\Delta_{(n)}^{(0)}(\infty;f)=\ {}_I\Delta_{(n)}^{(k)}(0;f)=1.$$

Hence

$$\infty\leq 5{}_I\Delta_{(n)}^{(0)}(\infty;f)+5{}_I\Delta_{(n)}^{(k)}(0;f)+1$$

which is contrary to the conclusion of Theorem 3.1.

Remark 3.2. *The condition $'a$, b and c are any three distinct complex numbers in Theorem 3.1$'$ is essential as we see from the following examples.*

Example 2. Let $f=\exp(2z)$ and $a=b=c=0$. Then we get that $N(r,f)=0$, So,

$$\begin{aligned} I(r,f) &= I(r,\exp(2z)) = \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right|d\theta = \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{e^{2re^{i\theta}}\cdot 2re^{i\theta}\cdot i}{e^{2re^{i\theta}}}\right|d\theta \\ &= \frac{r}{2\pi}\int_0^{2\pi}\left|2re^{i\theta}\cdot i\right|d\theta = \frac{r}{2\pi}\int_0^{2\pi}(2r)\,d\theta = \frac{r^2}{\pi}\int_0^{2\pi}d\theta = \frac{r^2}{\pi}\cdot 2\pi = 2r^2\neq 0 \end{aligned}$$

and

$$\rho = \limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r} = \limsup_{r\to\infty}\frac{\log^{[2]}e^{2r}}{\log r} = \limsup_{r\to\infty}\frac{\log(2r)}{\log r} = 1.$$

Now,

$${}_I\delta_{(n)}^{(0)}(a;f) = {}_I\delta_{(n)}^{(0)}(b;f) = {}_I\delta_{n}^{(0)}(c;f) = {}_I\delta_{n}^{(0)}(0;f) = {}_I\delta_{(n)}^{(k)}(\infty;f) = 1$$

and

$${}_I\Delta_{(n)}^{(0)}(\infty;f) = {}_I\Delta_{(n)}^{(k)}(0;f) = 1.$$

Hence

$$\begin{aligned} & 3{}_I\delta_{(n)}^{(0)}(a;f) + 2{}_I\delta_{(n)}^{(0)}(b;f) + {}_I\delta_{n}^{(0)}(c;f) + 5{}_I\delta_{(n)}^{(k)}(\infty;f) + \frac{1}{\pi\rho} \\ = & \; 3+2+1+5+\frac{1}{\pi} = 11+\frac{1}{\pi} \end{aligned}$$

and

$$5{}_I\Delta_{(n)}^{(0)}(\infty;f) + 5{}_I\Delta_{(n)}^{(k)}(0;f) + 1 = 5+5+1+11,$$

which is contrary to Theorem 3.1.

Example 3. Let $f=\exp(2z)$ and $a=b=c=\infty$. Then we see that $N(r,f)=0$, $I(r,f)=2r^2\neq 0$ and $\rho=1$.

So,

$${}_I\delta_{(n)}^{(0)}(a;f) = {}_I\delta_{(n)}^{(0)}(b;f) = {}_I\delta_{n}^{(0)}(c;f) = {}_I\delta_{n}^{(0)}(\infty;f) = {}_I\delta_{(n)}^{(k)}(\infty;f) = 1$$

and

$${}_I\Delta_{(n)}^{(0)}(\infty;f) = {}_I\Delta_{(n)}^{(k)}(0;f) = 1.$$

Thus,

$$\begin{aligned} & 3{}_I\delta_{(n)}^{(0)}(a;f) + 2{}_I\delta_{(n)}^{(0)}(b;f) + {}_I\delta_{n}^{(0)}(c;f) + 5{}_I\delta_{(n)}^{(k)}(\infty;f) + \frac{1}{\pi\rho} \\ = & \; 3+2+1+5+\frac{1}{\pi} = 11+\frac{1}{\pi} \end{aligned}$$

and

$$5{}_I\Delta_{(n)}^{(0)}(\infty;f) + 5{}_I\Delta_{(n)}^{(k)}(0;f) + 1 = 5+5+1+11,$$

which contradicts Theorem 3.1.

Theorem 3.2. *Let f be any entire function of finite order $'\rho'$ with no zeros in $\mathbb{C}$ satisfying the condition $m(r,f)=S(r,f)$. For any two positive integers k and n,*

$${}_I\delta_{(n)}^{(0)}(0;f) + {}_I\delta_{(n)}^{(k)}(\infty;f) + {}_I\delta_{n}^{(0)}(c;f) \leq {}_I\Delta_{(n)}^{(0)}(\infty;f) + 2{}_I\Delta_{(n)}^{(k)}(0;f)$$

where $'a'$ and $'c'$ are two nonzero finite complex numbers.

Proof. Let us consider the following identity

$$\frac{c}{f}=\left[\left\{1-\frac{f-c}{f^{(k)}}.\frac{f^{(k)}}{f}\right\}\left\{\frac{f^{(k)}}{f-a}.\frac{1}{f^{(k)}}\right\}\right].(f-a).$$

Since $m\left(r,\frac{1}{f}\right)\leq m\left(r,\frac{c}{f}\right)+O(1)$ and $m(r,f-a)\leq m(r,f)+O(1)$, we get from the above identity

$$m\left(r,\frac{c}{f}\right)\leq m\left(r,\frac{f-c}{f^{(k)}}\right)+m\left(r,\frac{1}{f^{(k)}}\right)+m(r,f)+S(r,f)+O(1)$$

$$\begin{aligned} i.e., m\left(r,\frac{1}{f}\right) \leq\ & T\left(r,\frac{f-c}{f^{(k)}}\right)-N\left(r,\frac{f-c}{f^{(k)}}\right)+T\left(r,\frac{1}{f^{(k)}}\right)-N\left(r,\frac{1}{f^{(k)}}\right)\\ & +m(r,f)+S(r,f)+O(1). \end{aligned} \tag{4}$$

Now by Nevanlinna's first fundamental theorem we get from Equation (4) that

$$\begin{aligned} m\left(r,\frac{1}{f}\right) \leq\ & T\left(r,\frac{f^{(k)}}{f-c}\right)-N\left(r,\frac{f-c}{f^{(k)}}\right)+T\left(r,f^{(k)}\right)-N\left(r,\frac{1}{f^{(k)}}\right)\\ & +m(r,f)+S(r,f)+O(1) \end{aligned}$$

$$\begin{aligned} i.e., m\left(r,\frac{1}{f}\right) \leq\ & N\left(r,\frac{f^{(k)}}{f-c}\right)-N\left(r,\frac{f-c}{f^{(k)}}\right)-N\left(r,\frac{1}{f^{(k)}}\right)+T\left(r,f^{(k)}\right)\\ & +m(r,f)+S(r,f)+O(1). \end{aligned} \tag{5}$$

In view of [5, p. 34], we obtain from Equation (5) that

$$\begin{aligned} m\left(r,\frac{1}{f}\right) \leq\ & N\left(r,f^{(k)}\right)+N\left(r,\frac{1}{f-c}\right)-N(r,f-c)-N\left(r,\frac{1}{f^{(k)}}\right)-N\left(r,\frac{1}{f^{(k)}}\right)\\ & +T\left(r,f^{(k)}\right)+m(r,f)+S(r,f)+O(1). \end{aligned} \tag{6}$$

In view of Lemma 2.5 and $m(r,f)=S(r,f)$, it follows from Equation (6) that

$$\begin{aligned} m\left(r,\frac{1}{f}\right) \leq\ & N\left(r,f^{(k)}\right)-N(r,f)-2N\left(r,\frac{1}{f^{(k)}}\right)+N\left(r,\frac{1}{f-c}\right)\\ & +T\left(r,f^{(k)}\right)+S(r,f) \end{aligned}$$

$$\begin{aligned} i.e., \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f}\right)}{I\left(r,f^{(n)}\right)} \leq\ & \liminf_{r\to\infty}\frac{N\left(r,f^{(k)}\right)}{I\left(r,f^{(n)}\right)}-\liminf_{r\to\infty}\frac{N(r,f)}{I\left(r,f^{(n)}\right)}-2\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(k)}}\right)}{I\left(r,f^{(n)}\right)}\\ & +\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-c}\right)}{I\left(r,f^{(n)}\right)}+\lim_{r\to\infty}\frac{T\left(r,f^{(k)}\right)}{I\left(r,f^{(n)}\right)} \end{aligned}$$

$$i.e., {}_I\delta_{(n)}^{(0)}(0;f)+{}_I\Delta_{(n)}^{(k)}(\infty;f)+{}_I\delta_{n}^{(0)}(c;f)\leq {}_I\Delta_{(n)}^{(0)}(\infty;f)+2{}_I\Delta_{(n)}^{(k)}(0;f).$$

Thus the theorem is established. □

REMARK 3.3. *The sign* $'\leq'$ *in Theorem* ***3.2*** *cannot be replaced by* $'<'$*only as is evident from the following example.*

Example 4. Let $f(z)=\exp z$ and $a=c=0$. Then we see that $N(r,f)=0$ and

$$\begin{aligned} T(r,f) &= N(r,f)+m(r,f)=\frac{1}{2\Pi}\int\limits_0^{2\Pi}\log^+\left|f(re^{i\theta})\right|d\theta=\frac{1}{2\Pi}\int\limits_0^{2\Pi}\log^+\left|e^{re^{i\theta}}\right| \\ &= \frac{1}{2\Pi}\int\limits_0^{2\Pi}\log^+(e^{r\cos\theta})d\theta=\frac{1}{2\Pi}\int\limits_{-\frac{\Pi}{2}}^{\frac{\Pi}{2}}r\cos\theta d\theta=\frac{r}{\Pi}. \end{aligned}$$

Now,

$$\begin{aligned} I(r,f) &= \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right|d\theta=\frac{r}{2\pi}\int_0^{2\pi}\left|\frac{e^{re^{i\theta}}\cdot re^{i\theta}\cdot i}{e^{re^{i\theta}}}\right|d\theta=\frac{r}{2\pi}\int_0^{2\pi}\left|re^{i\theta}\cdot i\right|d\theta \\ &= \frac{r}{2\pi}\int_0^{2\pi}(r)\,d\theta=\frac{r^2}{2\pi}\int_0^{2\pi}d\theta=\frac{r^2}{2\pi}\cdot 2\pi=r^2\neq 0 \end{aligned}$$

and

$$\rho=\limsup_{r\to\infty}\frac{\log T(r;f)}{\log r}=\limsup_{r\to\infty}\frac{\log\frac{r}{\pi}}{\log r}=1.$$

Thus,

$${}_I\delta^{(0)}_{(n)}(0;f)={}_I\Delta^{(k)}_{(n)}(\infty;f)={}_I\delta^{(0)}_{n}(c;f)=1$$

and

$${}_I\Delta^{(0)}_{(n)}(\infty;f)={}_I\Delta^{(k)}_{(n)}(0;f)=1$$

So,

$${}_I\Delta^{(0)}_{(n)}(\infty;f)+{}_I\Delta^{(k)}_{(n)}(0;f)+1=1+1+1=3$$

and

$${}_I\Delta^{(0)}_{(n)}(\infty;f)+2{}_I\Delta^{(k)}_{(n)}(0;f)=1+2=3.$$

Then

$${}_I\delta^{(0)}_{(n)}(0;f)+{}_I\Delta^{(k)}_{(n)}(\infty;f)+{}_I\delta^{(0)}_{n}(c;f)\leq{}_I\Delta^{(0)}_{(n)}(\infty;f)+2{}_I\Delta^{(k)}_{(n)}(0;f).$$

Theorem 3.3. *Let f be an entire function of finite order $'\rho'$ such that $m(r,f)=S(r,f)$. If a and d are two non zero finite complex numbers then for any two positive integers k and p with $0\leq k<p$,*

$${}_I\delta^{(0)}_{(n)}(d;f)+{}_I\Delta^{(p)}_{(n)}(\infty;f)+{}_I\delta^{(k)}_{n}(a;f)\leq{}_I\Delta^{(k)}_{(n)}(\infty;f)+{}_I\Delta^{(p)}_{(n)}(0;f)+{}_I\Delta^{(k)}_{(n)}(0;f)$$

where n is any positive integer.

Proof. Let us consider the following identity

$$\frac{1}{f-d}=\left[\frac{1}{a}\left\{\frac{f^{(k)}}{f-a}-\frac{f^{(k)}-a}{f^{(p)}}.\frac{f^{(p)}}{f-a}\right\}\left\{\frac{f^{(k)}}{f-d}.\frac{1}{f^{(k)}}\right\}\right].(f-a).$$

Since $m(r,f-a)\leq m(r,f)+O(1)$, we get from the above identity

$$m\left(r,\frac{1}{f-d}\right)\leq m\left(r,\frac{f^{(k)}-a}{f^{(p)}}\right)+m\left(r,\frac{1}{f^{(k)}}\right)+m(r,f)+S(r,f)+O(1). \quad (7)$$

Now by Nevanlinna's first fundamental theorem and Milloux's theorem [5, p. 55], we obtain from Equation (7) that

$$\begin{aligned} m\left(r,\frac{1}{f-d}\right) &\le T\left(r,\frac{f^{(k)}-a}{f^{(p)}}\right)-N\left(r,\frac{f^{(k)}-a}{f^{(p)}}\right)+T\left(r,\frac{1}{f^{(k)}}\right) \\ &\quad -N\left(r,\frac{1}{f^{(k)}}\right)+m\left(r,f\right)+S\left(r,f\right)+O\left(1\right) \end{aligned}$$

$$\begin{aligned} i.e., m\left(r,\frac{1}{f-d}\right) &\le T\left(r,\frac{f^{(p)}}{f^{(k)}-a}\right)-N\left(r,\frac{f^{(k)}-a}{f^{(p)}}\right)+T\left(r,f^{(k)}\right) \\ &\quad -N\left(r,\frac{1}{f^{(k)}}\right)+m\left(r,f\right)+S\left(r,f\right)+O\left(1\right) \end{aligned}$$

$$\begin{aligned} i.e., m\left(r,\frac{1}{f-d}\right) &\le N\left(r,\frac{f^{(p)}}{f^{(k)}-a}\right)-N\left(r,\frac{f^{(k)}-a}{f^{(p)}}\right)+T\left(r,f^{(k)}\right) \\ &\quad -N\left(r,\frac{1}{f^{(k)}}\right)+m\left(r,f\right)+S\left(r,f\right)+O\left(1\right). \qquad (8) \end{aligned}$$

In view of [5, p. 34], we get from Equation (8) that

$$\begin{aligned} m\left(r,\frac{1}{f-d}\right) &\le N\left(r,f^{(p)}\right)+N\left(r,\frac{1}{f^{(k)}-a}\right)-N\left(r,f^{(k)}-a\right)-N\left(r,\frac{1}{f^{(p)}}\right) \\ &\quad +T\left(r,f^{(k)}\right)-N\left(r,\frac{1}{f^{(k)}}\right)+m\left(r,f\right)+S\left(r,f\right)+O\left(1\right). \qquad (9) \end{aligned}$$

In view of Lemma 2.5 and $m\left(r,f\right)=S\left(r,f\right)$, it follows from Equation (9) that

$$\begin{aligned} m\left(r,\frac{1}{f-d}\right) &\le N\left(r,f^{(p)}\right)+N\left(r,\frac{1}{f^{(k)}-a}\right)-N\left(r,f^{(k)}\right)-N\left(r,\frac{1}{f^{(p)}}\right) \\ &\quad -N\left(r,\frac{1}{f^{(k)}}\right)+T\left(r,f^{(k)}\right)+S\left(r,f\right) \end{aligned}$$

$$\begin{aligned} i.e., \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-d}\right)}{I\left(r,f^{(n)}\right)} &\le \liminf_{r\to\infty}\frac{N\left(r,f^{(p)}\right)}{I\left(r,f^{(n)}\right)}-\liminf_{r\to\infty}\frac{N\left(r,f^{(k)}\right)}{I\left(r,f^{(n)}\right)}-\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(p)}}\right)}{I\left(r,f^{(n)}\right)} \\ &\quad -\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(k)}}\right)}{I\left(r,f^{(n)}\right)}+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(k)}-a}\right)}{I\left(r,f^{(n)}\right)}+\lim_{r\to\infty}\frac{T\left(r,f^{(k)}\right)}{I\left(r,f^{(n)}\right)} \end{aligned}$$

$$\begin{aligned} i.e.,{}_I\,\delta^{(0)}_{(n)}\left(d;f\right)-\left(1-\frac{1}{\pi\rho}\right) &\le \left\{1-{}_I\Delta^{(p)}_{(n)}\left(\infty;f\right)\right\}-\left\{1-{}_I\Delta^{(k)}_{(n)}\left(\infty;f\right)\right\} \\ &\quad -\left\{1-{}_I\Delta^{(p)}_{(n)}\left(0;f\right)\right\}-\left\{1-{}_I\Delta^{(k)}_{(n)}\left(0;f\right)\right\} \\ &\quad +\left\{1-{}_I\delta^{(k)}_{n}\left(a;f\right)\right\}+\frac{1}{\pi\rho} \end{aligned}$$

$$i.e.,{}_I\,\delta^{(0)}_{(n)}\left(d;f\right)+{}_I\Delta^{(p)}_{(n)}\left(\infty;f\right)+{}_I\delta^{(k)}_{n}\left(a;f\right)\le {}_I\Delta^{(k)}_{(n)}\left(\infty;f\right)+{}_I\Delta^{(p)}_{(n)}\left(0;f\right)+{}_I\Delta^{(k)}_{(n)}\left(0;f\right).$$

This proves the theorem. □

Remark 3.4. *In Theorem 3.3, the inequality* $' \leq '$ *cannot be removed by* $'' <'$*only which can be seen from the following example.*

Example 5. Let $f = \exp(z^2)$ and $a = d = 0$.Then $N(r, f) = 0$,

$$\begin{aligned} T(r,f) &= m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|e^{r^2 e^{2i\theta}}\right| d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|e^{r^2(\cos 2\theta + i\sin 2\theta)}\right| d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left(e^{r^2\cos 2\theta}\right) d\theta = \frac{1}{2\pi}\int_0^{2\pi} r^2 \cos 2\theta d\theta = \frac{r^2}{\pi} \end{aligned}$$

and

$$\rho = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log^{[2]} e^{r^2}}{\log r} = \limsup_{r\to\infty} \frac{2\log r}{\log r} = 2.$$

Thus,

$$\begin{aligned} I(r,f) &= \frac{r}{2\pi}\int_0^{2\pi} \left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi} \frac{\left|e^{r^2 e^{2i\theta}}\right| \cdot \left|2ir^2 e^{2i\theta}\right|}{\left|e^{r^2 e^{2i\theta}}\right|} d\theta \\ &= \frac{r}{2\pi} \cdot 2r^2 \int_0^{2\pi} \frac{e^{r^2\cos 2\theta} \cdot e^{c\cos 2\theta}}{e^{r^2\cos 2\theta}} d\theta = \frac{r^3}{\pi}\int_0^{2\pi} e^{\cos 2\theta} d\theta \\ &= \frac{r^3}{\pi} \cdot \frac{1}{2}\int_0^{4\pi} e^{\cos\eta} d\eta = \frac{r^3}{2\pi} \cdot 4\pi I_0(1) = \frac{2r^3}{\pi} \cdot I_0(1) \neq 0 \end{aligned}$$

where $I_n(z)$ is the Modified Bessel Function of the first kind such that

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta} \cdot \cos n\theta d\theta.$$

Now,

$${}_I\delta_{(n)}^{(0)}(d; f) = {}_I\delta_{(n)}^{(0)}(0; f) = 1,\ {}_I\delta_{n}^{(k)}(a; f) = {}_I\delta_{n}^{(k)}(0; f) = 1$$

and

$${}_I\Delta_{(n)}^{(k)}(\infty; f) = {}_I\Delta_{(n)}^{(p)}(\infty; f) = {}_I\Delta_{(n)}^{(p)}(0; f) = {}_I\Delta_{(n)}^{(k)}(0; f) = 1.$$

Hence,

$${}_I\delta_{(n)}^{(0)}(d; f) + {}_I\Delta_{(n)}^{(p)}(\infty; f) + {}_I\delta_{n}^{(k)}(a; f) = 3 = {}_I\Delta_{(n)}^{(k)}(\infty; f) + {}_I\Delta_{(n)}^{(p)}(0; f) + {}_I\Delta_{(n)}^{(p)}(0; f).$$

Future Prospect: In the line of the works as carried out in the paper one may think of relative deficiencies of higher index in case of meromorphic functions with respect to another one on the basis of sharing of values of them. As a consequence, the derivation of relevant results in this field may be a virgin area of research to the future workers of this branch.

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