Korean J. Math. **19** (2011), No. 4, pp. 465–480

IDENTITIES ABOUT INFINITE SERIES CONTAINING HYPERBOLIC FUNCTIONS AND TRIGONOMETRIC FUNCTIONS

Sung-Geun Lim

Abstract. B. C. Berndt established many identities about infinite series. In this paper, continuing his work, we find new identities about infinite series containing hyperbolic functions and trigonometric functions.

1. Introduction and preliminaries

B. C. Berndt [2, 3] found a lot of identities about infinite series using a certain modular transformation formula that originally stems from the generalized Eisenstein series. It seems that all his findings on infinite series look like those found in the Notebooks of Ramanujan [6]. In fact, some of Berndt's results are stated in the Notebooks and others are generalizations of formulas of Ramanujan. Recently he gave a suggestion that analogous results of his work could be found from the modular transformation formula in [3]. Following his suggestion, the author derived a lot of new series relation between infinite series [4, 5]. In this paper, we find more of new series relations between infinite series, some of which are compared with series relations in [2, 3, 4]. For example, we find that, for $k < -1$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n \operatorname{csch}((2n+1)\pi/2)}{(2n+1)^{2k+2}} = (-1)^{k+1} 2^{-2k-2} \sum_{n=1}^{\infty} \frac{(-1)^n \operatorname{sech}(n\pi)}{n^{2k+2}}$$

Received November 5, 2011. Revised December 12, 2011. Accepted December 14, 2011.

2000 Mathematics Subject Classification: 34A25, 11M36.

Key words and phrases: infinite series, modular transformation, Eisenstein series.

and, for $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$,

$$\alpha^{1/2}\sum_{n=0}^{\infty}\operatorname{sech}((\alpha-\pi i)(2n+1)/(4c))$$
$$=(-\beta)^{1/2}\sum_{n=0}^{\infty}\operatorname{sech}((\beta+\pi i)(2n+1)/(4c)),$$

where c is a positive integer(See Corollary 2.9 and Corollary 2.19).

In this paper, we use the following notations. Let $e(w) = e^{2\pi i w}$. We choose the branch of the argument for a complex w with $-\pi \leq \arg w < \pi$. $V\tau = V(\tau) = (a\tau + b)/(c\tau + d)$ always denote a modular transformation with $c > 0$ for every complex τ. Let $r = (r_1, r_2)$ and $h = (h_1, h_2)$ denote real vectors, and the associated vectors R and H are defined by $R = (R_1, R_2) = (ar_1 + cr_2, br_1 + dr_2)$ and $H = (H_1, H_2) = (dh_1 - bh_2, -ch_1 + ah_2)$. Let λ denote the characteristic function of the integers. For a real number x, $[x]$ denotes the greatest integer less than or equal to x and $\{x\} := x - [x]$. For real α, x and $\mathrm{Re}(s) > 1$, let

$$\psi(x,\alpha,s) := \sum_{n+\alpha>0}\frac{e(nx)}{(n+\alpha)^s}. \tag{1.1}$$

If x is an integer and α is not an integer, then $\psi(x, \alpha, s) = \zeta(s, \{\alpha\})$, where $\zeta(s, x)$ is the Hurwitz zeta-function. The function $\psi(x, \alpha, s)$ can be analytically continued to the entire s-plane [1] except for a possible simple pole at $s = 1$ when x is an integer. Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, the upper half-plane. For $\tau \in \mathbb{H}$ and an arbitrary complex numbers s, define

$$A(\tau,s;r,h) := \sum_{m+r_1>0}\sum_{n-h_2>0}\frac{e\left(mh_1+((m+r_1)\tau+r_2)(n-h_2)\right)}{(n-h_2)^{1-s}}.$$

Let

$$H(\tau,s;r,h) := A(\tau,s;r,h) + e\left(s/2\right)A(\tau,s;-r,-h).$$

We now state the theorem which is important for our results.

Theorem 1.1. [2]. *Let* $Q = \{\tau \in \mathbb{C} \mid \mathrm{Re}(\tau) > -d/c\}$ *and* $\varrho = c\{R_2\} - d\{R_1\}$. *Then for* $\tau \in Q$ *and all* s,

$$\begin{aligned}&(c\tau+d)^{-s}H(V\tau,s;r,h) = H(\tau,s;R,H)\\&-\lambda(r_1)e(-r_1h_1)(c\tau+d)^{-s}\Gamma(s)(-2\pi i)^{-s}\left(\psi(h_2,r_2,s)+e\left(s/2\right)\psi(-h_2,-r_2,s)\right)\\&+\lambda(R_1)e(-R_1H_1)\Gamma(s)(-2\pi i)^{-s}\left(\psi(H_2,R_2,s)+e\left(-s/2\right)\psi(-H_2,-R_2,s)\right)\\&+(2\pi i)^{-s}L(\tau,s;R,H),\end{aligned}$$

where

$$
\begin{aligned}
&L(\tau, s; R, H)\\
&:= \sum_{j=1}^{c'} e(-H_1(j+[R_1]-c) - H_2([R_2]+1+[(jd+\varrho)/c]-d))\\
&\quad \cdot \int_C u^{s-1} \frac{e^{-(c\tau+d)(j-\{R_1\})u/c}}{e^{-(c\tau+d)u} - e(cH_1+dH_2)} \frac{e^{\{(jd+\varrho)/c\}u}}{e^u - e(-H_2)} du,
\end{aligned}
$$

where C is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in the positive direction so that $u = 0$ is the only zero of

$$\left(e^{-(c\tau+d)u} - e(cH_1+dH_2)\right)\left(e^u - e(-H_2)\right)$$

lying "inside" the loop, and then returning to $+\infty$ in the lower half plane. Here, we choose the branch of u^s with $0 < \arg u < 2\pi$.

Remark 1.2. Theorem 1.1 is true for $\tau \in Q$. But, after the evaluation of $L(\tau, s; R, H)$ for an integer s, it will be valid for all $\tau \in \mathbb{H}$ by analytic continuation.

We shall use two polynomials. One is the Bernoulli polynomials $B_n(x)$, $n \geq 0$, defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \ (|t| < 2\pi).$$

The n-th Bernoulli number B_n, $n \geq 0$, is defined by $B_n = B_n(0)$. Put $\bar{B}_n(x) = B_n(\{x\}),\ n \geq 0$. The other is the Euler polynomials $E_n(x)$, $n \geq 0$, defined by

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \ (|t| < \pi).$$

The Euler numbers E_n are defined by

$$E_n := 2^n E_n\left(\frac{1}{2}\right),\ n \geq 0.$$

Put $\bar{E}_n(x) = E_n(\{x\}),\ n \geq 0$.

2. Infinite series identities

From now on, we let V be a modular transformation corresponding to

$$\begin{pmatrix} 1 & -1 \\ c & 1-c \end{pmatrix}$$

for $c > 0$. Put $r = (r_1, r_2/c)$. Then

$$R_1 = r_1 + r_2, \ R_2 = -r_1 - r_2 + \frac{r_2}{c}.$$

Replacing $c\tau + 1 - c$ by z, we have

$$V\tau = \frac{1}{c} - \frac{1}{cz}, \ \tau = 1 - \frac{1}{c} + \frac{1}{c}z.$$

If $\tau \in Q$, then Re $z > 0$ and $z \in \mathbb{H}$. By Remark 1.2, we shall put $z = \pi i/\alpha$ for a positive real number α. In this section, we consider three cases of $h = (h_1, h_2)$, i.e., $h = (1/2, 1/2)$, $(1/2, 0)$ and $(0, 1/2)$. We also suppose that r_1 and r_2 are not integers. In this case, $\lambda(r_1) = \lambda(R_1) = 0$. By Theorem 1.1, for any integer m and $z \in \mathbb{H}$ with Re $z > 0$,

$$z^m H(V\tau, -m; r, h) = H(\tau, -m; R, H) + (2\pi i)^m L(\tau, -m; R, H). \tag{2.1}$$

For r_1 not an integer,

$$\begin{aligned} H(V\tau, s; r, h) = e(-[r_1]h_1) &\sum_{n-h_2>0} \frac{e((\{r_1\}V\tau + r_2/c)(n-h_2))}{(n-h_2)^{1-s}(1-e(h_1+V\tau(n-h_2)))} \\ +e^{\pi is}e(-([r_1]+1)h_1) &\sum_{n+h_2>0} \frac{e((((1-\{r_1\})V\tau - r_2/c)(n+h_2))}{(n+h_2)^{1-s}(1-e(-h_1+V\tau(n+h_2)))}, \end{aligned} \tag{2.2}$$

and, for R_1 not an integer,

$$\begin{aligned} H(\tau, s; R, H) = e(-[R_1]H_1) &\sum_{n-H_2>0} \frac{e((\{R_1\}\tau + R_2)(n-H_2))}{(n-H_2)^{1-s}(1-e(H_1+\tau(n-H_2)))} \\ +e^{\pi is}e(-([R_1]+1)H_1) &\sum_{n+H_2>0} \frac{e(((1-\{R_1\})\tau - R_2)(n+H_2))}{(n+H_2)^{1-s}(1-e(-H_1+\tau(n+H_2)))}. \end{aligned} \tag{2.3}$$

THEOREM 2.1. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. Let r_1 and r_2 be real numbers such that r_1 and $r_1 + r_2$ are not integers. Then, for any integer k and for any positive even integer c,*

$$\begin{aligned} &(-1)^{[r_1]}\alpha^{-k} \sum_{n=0}^{\infty} \frac{\sinh(((2\{r_1\}-1)(\alpha-\pi i) - 2\pi i r_2)(2n+1)/(2c))}{(2n+1)^{2k+1}\cosh((\alpha-\pi i)(2n+1)/(2c))} \\ &= (-1)^{[r_1+r_2]}(-\beta)^{-k} \sum_{n=0}^{\infty} \frac{\sinh(((2\{r_1+r_2\}-1)(\beta+\pi i) - 2\pi i r_2)(2n+1)/(2c))}{(2n+1)^{2k+1}\cosh((\beta+\pi i)(2n+1)/(2c))} \end{aligned}$$

$$+\frac{(-1)^{[r_1+r_2]}}{4}\sum_{j=1}^{c}(-1)^{j+[(j+r_2-\{r_1+r_2\})/c]}\sum_{\ell=0}^{2k}\frac{E_\ell((j-\{r_1+r_2\})/c)}{\ell!}$$
$$\cdot\frac{\bar{E}_{2k-\ell}((j+r_2-\{r_1+r_2\})/c)}{(2k-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell},$$

and, for any positive odd integer c,

$$(-1)^{[r_1]}\alpha^{-k}\sum_{n=0}^{\infty}\frac{\sinh\left(((2\{r_1\}-1)(\alpha-\pi i)-2\pi i r_2)(2n+1)/(2c)\right)}{(2n+1)^{2k+1}\cosh\left((\alpha-\pi i)(2n+1)/(2c)\right)}$$
$$=\frac{(-1)^{[r_1+r_2]}}{2^{2k+1}}(-\beta)^{-k}\sum_{n=1}^{\infty}\frac{\sinh\left(((2\{r_1+r_2\}-1)(\beta+\pi i)-2\pi i r_2)n/c\right)}{n^{2k+1}\cosh\left((\beta+\pi i)n/c\right)}$$
$$+\frac{(-1)^{[r_1+r_2]}}{2}\sum_{j=1}^{c}(-1)^{j+1}\sum_{\ell=0}^{2k+1}\frac{E_\ell((j-\{r_1+r_2\})/c)}{\ell!}$$
$$\cdot\frac{\bar{B}_{2k+1-\ell}((j+r_2-\{r_1+r_2\})/c)}{(2k+1-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell}.$$

Proof. Let $h=(1/2,1/2)$ and $m=2k$ in (2.1). Then we have from (2.2) that

$$H(V\tau,-2k;r,h)$$
$$=(-1)^{[r_1]}2^{2k+1}\sum_{n=1}^{\infty}\frac{e((\{r_1\}(1-1/z)+r_2)(2n-1)/(2c))}{(2n-1)^{2k+1}(1+e((1-1/z)(2n-1)/(2c)))}$$
$$-(-1)^{[r_1]}2^{2k+1}\sum_{n=1}^{\infty}\frac{e(((1-\{r_1\})(1-1/z)-r_2)(2n-1)/(2c))}{(2n-1)^{2k+1}(1+e((1-1/z)(2n-1)/(2c)))}$$
$$=(-1)^{[r_1]}2^{2k+1}\sum_{n=1}^{\infty}\frac{\sinh(\pi i((2\{r_1\}-1)(1-1/z)+2r_2)(2n-1)/(2c))}{(2n-1)^{2k+1}\cosh(\pi i(1-1/z)(2n-1)/(2c))}. \tag{2.4}$$

If c is even, then $\{H_1\}=0$ and $\{H_2\}=1/2$. Thus, for c even, it follows from (2.3) that

$$H(\tau,-2k;R,H)$$
$$=2^{2k+1}\sum_{n=0}^{\infty}\frac{e^{\pi i(\{R_1\}\tau+R_2)(2n+1)}+e^{-\pi i(\{R_1\}\tau+R_2)(2n+1)}e^{\pi i\tau(2n+1)}}{(2n+1)^{2k+1}(1-e^{\pi i\tau(2n+1)})}$$
$$=(-1)^{[r_1+r_2]}2^{2k+1}\sum_{n=0}^{\infty}\frac{\sinh(\pi i((2\{r_1+r_2\}-1)(z-1)+2r_2)(2n+1)/(2c))}{(2n+1)^{2k+1}\cosh(\pi i(z-1)(2n+1)/(2c))}. \tag{2.5}$$

If c is odd, then $\{H_1\}=1/2$ and $\{H_2\}=0$. So, for c odd, (2.3) gives

$$H(\tau,-2k;R,H)$$
$$=(-1)^{[R_1]}\sum_{n=1}^{\infty}\frac{e((\{R_1\}\tau+R_2)n))-e(-(\{R_1\}\tau+R_2)n))e(\tau n)}{n^{2k+1}(1+e(\tau n))}$$

$$= (-1)^{[r_1+r_2]} \sum_{n=1}^{\infty} \frac{\sinh(\pi i((2\{r_1+r_2\}-1)(z-1)+2r_2)n/c)}{n^{2k+1}\cosh(\pi i(z-1)n/c)}. \tag{2.6}$$

We see that

$$\frac{e^{-zu(j-\{R_1\})/c}}{e^{-zu}+1} = \frac{1}{2}\sum_{n=0}^{\infty} E_n\left(\frac{j-\{R_1\}}{c}\right)\frac{(-zu)^n}{n!},$$
$$\frac{e^{\{(j(1-c)+\varrho)/c\}u}}{e^u+1} = \frac{1}{2}\sum_{n=0}^{\infty} \bar{E}_n\left(\frac{j+\varrho}{c}\right)\frac{u^n}{n!},$$
$$\frac{e^{\{(j(1-c)+\varrho)/c\}u}}{e^u-1} = u^{-1}\sum_{n=0}^{\infty} \bar{B}_n\left(\frac{j+\varrho}{c}\right)\frac{u^n}{n!},$$

and

$$\left[\frac{j(1-c)+\varrho}{c}\right] = -j-[R_1]-[R_2]+\left[\frac{j+r_2-\{R_1\}}{c}\right].$$

Then, in case of c even, we have that

$$\begin{aligned} &L(\tau,-2k;R,H)\\ &= \frac{1}{4}\sum_{j=1}^{c} e\left(-\frac{1}{2}\left([R_2]+c+\left[\frac{j(1-c)+\varrho}{c}\right]\right)\right)\\ &\qquad\cdot\int_C u^{-2k-1}\sum_{n=0}^{\infty}E_n\left(\frac{j-\{R_1\}}{c}\right)\frac{(-zu)^n}{n!}\sum_{m=0}^{\infty}\bar{E}_m\left(\frac{j+\varrho}{c}\right)\frac{u^m}{m!}\,du\\ &= \frac{(-1)^{[r_1+r_2]}}{2}\pi i\sum_{j=1}^{c}(-1)^{j+[(j+r_2-\{r_1+r_2\})/c]}\sum_{\ell=0}^{2k}\frac{E_\ell((j-\{r_1+r_2\})/c)}{\ell!}\\ &\qquad\cdot\frac{\bar{E}_{2k-\ell}((j+r_2-\{r_1+r_2\})/c)}{(2k-\ell)!}(-z)^\ell \end{aligned} \tag{2.7}$$

and, in case of c odd,

$$\begin{aligned} &L(\tau,-2k;R,H)\\ &= \frac{1}{2}\sum_{j=1}^{c} e\left(-\frac{1}{2}(j+[R_1]-c)\right)\\ &\qquad\cdot\int_C u^{-2k-2}\sum_{n=0}^{\infty}E_n\left(\frac{j-\{R_1\}}{c}\right)\frac{(-zu)^n}{n!}\sum_{m=0}^{\infty}\bar{B}_m\left(\frac{j+\varrho}{c}\right)\frac{u^m}{m!}\,du\\ &= (-1)^{[r_1+r_2]+1}\pi i\sum_{j=1}^{c}(-1)^j\sum_{\ell=0}^{2k+1}\frac{E_\ell((j-\{r_1+r_2\})/c)}{\ell!}\\ &\qquad\cdot\frac{\bar{B}_{2k+1-\ell}((j+r_2-\{r_1+r_2\})/c)}{(2k+1-\ell)!}(-z)^\ell. \end{aligned} \tag{2.8}$$

Now, plugging (2.4), (2.5), (2.6), (2.7) and (2.8) into (2.1) and letting $z = \pi i/\alpha$, we prove the theorem. □

COROLLARY 2.2. *Let* r_1 *be a real number with* $0 < r_1 < 1$. *Then*

$$\alpha^{-k} \sum_{n=0}^{\infty} \frac{\cosh((2n+1)(2r_1-1)\alpha/2)\cos((2n+1)\pi r_1)}{(2n+1)^{2k+1}\sinh((2n+1)\alpha/2)}$$
$$= -2^{-2k-1}(-\beta)^{-k} \sum_{n=1}^{\infty} \frac{\sinh((2r_1-1)n\beta)\cos(2\pi n r_1)}{n^{2k+1}\cosh(n\beta)}$$
$$-\frac{1}{2}\sum_{\ell=0}^{k} \frac{E_{2\ell+1}(1-r_1)B_{2k-2\ell}(1-r_1)}{(2\ell+1)!(2k-2\ell)!}\alpha^{k-\ell+1}(-\beta)^{\ell+1}.$$

Proof. Put $c = 1$, $r_2 = 0$ and let $0 < r_1 < 1$ in Theorem 2.1 and equate the real parts. □

COROLLARY 2.3. *Let* r_1 *be a real number with* $0 < r_1 < 1$. *Then*

$$\alpha^{-k} \sum_{n=0}^{\infty} \frac{\sinh((2n+1)(2r_1-1)\alpha/2)\sin((2n+1)\pi r_1)}{(2n+1)^{2k+1}\sinh((2n+1)\alpha/2)}$$
$$= 2^{-2k-1}(-\beta)^{-k} \sum_{n=1}^{\infty} \frac{\cosh((2r_1-1)n\beta)\sin(2\pi n r_1)}{n^{2k+1}\cosh(n\beta)}$$
$$-\frac{\pi}{2}\sum_{\ell=0}^{k} \frac{E_{2\ell}(1-r_1)B_{2k+1-2\ell}(1-r_1)}{(2\ell)!(2k+1-2\ell)!}\alpha^{k-\ell}(-\beta)^{\ell}.$$

Proof. Put $c = 1$, $r_2 = 0$ and let $0 < r_1 < 1$ in Theorem 2.1 and equate the imaginary parts. □

THEOREM 2.4. *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2$. *Let* r_1 *and* r_2 *be real numbers such that* r_1 *and* $r_1 + r_2$ *are not integers. Then, for any integer* k *and for any positive even integer* c,

$$(-1)^{[r_1]}\alpha^{-k-1/2} \sum_{n=0}^{\infty} \frac{\cosh(((2\{r_1\}-1)(\alpha-\pi i)-2\pi i r_2)(2n+1)/(2c))}{(2n+1)^{2k+2}\cosh((\alpha-\pi i)(2n+1)/(2c))}$$
$$= (-1)^{[r_1+r_2]}(-\beta)^{-k-1/2}$$
$$\cdot\sum_{n=0}^{\infty} \frac{\cosh(((2\{r_1+r_2\}-1)(\beta+\pi i)-2\pi i r_2)(2n+1)/(2c))}{(2n+1)^{2k+2}\cosh((\beta+\pi i)(2n+1)/(2c))}$$
$$-\frac{(-1)^{[r_1+r_2]}}{4}\sum_{j=1}^{c}(-1)^{j+[(j+r_2-\{r_1+r_2\})/c]}\sum_{\ell=0}^{2k+1}\frac{E_\ell((j-\{r_1+r_2\})/c)}{\ell!}$$
$$\cdot\frac{\bar{E}_{2k+1-\ell}((j+r_2-\{r_1+r_2\})/c)}{(2k+1-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell+1/2},$$

and, for any positive odd integer c,

$$(-1)^{[r_1]}\alpha^{-k-1/2} \sum_{n=0}^{\infty} \frac{\cosh(((2\{r_1\}-1)(\alpha-\pi i)-2\pi i r_2)(2n+1)/(2c))}{(2n+1)^{2k+2}\cosh((\alpha-\pi i)(2n+1)/(2c))}$$

$$= \frac{(-1)^{[r_1+r_2]}}{2^{2k+2}}(-\beta)^{-k-1/2}\sum_{n=1}^{\infty}\frac{\cosh\left(((2\{r_1+r_2\}-1)(\beta+\pi i)-2\pi i r_2)n/c\right)}{n^{2k+2}\cosh\left((\beta+\pi i)n/c\right)}$$
$$-\frac{(-1)^{[r_1+r_2]}}{2}\sum_{j=1}^{c}(-1)^{j+1}\sum_{\ell=0}^{2k+2}\frac{E_\ell((j-\{r_1+r_2\})/c)}{\ell!}$$
$$\cdot\frac{\bar{B}_{2k+2-\ell}((j+r_2-\{r_1+r_2\})/c)}{(2k+2-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell+1/2}.$$

Proof. Let $h=(1/2,1/2)$ and $m=2k+1$ in (2.1). By the same way as we derived equations (2.4), (2.5), (2.6), (2.7) and (2.8), we obtain the followings;

$$H(V\tau,-2k-1;r,h) = (-1)^{[r_1]}2^{2k+2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2k+2}}$$
$$\cdot\frac{\cosh(\pi i((2\{r_1\}-1)(1-1/z)+2r_2)(2n-1)/(2c))}{\cosh(\pi i(1-1/z)(2n-1)/(2c))}, \tag{2.9}$$

for c even,

$$H(\tau,-2k-1;R,H) = (-1)^{[r_1+r_2]}2^{2k+2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{2k+2}}$$
$$\cdot\frac{\cosh(\pi i((2\{r_1+r_2\}-1)(z-1)+2r_2)(2n+1)/(2c))}{\cosh(\pi i(z-1)(2n+1)/(2c))}, \tag{2.10}$$

$$L(\tau,-2k-1;R,H) = \frac{(-1)^{[r_1+r_2]}}{2}\pi i\sum_{j=1}^{c}(-1)^{j+[(j+r_2-\{r_1+r_2\})/c]}$$
$$\cdot\sum_{\ell=0}^{2k+1}\frac{E_\ell((j-\{r_1+r_2\})/c)\bar{E}_{2k+1-\ell}((j+r_2-\{r_1+r_2\})/c)}{\ell!(2k+1-\ell)!}(-z)^\ell, \tag{2.11}$$

for c odd,

$$H(\tau,-2k-1;R,H) \tag{2.12}$$
$$= (-1)^{[r_1+r_2]}\sum_{n=1}^{\infty}\frac{\cosh(\pi i((2\{r_1+r_2\}-1)(z-1)+2r_2)n/c)}{n^{2k+2}\cosh(\pi i(z-1)n/c)},$$

$$L(\tau,-2k;R,H) = (-1)^{[r_1+r_2]+1}\pi i\sum_{j=1}^{c}(-1)^j\sum_{\ell=0}^{2k+2}\frac{E_\ell((j-\{r_1+r_2\})/c)}{\ell!}$$
$$\cdot\frac{\bar{B}_{2k+2-\ell}((j+r_2-\{r_1+r_2\})/c)}{(2k+2-\ell)!}(-z)^\ell. \tag{2.13}$$

Now let $z=\pi i/\alpha$, put (2.9), (2.10), (2.11), (2.12) and (2.13) into (2.1), and obtain the desired results. □

COROLLARY 2.5. *Let r_1 be a real number with $0 < r_1 < 1$. Then*

$$\begin{aligned}
&\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{\sinh\left((2n+1)(2r_1-1)\alpha/2\right)\cos((2n+1)\pi r_1)}{(2n+1)^{2k+2}\,\sinh\left((2n+1)\alpha/2\right)}\\
&=(-1)^{k+1}2^{-2k-2}\beta^{-k-1/2}\sum_{n=1}^{\infty}\frac{\sinh\left((2r_1-1)n\beta\right)\sin(2\pi n r_1)}{n^{2k+2}\,\cosh(n\beta)}\\
&\quad+\frac{1}{2}\sum_{\ell=0}^{k}\frac{E_{2\ell+1}(1-r_1)E_{2k+1-2\ell}(1-r_1)}{(2\ell+1)!(2k+1-2\ell)!}\alpha^{k-\ell+1/2}(-\beta)^{\ell+1}.
\end{aligned}$$

Proof. Put $c = 1$, $r_2 = 0$ and let $0 < r_1 < 1$ in Theorem 2.4 and equate the real parts. □

COROLLARY 2.6. *Let r_1 be a real number with $0 < r_1 < 1$. Then*

$$\begin{aligned}
&\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{\cosh\left((2n+1)(2r_1-1)\alpha/2\right)\sin((2n+1)\pi r_1)}{(2n+1)^{2k+2}\,\sinh\left((2n+1)\alpha/2\right)}\\
&=(-1)^{k+1}2^{-2k-2}\beta^{-k-1/2}\sum_{n=1}^{\infty}\frac{\cosh\left((2r_1-1)n\beta\right)\cos(2\pi n r_1)}{n^{2k+2}\,\cosh(n\beta)}\\
&\quad+\frac{\pi}{2}\sum_{\ell=0}^{k}\frac{E_{2\ell}(1-r_1)E_{2k+2-2\ell}(1-r_1)}{(2\ell)!(2k+2-2\ell)!}\alpha^{k-\ell+1/2}(-\beta)^{\ell}.
\end{aligned}$$

Proof. Put $c = 1$, $r_2 = 0$ and let $0 < r_1 < 1$ in Theorem 2.4 and equate the imaginary parts. □

COROLLARY 2.7. *For any positive even integer c,*

$$\begin{aligned}
&\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{\operatorname{sech}\left((\alpha-\pi i)(2n+1)/(2c)\right)}{(2n+1)^{2k+2}}\\
&=(-\beta)^{-k-1/2}\sum_{n=0}^{\infty}\frac{\operatorname{sech}\left((\beta+\pi i)(2n+1)/(2c)\right)}{(2n+1)^{2k+2}}\\
&\quad-\frac{1}{4}\sum_{j=1}^{c}(-1)^j\sum_{\ell=0}^{2k+1}\frac{E_\ell((j-1/2)/c)E_{2k+1-\ell}((j-1/2)/c)}{\ell!(2k+1-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell+1/2},
\end{aligned}$$

and, for any positive odd integer c,

$$\begin{aligned}
&\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{\operatorname{sech}\left((\alpha-\pi i)(2n+1)/(2c)\right)}{(2n+1)^{2k+2}}\\
&=2^{-2k-2}(-\beta)^{-k-1/2}\sum_{n=1}^{\infty}\frac{\operatorname{sech}\left((\beta+\pi i)n/c\right)}{n^{2k+2}}\\
&\quad+\frac{1}{2}\sum_{j=1}^{c}(-1)^j\sum_{\ell=0}^{2k+2}\frac{E_\ell((j-1/2)/c)E_{2k+2-\ell}((j-1/2)/c)}{\ell!(2k+2-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell+1/2}.
\end{aligned}$$

Proof. Put $r_1 = 1/2$ and $r_2 = 0$ in Theorem 2.4. □

COROLLARY 2.8.

$$\begin{aligned}&\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{(-1)^n\operatorname{csch}((2n+1)\alpha/2)}{(2n+1)^{2k+2}}\\&=(-1)^{k+1}2^{-2k-2}\beta^{-k-1/2}\sum_{n=1}^{\infty}\frac{(-1)^n\operatorname{sech}(n\beta)}{n^{2k+2}}\\&\quad+\frac{\pi}{2}\sum_{\ell=0}^{k+1}\frac{E_{2\ell}(1/2)E_{2k+2-2\ell}(1/2)}{(2\ell)!(2k+2-2\ell)!}\alpha^{k-\ell+1/2}(-\beta)^{\ell}.\end{aligned}$$

Proof. Put $c=1$ in Corollary 2.7 and apply $E_{2n+1}\left(\frac{1}{2}\right)=0,\ n\geq 0$. □

COROLLARY 2.9. *For $k<-1$,*

$$\sum_{n=0}^{\infty}\frac{(-1)^n\operatorname{csch}((2n+1)\pi/2)}{(2n+1)^{2k+2}}=(-1)^{k+1}2^{-2k-2}\sum_{n=1}^{\infty}\frac{(-1)^n\operatorname{sech}(n\pi)}{n^{2k+2}}$$

Proof. Put $c=1,\ \alpha=\beta=\pi$ in Corollary 2.7 and let $k<-1$. □

COROLLARY 2.10.

$$\sum_{n=0}^{\infty}(-1)^n\operatorname{csch}((2n+1)\pi/2)=\sum_{n=1}^{\infty}(-1)^n\operatorname{sech}(n\pi)+\frac{1}{2}$$

Proof. Put $c=1,\ k=-1$ and $\alpha=\beta=\pi$ in Corollary 2.7 . □

THEOREM 2.11. *Let $\alpha,\ \beta>0$ with $\alpha\beta=\pi^2$. Let r_1 and r_2 be real numbers such that r_1 and r_1+r_2 are not integers. Then, for any integer k and for any positive even integer c,*

$$\begin{aligned}&(-1)^{[r_1]}\alpha^{-k}\sum_{n=1}^{\infty}\frac{\sinh\left(((2\{r_1\}-1)(\alpha-\pi i)-2\pi i r_2)n/c\right)}{n^{2k+1}\cosh\left((\alpha-\pi i)n/c\right)}\\&=(-1)^{[r_1+r_2]}(-\beta)^{-k}\sum_{n=1}^{\infty}\frac{\sinh\left(((2\{r_1+r_2\}-1)(\beta+\pi i)-2\pi i r_2)n/c\right)}{n^{2k+1}\cosh\left((\beta+\pi i)n/c\right)}\\&\quad+(-1)^{[r_1+r_2]}2^{2k+1}\sum_{j=1}^{c}(-1)^j\sum_{\ell=0}^{2k+2}\frac{B_\ell((j-\{r_1+r_2\})/c)}{\ell!}\\&\qquad\cdot\frac{\bar{B}_{2k+2-\ell}((j+r_2-\{r_1+r_2\})/c)}{(2k+2-\ell)!}(-\pi i)^{\ell}\alpha^{k-\ell+1}.\end{aligned}$$

Proof. Let $h=(1/2,0)$ and $m=2k$ in (2.1). Then it follows from (2.2) that

$$\begin{aligned}&H(V\tau,-2k;r,h)\\&=(-1)^{[r_1]}\sum_{n=1}^{\infty}\frac{e((\{r_1\}(1-1/z)+r_2)n/c)-e(((1-\{r_1\})(1-1/z)-r_2)n/c)}{n^{2k+1}(1+e((1-1/z)n/c))}\\&=(-1)^{[r_1]}\sum_{n=1}^{\infty}\frac{\sinh(\pi i((2\{r_1\}-1)(1-1/z)+2r_2)n/c)}{n^{2k+1}\cosh(\pi i(1-1/z)n/c)},\end{aligned}\tag{2.14}$$

$$
\begin{aligned}
&H(\tau,-2k;R,H)\\
&= (-1)^{[r_1+r_2]}\sum_{n=1}^{\infty}\frac{e((\{r_1+r_2\}(z-1)+r_2)n/c)-e(-(\{r_1+r_2\}(z-1)+r_2)n/c)}{n^{2k+1}(1+e((z-1)n/c))}\\
&= (-1)^{[r_1+r_2]}\sum_{n=1}^{\infty}\frac{\sinh(\pi i((2\{r_1+r_2\}-1)(z-1)+2r_2)n/c)}{n^{2k+1}\cosh(\pi i(z-1)n/c)}. \qquad (2.15)
\end{aligned}
$$

Since c is even, $cH_1+(1-c)H_2\equiv H_2\equiv 0 \pmod 1$. We use that

$$
\begin{aligned}
\frac{e^{-zu(j-\{R_1\})/c}}{e^{-zu}-1} &= (-zu)^{-1}\sum_{m=0}^{\infty}B_m\left(\frac{j-\{R_1\}}{c}\right)\frac{(-zu)^m}{m!},\\
\frac{e^{\{(j(1-c)+\varrho)/c\}u}}{e^{u}-1} &= u^{-1}\sum_{m=0}^{\infty}\bar{B}_m\left(\frac{j(1-c)+\varrho}{c}\right)\frac{u^m}{m!}
\end{aligned}
$$

and

$$
\left\{\frac{j(1-c)+\varrho}{c}\right\}=\left\{\frac{j+r_2-\{r_1+r_2\}}{c}\right\}.
$$

Then by the residue theorem we have

$$
\begin{aligned}
&L(\tau,-2k;R,H)\\
&= (-z)^{-1}\sum_{j=1}^{c}e^{-\pi i(j+[R_1]-c)}\int_C u^{-2k-3}\sum_{m=0}^{\infty}B_m\left(\frac{j-\{R_1\}}{c}\right)\frac{(-zu)^m}{m!}\\
&\qquad\qquad \cdot\sum_{n=0}^{\infty}\bar{B}_n\left(\frac{j(1-c)+\varrho}{c}\right)\frac{u^n}{n!}du\\
&= (-1)^{[r_1+r_2]}2\pi i\sum_{j=1}^{c}(-1)^j\sum_{\ell=0}^{2k+2}\frac{B_\ell((j-\{r_1+r_2\})/c)}{\ell!}\\
&\qquad\qquad \cdot\frac{\bar{B}_{2k+2-\ell}((j+r_2-\{r_1+r_2\})/c)}{(2k+2-\ell)!}(-z)^{\ell-1}. \qquad (2.16)
\end{aligned}
$$

Employing (2.14), (2.15) and (2.16) in (2.1) with $z=\pi i/\alpha$, we complete the proof. □

For c odd, if we put $h=(1/2,0)$, $m=2k$ and $z=\pi i/\alpha$ into (2.1), then we obtain the complex conjugate of the second series identity in Theorem 2.1.

THEOREM 2.12. *Let α, $\beta>0$ with $\alpha\beta=\pi^2$. Let r_1 and r_2 be real numbers such that r_1 and r_1+r_2 are not integers. Then, for any integer k and for any positive even integer c,*

$$
(-1)^{[r_1]}\alpha^{-k-1/2}\sum_{n=1}^{\infty}\frac{\cosh(((2\{r_1\}-1)(\alpha-\pi i)-2\pi i r_2)n/c)}{n^{2k+2}\cosh((\alpha-\pi i)n/c)}
$$

$$= (-1)^{[r_1+r_2]}(-\beta)^{-k-1/2}\sum_{n=1}^{\infty}\frac{\cosh\left(((2\{r_1+r_2\}-1)(\beta+\pi i)-2\pi i r_2)n/c\right)}{n^{2k+2}\cosh\left((\beta+\pi i)n/c\right)}$$
$$-(-1)^{[r_1+r_2]}2^{2k+2}\sum_{j=1}^{c}(-1)^j\sum_{\ell=0}^{2k+3}\frac{B_\ell((j-\{r_1+r_2\})/c)}{\ell!}$$
$$\cdot\frac{\bar{B}_{2k+3-\ell}((j+r_2-\{r_1+r_2\})/c)}{(2k+3-\ell)!}(-\pi i)^\ell\alpha^{k-\ell+3/2}.$$

Proof. Let $h=(1/2,0)$ and let $m=2k+1$ in (2.1). In similar to (2.14), (2.15) and (2.16), we obtain that

$$H(V\tau,-2k-1;r,h) = (-1)^{[r_1]}\sum_{n=1}^{\infty}\frac{\cosh(\pi i((2\{r_1\}-1)(1-1/z)+2r_2)n/c)}{n^{2k+1}\cosh(\pi i(1-1/z)n/c)}, \tag{2.17}$$

$$H(\tau,-2k-1;R,H) = (-1)^{[r_1+r_2]}\sum_{n=1}^{\infty}\frac{\cosh(\pi i((2\{r_1+r_2\}-1)(z-1)+2r_2)n/c)}{n^{2k+1}\cosh(\pi i(z-1)n/c)} \tag{2.18}$$

and

$$L(\tau,-2k;R,H) = (-1)^{[r_1+r_2]}2\pi i\sum_{j=1}^{c}(-1)^j\sum_{\ell=0}^{2k+3}\frac{B_\ell((j-\{r_1+r_2\})/c)}{\ell!} \cdot\frac{\bar{B}_{2k+3-\ell}((j+r_2-\{r_1+r_2\})/c)}{(2k+3-\ell)!}(-z)^{\ell-1}. \tag{2.19}$$

Applying (2.14), (2.15) and (2.16) to (2.1), we arrive at the desired results. □

If $h=(1/2,0)$, $m=2k+1$ and $z=\pi i/\alpha$ in (2.1) when c is odd, then we obtain the complex conjugate of the second series identity in Theorem 2.4. If $r_1=1/2$ and $r_2=0$ in Theorem 2.12, then we obtain Corollary 3.10 in [4].

THEOREM 2.13. *Let α, $\beta>0$ with $\alpha\beta=\pi^2$. Let r_1 and r_2 be real numbers such that r_1 and r_1+r_2 are not integers. Then, for any integer k and for any positive integer c,*

$$\alpha^{-k}\sum_{n=0}^{\infty}\frac{\cosh\left(((2\{r_1\}-1)(\alpha-\pi i)-2\pi i r_2)(2n+1)/(2c)\right)}{(2n+1)^{2k+1}\sinh\left((\alpha-\pi i)(2n+1)/(2c)\right)}$$
$$= (-\beta)^{-k}\sum_{n=0}^{\infty}\frac{\cosh\left(((2\{r_1+r_2\}-1)(\beta+\pi i)-2\pi i r_2)(2n+1)/(2c)\right)}{(2n+1)^{2k+1}\sinh\left((\beta+i\pi)(2n+1)/(2c)\right)}$$
$$-\frac{1}{4}\sum_{j=1}^{c}(-1)^{[(j+r_2-\{r_1+r_2\})/c]}\sum_{\ell=0}^{2k}\frac{E_\ell((j-\{r_1+r_2\})/c)}{\ell!}$$

$$\cdot \frac{\bar{E}_{2k-\ell}((j+r_2-\{r_1+r_2\})/c)}{(2k-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell}.$$

Proof. Let $z=\pi i/\alpha$, $h=(0,1/2)$ and $m=2k$ in (2.1). We obtain from (2.2) and (2.3) that

$$\begin{aligned} &H(V\tau,-2k;r,h)\\ &= 2^{2k+1}\sum_{n=0}^{\infty}\frac{e^{\pi i(\{r_1\}(1-1/z)+r_2)(2n+1)/c}+e^{\pi i((1-\{r_1\})(1-1/z)-r_2)(2n+1)/c}}{(2n+1)^{2k+1}(1-e((1-1/z)(2n+1)/(2c)))}\\ &= 2^{2k+1}\sum_{n=0}^{\infty}\frac{\cosh(\pi i((2\{r_1\}-1)(1-1/z)+2r_2)(2n+1)/(2c))}{(2n+1)^{2k+1}\sinh(\pi i(1/z-1)(2n+1)/(2c))} \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} &H(\tau,-2k;R,H)\\ &= 2^{2k+1}\sum_{n=0}^{\infty}\frac{e^{\pi i(\{r_1+r_2\}(z-1)+r_2)(2n+1)/c}+e^{\pi i((1-\{r_1+r_2\})(z-1)-r_2)(2n+1)/c}}{(2n+1)^{2k+1}(1-e((z-1)(2n+1)/(2c)))}\\ &= 2^{2k+1}\sum_{n=0}^{\infty}\frac{\cosh(\pi i((2\{r_1+r_2\}-1)(z-1)+2r_2)(2n+1)/(2c))}{(2n+1)^{2k+1}\sinh(\pi i(1-z)(2n+1)/(2c))}. \end{aligned} \tag{2.21}$$

Since $H_1\equiv H_2\equiv 1/2 \pmod 1$,

$$\begin{aligned} L(\tau,-2k;R,H) &= \sum_{j=1}^{c} e\left(-\frac{1}{2}\left[\frac{j+r_2-\{r_1+r_2\}}{c}\right]\right)\\ &\quad\cdot\int_C u^{-2k-1}\sum_{m=0}^{\infty}\bar{E}_m\left(\frac{j+\varrho}{c}\right)\frac{u^m}{m!}\sum_{n=0}^{\infty}E_n\left(\frac{j-\{R_1\}}{c}\right)\frac{(-zu)^n}{n!}\\ &= \frac{1}{2}\pi i\sum_{j=1}^{c}(-1)^{[(j+r_2-\{r_1+r_2\})/c]}\sum_{\ell=0}^{2k}\frac{E_\ell((j-\{r_1+r_2\})/c)}{\ell!}\\ &\qquad\cdot\frac{\bar{E}_{2k-\ell}((j+r_2-\{r_1+r_2\})/c)}{(2k-\ell)!}(-z)^\ell. \end{aligned} \tag{2.22}$$

Put $z=\pi i/\alpha$ and apply (2.20), (2.21) and (2.22) to (2.1). Then we deduce Theorem 2.13. □

COROLLARY 2.14. *For any positive integer c,*

$$\begin{aligned} &\alpha^{-k}\sum_{n=0}^{\infty}\frac{\operatorname{csch}((\alpha-\pi i)(2n+1)/(2c))}{(2n+1)^{2k+1}}\\ &= (-\beta)^{-k}\sum_{n=0}^{\infty}\frac{\operatorname{csch}((\beta+\pi i)(2n+1)/(2c))}{(2n+1)^{2k+1}}\\ &\quad -\frac{1}{4}\sum_{j=1}^{c}\sum_{\ell=0}^{2k}\frac{E_\ell((j-1/2)/c)E_{2k-\ell}((j-1/2)/c)}{\ell!(2k-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell}. \end{aligned}$$

Proof. Put $r_1=1/2$ and $r_2=0$ in Theorem 2.13. □

Corollary 2.14 should be compared with Corollary 3.3 in [4].

Corollary 2.15.

$$\alpha^{-k}\sum_{n=0}^{\infty}\frac{(-1)^n\operatorname{sech}((2n+1)\alpha/2)}{(2n+1)^{2k+1}} = -(-\beta)^{-k}\sum_{n=0}^{\infty}\frac{(-1)^n\operatorname{sech}((2n+1)\beta/2)}{(2n+1)^{2k+1}} + \frac{\pi}{4}\sum_{\ell=0}^{k}\frac{E_{2\ell}(1/2)E_{2k-2\ell}(1/2)}{(2\ell)!(2k-2\ell)!}\alpha^{k-\ell}(-\beta)^{\ell}.$$

Proof. Put $c = 1$ in Corollary 2.14 and apply $E_{2n+1}\left(\frac{1}{2}\right) = 0$, $n \geq 0$. □

Corollary 2.15 has been stated in Ramanujan's Notebook [6].

Corollary 2.16. *For any positive integer* M,

$$\alpha^{2M-1}\sum_{n=0}^{\infty}\frac{(-1)^n\operatorname{sech}((2n+1)\alpha/2)}{(2n+1)^{-4M+3}} = \beta^{2M-1}\sum_{n=0}^{\infty}\frac{(-1)^n\operatorname{sech}((2n+1)\beta/2)}{(2n+1)^{-4M+3}}.$$

Proof. Put $c = 1$ in Corollary 2.14 and let $k = -2M + 1$ for $M > 0$. □

Theorem 2.17. *Let* α, $\beta > 0$ *with* $\alpha\beta = \pi^2$. *Let* r_1 *and* r_2 *be real numbers such that* r_1 *and* $r_1 + r_2$ *are not integers. Then, for any integer* k *and for any positive integer* c,

$$\begin{aligned}
&\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{\sinh(((2\{r_1\}-1)(\alpha-\pi i)-2\pi i r_2)(2n+1)/(2c))}{(2n+1)^{2k+2}\sinh((\alpha-\pi i)(2n+1)/(2c))}\\
&= (-\beta)^{-k-1/2}\sum_{n=0}^{\infty}\frac{\sinh(((2\{r_1+r_2\}-1)(\beta+\pi i)-2\pi i r_2)(2n+1)/(2c))}{(2n+1)^{2k+2}\sinh((\beta+i\pi)(2n+1)/(2c))}\\
&\quad+\frac{1}{4}\sum_{j=1}^{c}(-1)^{[(j+r_2-\{r_1+r_2\})/c]}\sum_{\ell=0}^{2k+1}\frac{E_\ell((j-\{r_1+r_2\})/c)}{\ell!}\\
&\qquad\cdot\frac{\bar{E}_{2k+1-\ell}((j+r_2-\{r_1+r_2\})/c)}{(2k+1-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell+1/2}.
\end{aligned}$$

Proof. Let $h = (0, 1/2)$ and $m = 2k+1$ in (2.1). By the same matter in (2.20), (2.21) and (2.22), we have

$$\begin{aligned}
&H(V\tau,-2k-1;r,h)\\
&= 2^{2k+2}\sum_{n=0}^{\infty}\frac{\sinh(\pi i((2\{r_1\}-1)(1-1/z)+2r_2)(2n+1)/(2c))}{(2n+1)^{2k+2}\sinh(\pi i(1/z-1)(2n+1)/(2c))},
\end{aligned}\tag{2.23}$$

$$\begin{aligned}
&H(\tau,-2k-1;R,H)\\
&= 2^{2k+2}\sum_{n=0}^{\infty}\frac{\sinh(\pi i((2\{r_1+r_2\}-1)(z-1)+2r_2)(2n+1)/(2c))}{(2n+1)^{2k+2}\sinh(\pi i(1-z)(2n+1)/(2c))}
\end{aligned}\tag{2.24}$$

and

$$L(\tau,-2k-1;R,H)=\frac{1}{2}\pi i\sum_{j=1}^{c}(-1)^{[(j+r_2-\{r_1+r_2\})/c]}\sum_{\ell=0}^{2k+1}\frac{E_\ell((j-\{r_1+r_2\})/c)}{\ell!}$$

$$\cdot\frac{\bar{E}_{2k+1-\ell}((j+r_2-\{r_1+r_2\})/c)}{(2k+1-\ell)!}(-z)^\ell. \tag{2.25}$$

Take $z=\pi i/\alpha$ and plug (2.23), (2.24) and (2.25) into (2.1). Then the desired results follow. □

Corollary 2.18.

$$\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{\operatorname{sech}((\alpha-\pi i)(2n+1)/(4c))}{(2n+1)^{2k+2}}$$
$$=(-\beta)^{-k-1/2}\sum_{n=0}^{\infty}\frac{\operatorname{sech}((\beta+\pi i)(2n+1)/(4c))}{(2n+1)^{2k+2}}$$
$$-\frac{1}{4}\sum_{j=1}^{c}\sum_{\ell=0}^{2k+1}\frac{E_\ell((j-1/4)/c)}{\ell!}\frac{\bar{E}_{2k+1-\ell}((j-1/4)/c)}{(2k+1-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell+1/2}.$$

Proof. Let $r_1=1/4$ and $r_2=0$ in Theorem 2.17 □

Corollary 2.18 should be compared with Corollary 3.10 in [4].

Corollary 2.19.

$$\alpha^{1/2}\sum_{n=0}^{\infty}\operatorname{sech}((\alpha-\pi i)(2n+1)/(4c))$$
$$=(-\beta)^{1/2}\sum_{n=0}^{\infty}\operatorname{sech}((\beta+\pi i)(2n+1)/(4c)).$$

Proof. Let $k=-1$ in Corollary 2.18. □

References

[1] B.C. Berndt, *Two new proofs of Lerch's functional equation*, Proc. Amer. Math. Soc. **32** (1972), 403–408.

[2] B.C. Berndt, *Modular transformations and generalizations of several formulae of Ramanujan*, Rocky Mountain J. Math. **7** (1) (1977), 147–189.

[3] B.C. Berndt, *Analytic Eisenstein series, theta-functions, and series relations in the spirit of Ramanujan*, J. Reine Angew. Math. **304** (1978), 332–365.

[4] S. Lim, *Infinite series Identities from modular transformation formulas that stem from generalized Eisenstein series*, Acta Arith. **141** (3) (2010), 241–273.

[5] S. Lim, *Series relations from certain modular transformation formula*, J. Chungcheong Math. Soc. **24** (3) (2011), 481–502.

[6] S. Ramanujan, *Notebooks of Srinivasa Ramanujan* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.

Department of Mathematics
Mokwon University
Mokwon Gil 21, Seo-gu, Daejeon
302-318, Republic of Korea
E-mail: sglimj@mokwon.ac.kr