Korean J. Math. **28** (2020), No. 4, pp. 775–789
http://dx.doi.org/10.11568/kjm.2020.28.4.775

e-FUZZY FILTERS OF STONE ALMOST DISTRIBUTIVE LATTICES

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Abstract. In this paper the concept of e-fuzzy filters is introduced in a Stone Almost Distributive Lattice. Several properties are derived on e-fuzzy filters with the help of maximal fuzzy filters. It is proved that the set of all e-fuzzy filters forms a complete distributive lattice.

1. Introduction

U. M. Swamy and G. C. Rao [9] introduced the notion of an Almost Distributive Lattice (ADL). An ADL $(A, \wedge, \vee, 0)$ satisfies all the axioms of distributive lattice, except possibly the commutativity of the operations $\wedge$ and $\vee$. It is known that, in any ADL the commutativity of $\vee$ is equivalent to that of $\wedge$ and also to the right distributivity of $\vee$ over $\wedge$. U.M. Swamy, G.C. Rao, and G. Nanaji Rao [10] introduced pseudo-complementation on almost distributive lattices. U.M. Swamy, G.C. Rao, and G. Nanaji Rao [11] studied Stone Almost Distributive Lattices. In addition to this N. Rafi, Ravi Kumar Bandaru and G.C. Rao [6] studided e-filters in Stone Almost Distributive Lattices. On the other hand, fuzzy set theory was introduced by Zadeh [15]. Next, fuzzy groups were studied by Rosenfield [7]. Many scholars have used this idea

Received June 24, 2020. Revised September 17, 2020. Accepted September 28, 2020.

2010 Mathematics Subject Classification: 06D99, 06D05, 06D30.

Key words and phrases: ADL, Stone ADL, e-fuzzy filters, , prime e-fuzzy filters, maximal e-fuzzy filters.

to different mathematical branches such as semi-group, ring, semi-ring, near-ring, lattice etc. For instance Yuan and Wu [14] introduced the notion of fuzzy sublattice and fuzzy ideals of lattice, Swamy and Raju [8] fuzzy ideals and congruences of lattices, Kumar [5], topologized the set of all fuzzy prime ideals of a commutative ring with unity and studied some properties of the space, Kumar [5], studied about the space of prime fuzzy ideals of a ring in different way and Hadji-Abadi and Zahedi [3] extended the result of Kumar.

More recently, U. M. Swamy et al. [12] Introduced fuzzy ideals of ADLs. In adition to this B. A. Alaba and G. M. Addis [1] studied fuzzy congruence relations on almost distributive lattices. U. M. Swamy et al. [13] studied L-Fuzzy Filters of Almost Distributive Lattices. B. A. Alaba and T.G. Alemayehu [2] introduce e-fuzzy filters of MS-algebras.

In this article our aim is to present e-fuzzy filters of a Stone Almost Distributive Lattice.

2. PRELIMINARIES

In this section, we recall basic definitions and results which will be used in this article. For further detail on e-filters of a Stone ADL, we refer to [6].

Definition 2.1. [9] An algebra $L = (L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL), if it satisfies the following conditions for all a, b and $c \in L$:

1. $0 \wedge a = 0$,
2. $a \vee 0 = a$,
3. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
4. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
5. $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$,
6. $(a \vee b) \wedge b = b$.

[9] Every nonempty set X can be regarded as an ADL as follows. Let $x_0 \in X$. Define the binary operations $\vee, \wedge$ on X by

$$x \vee y = \begin{cases} x & if\ x \neq x_0 \\ y & if x = x_0 \end{cases}$$

.

$$x \wedge y = \begin{cases} x & if\ y \neq x_0 \\ x_0 & if x = x_0 \end{cases}.$$

Then $(X, \vee, \wedge, x_0)$ is an ADL (where x_0 is the zero) and is called a discrete ADL.

If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b \in L$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on L.

Definition 2.2. [9]

If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in L$, we have the following:

1. $a \vee b = a \Leftrightarrow a \wedge b = b$,
2. $a \vee b = b \Leftrightarrow a \wedge b = a$,
3. $\wedge$ is associative in L,
4. $a \wedge b \wedge c = b \wedge a \wedge c$,
5. $(a \vee b) \wedge c = (b \vee a) \wedge c$
6. $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$,
7. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
8. $a \wedge (a \vee b) = a$, $(a \wedge b) \vee b = b$ and $a \vee (b \wedge a) = a$,
9. $a \leq a \vee b$ and $a \wedge b \leq b$,
10. $a \wedge a = a$ and $a \vee a = a$,
11. $0 \vee a = a$ and $a \wedge 0 = 0$,
12. If $a \leq c, b \leq c$ then $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$,

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL L a distributive lattice.

As usual, an element $m \in L$ is called maximal if it is a maximal element in the partially ordered set $(L, \leq)$. That is, for any $a \in L, m \leq a \Rightarrow m = a$.

Theorem 2.3. [9] *Let L be an ADL and $m \in L$. Then the following are equivalent:*

1. *m is maximal with respect to $\leq$,*
2. *$m \vee a = m$, for all $a \in L$,*
3. *$m \wedge a = a$, for all $a \in L$,*
4. *$a \vee m$ is maximal, for all $a \in L$.*

As in distributive lattices [9], a non-empty subset I of an ADL L is called an ideal of L if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in L$.

Also, a non-empty subset F of L is said to be a filter of L if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in L$. The set $I(L)$ of all ideals of L is a bounded distributive lattice with least element $\{0\}$ and greatest element L under set inclusion in which, for any $I, J \in I(L)$, $I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J = \{a \vee b : a \in I, b \in J\}$. A proper ideal P of L is called a prime ideal if, for any $x, y \in L, x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. A proper ideal M of L is said to be maximal if it is not properly contained in any proper ideal of L. It can be observed that every maximal ideal of L is a prime ideal. Every proper ideal of L is contained in a maximal ideal.

For any $A \subseteq L$, $Ann\{A\} = \{x \in L : a \wedge x = 0$ for all $a \in A\}$ is an ideal of L. We write $Ann\{(a]\}$ for $Ann\{a\}$. Then clearly $Ann\{(0]\} = L$ and $Ann\{L\} = (0]$.

Definition 2.4. [6] Let L be an ADL and $x \in L$. Then define $Ann\{x\} = \{y \in L : x \wedge y = 0\}$. Clearly, $Ann\{x\}$ is an ideal in L and hence an annihilator ideal.

Definition 2.5. [10] Let $(L, \vee, \wedge, 0)$ be an ADL. Then a unary operation $a \mapsto a^*$ on L is called a pseudo-complementation on L if, for any $a, b \in L$, it satisfies the following conditions:

1. $a \wedge b = 0 \Rightarrow a^* \wedge b = b$,
2. $a \wedge a^* = 0$,
3. $(a \vee b)^* = a^* \wedge b^*$,

Then $(L, \vee, \wedge, ^*, 0)$ is called a pseudo-complemented ADL.

Here, the unary operation * is called a pseudo-complementation on L and a^* is called a pseudo-complement of a in L. An element a of a pseudo-complemnted ADL L is called a dense element if $a^* = 0$.

Let us denote the set of all dense elements of L by D.

Now we list some results of pseudo-complementation.

Theorem 2.6. [10] *Let L be an ADL and * be a pseudo-complementation on L. Then, for any $a, b \in L$, we have the following:*

1. *0^* is amaximal,*
2. *If a is maximal, then $a^* = 0$,*
3. *$0^{**} = 0$,*
4. *$a^{**} \wedge a = a$,*
5. *$a^{***} = a^*$,*
6. *$a \leq b \Rightarrow b^* \leq a^*$,*

7. $a^* \wedge b^* = b^* \wedge a^*$,
8. $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

Definition 2.7. [11] Let L be an ADL and * a pseudo-complementation on L. Then L is called Stone ADL if, for any $x \in L$, $x^* \vee x^{**} = 0^*$.

Lemma 2.8. [11] *Let L be a Stone ADL and $a, b \in L$. Then* $(a \wedge b)^* = a^* \vee b^*$

Definition 2.9. [6] For any filter F of a Stone ADL L, define an extenstion of F as the set $F^e = \{x \in L/x^* \in Ann\{a\}$ for some $a \in F\}$

Definition 2.10. [6] A filter F of a Stone ADL L is called an e-filter of L if $F = F^e$

Remember that, for any set S a function $\mu : S \longrightarrow ([0, 1], \wedge, \vee)$ is called a fuzzy subset of S, where $[0, 1]$ is a unit interval, $\alpha \wedge \beta = min\{\alpha, \beta\}$ and $\alpha \vee \beta = max\{\alpha, \beta\}$ for all $\alpha, \beta \in [0, 1]$.

Definition 2.11. [13] Let λ be a fuzzy subset of an ADL L. For any $\alpha \in [0, 1]$, we denote the level subset λ_α, i.e

$$\lambda_\alpha = \{x \in L : \alpha \leq \lambda(x)\}.$$

U.M. Swamy et.al [13] $\mu : L \longrightarrow L'$, where L is an ADL and L' is a complete lattice satisfing infinate meet distiributive law. Now in our cases take L' as $[0, 1]$.

λ is said to be a fuzzy filter of an ADL L if λ_α is a filter of L for all $\alpha \in L$.

Theorem 2.12. [13]
Let λ be a fuzzy subset of an ADL L. Then the following are equivalent to each other.

1. *λ is a fuzzy filter of L,*
2. *$\lambda(m) = 1$ for all maximal element m and $\lambda(x \wedge y) = \lambda(x) \wedge \lambda(y)$, for all $x, y \in L$,*
3. *$\lambda(m) = 1$ for all maximal element m and $\lambda(x \vee y) \geq \lambda(x) \vee \lambda(y)$ and $\lambda(x \wedge y) \geq \lambda(x) \wedge \lambda(y)$, for all $x, y \in L$.*

We define the binary operations " + " and "." on all fuzzy subsets of an ADL L as: $(\mu + \theta)(x) = \sup\{\mu(a) \wedge \theta(b) : a, b \in L, a \vee b = x\}$ and $(\mu.\theta)(x) = \sup\{\mu(a) \wedge \theta(b) : a, b \in L, a \wedge b = x\}$.

The intersection of fuzzy filters of L is a fuzzy filter. However the union of fuzzy filters may not be fuzzy filter. The least upper bound of a fuzzy filters μ and θ of L is denoted as $\mu \vee \theta = \cap\{\sigma \in FF(L) : \mu \cup \theta \subseteq \sigma\}$.

If μ and θ are fuzzy filters of L, then $\mu.\theta = \mu \vee \theta$ and $\mu + \theta = \mu \cap \theta$

In the next sections L stands for a Stone ADL unless otherwise mentioned.

3. e-Fuzzy Filters of Stone Almost Distributive Lattices

In [6], N. Rafi, Ravi Kumar Bandaru and G.C. Rao introduced the concept of e-filters in Stone ADL and studied their properties. In this paper, we extend this concept to e-fuzzy filters of a Stone ADL. Some basic properties of e-fuzzy filters are observed in terms of maximal fuzzy filters. We proved that every maximal fuzzy filter of Stone ADL is always an e-fuzzy filter and also observed that every minimal prime fuzzy filter containing a given e-fuzzy filter is an e-fuzzy filter.

Definition 3.1. For any fuzzy filter λ of a Stone ADL L, define an extension of λ as the fuzzy subset $\lambda^e(x) = sup\{\lambda(a) : x^* \wedge a = 0,\ a \in L\}$ for all $x \in L$.

The following Lemma reveals some basic properties of λ^e

Lemma 3.2. *Let L be a Stone ADL. For any two fuzzy filters λ and ν of L, the following holds true.*

(1) λ^e *is a fuzzy filter of* L,
(2) $\lambda \subseteq \lambda^e$,
(3) $\lambda \subseteq \nu \Rightarrow \lambda^e \subseteq \nu^e$,
(4) $(\lambda \cap \nu)^e = \lambda^e \cap \nu^e$,
(5) $(\lambda^e)^e = \lambda^e$.

Proof. For any elements $x, y, a, b \in L$ and for any maximal element L,

(1) $\lambda^e(m) = sup\{\lambda(a) : m^* \wedge a = 0,\ a \in L\} \geq \lambda(m) = 1$. Hence $\lambda^e(m) = 1$.

Next,

$$\begin{aligned}\lambda^e(x)\vee\lambda^e(y) &= sup\{\lambda(a): x^*\wedge a=0,\ a\in L\}\vee sup\{\lambda(b): y^*\wedge b=0,\ b\in L\}\\ &= sup\{\lambda(a)\vee\lambda(b): x^*\wedge a=0,\ y^*\wedge b=0,\ a,\ b\in L\}\\ &\leq sup\{\lambda(a\vee b): (x\vee y)^*\wedge(a\vee b)=0\}\\ &= \lambda^e(x\vee y)\end{aligned}$$

and

$$\begin{aligned}\lambda^e(x)\wedge\lambda^e(y) &= sup\{\lambda(a): x^*\wedge a=0,\ a\in L\}\wedge sup\{\lambda(b): y^*\wedge b=0,\ b\in L\}\\ &= sup\{\lambda(a)\wedge\lambda(b): x^*\wedge a=0,\ y^*\wedge b=0,\ a,\ b\in L\}\\ &\leq sup\{\lambda(a\wedge b): (x\wedge y)^*\wedge(a\wedge b)=0,\ a,b\in L\}\\ &= \lambda^e(x\wedge y)\end{aligned}$$

Thus λ^e is a fuzzy filter of L.

(2) $\lambda^e(x)=sup\{\lambda(a): x^*\wedge a=0\}\geq\lambda(x)$. Hence $\lambda\subseteq\lambda^e$.

(3) Suppose that $\lambda\subseteq\nu$, then
$\nu^e(x)=sup\{\nu(a): x^*\wedge a=0,\ a\in L\}\geq sup\{\lambda(a): x^*\wedge a=0,\ a\in L\}=\lambda^e(x)$.
Hence $\lambda^e\subseteq\nu^e$

(4) By (3) $(\lambda\cap\nu)^e\subseteq\lambda^e\cap\nu^e$.

Conversely,

$$\begin{aligned}(\lambda^e\cap\nu^e)(x) &= \lambda^e(x)\wedge\nu^e(x)\\ &= \sup\{\lambda(a): x^*\wedge a=0,\ a\in L\}\wedge sup\{\nu(b): x^*\wedge b=0,\ b\in L\}\\ &= \sup\{\lambda(a)\wedge\nu(b): x^*\wedge a=0,\ x^*\wedge b=0,\ a,b\in L\}\\ &\leq \sup\{\lambda(a\vee b)\wedge\nu(a\vee b): x^*\wedge(a\vee b)=0,\ a,b\in L\}\\ &= \sup\{(\lambda\cap\nu)(a\vee b): x^*\wedge(a\vee b)=0,\ a,b\in L\}\\ &= (\lambda\cap\nu)^e(x)\end{aligned}$$

Hence $(\lambda^e\cap\nu^e)=(\lambda\cap\nu)^e$.

(5) If $x^*\wedge a=0$ and $a^*\wedge z=0$, then $a^*\wedge x^*=x^*$ and also we have $x^*\wedge z=a^*\wedge x^*\wedge z=x^*\wedge a^*\wedge z=x^*\wedge 0=0$

$$\begin{aligned}(\lambda^e)^e(x) &= sup\{\lambda^e(a): x^*\wedge a=0,\ a\in L\}\\ &= sup\{sup\{\lambda(z): a^*\wedge z=0,\ z\in L\}: x^*\wedge a=0,\ a,x\in L\}\\ &\leq sup\{\lambda(z): x^*\wedge z=0,\ z\in L\}\\ &= \lambda^e(x)\end{aligned}$$

Clearly $\lambda^e\subseteq(\lambda^e)^e$. Hence $(\lambda^e)^e=\lambda^e$. □

Now we define e-fuzzy filter in Stone ADL L.

Definition 3.3. A fuzzy filter λ of a Stone ADL L is called an e-fuzzy filter of L if $\lambda = \lambda^e$.

Theorem 3.4. *λ is an e-fuzzy filter of a Stone ADL L if and only if λ_α is an e-filter of L, for all $\alpha \in [0,1]$.*

Corollary 3.5. *F is an e-filter of a Stone ADL L if and only if χ_F is an e-fuzzy filter of L.*

Lemma 3.6. *Let D be the set of all dense elements of L. Then χ_D is the smallest e-fuzzy filter.*

Proof. Since D is an e-fuzzy filter of L. By Corrollary 3.5 χ_D is an e-fuzzy filter of L. Suppose λ is any e-fuzzy filter of L. If $\chi_D(x) = 1$. This implies $x^* = 0$. Now $\lambda(x) = \sup\{\lambda(a) : x^* \wedge a = 0,\ a \in L\} \geq \lambda(m) = 1$, for any maximal element m. Since $x^* \wedge m = 0$. In this case $\chi_D(x) \leq \lambda(x)$. If $\chi_D(x) = 0$, then $\chi_D(x) = 0 \leq \lambda(x)$. This implies $\chi_D(x) \leq \lambda(x)$ for all $x \in L$. Hence χ_D is the smallest e-fuzzy filter of L. □

In Lemma 3.2(4), we can mention that the intersection of two e-fuzzy filters of a Stone ADL L is an e-fuzzy filter. But the union of two e-fuzzy filters may not be an e-fuzzy filter.

Corollary 3.7. *Let $\{\lambda_i : i \in \Omega\}$ be a family of e-fuzzy filters of a Stone ADL L. Then $\cap_{i\in\Omega}\lambda_i$ is an e-fuzzy filter of L.*

We denote the class of all e-fuzzy filters of a Stone ADL L by $\mathcal{FF}^e(L)$

Theorem 3.8. *Let L be a Stone ADL L. Then the class $\mathcal{FF}^e(L)$ of all e-fuzzy filters forms a complete distributive lattice with relation $\subseteq$.*

Proof. Since $\chi_D, \chi_L \in \mathcal{FF}^e(L)$, $\mathcal{FF}^e(L) \neq \emptyset$. Clearly $(\mathcal{FF}^e(L), \subseteq)$ is a partially order set. Now for any $\lambda, \sigma \in \mathcal{FF}^e(L)$, define $\lambda \wedge \sigma = \lambda \cap \sigma$ and $\lambda \cup \sigma = (\lambda \vee \sigma)^e$, where $(\lambda \vee \sigma)^e(x) = sup\{\lambda(a) \wedge \lambda(b) : x^* \wedge (a \wedge b) = 0,\ a, b \in L\}\ \forall\ x \in L$. It can be easily verified that $\lambda \cap \sigma, (\mu \vee \sigma)^e \in \mathcal{FF}^e(L)$ and $\lambda \cap \sigma$ is the greatest lower bound of λ and σ. We prove that $\lambda \cup \sigma$ is the least upper bound of λ and σ. Since $\lambda, \sigma \subseteq \lambda \vee \sigma \subseteq (\lambda \vee \sigma)^e$, $(\lambda \vee \sigma)^e$ is an upper bound of λ and σ. Let γ be any e-fuzzy filter of L such that $\lambda \subseteq \gamma$ and $\sigma \subseteq \gamma$.

$$\begin{aligned}
(\lambda \vee \sigma)^e(x) &= Sup\{\lambda(a) \wedge \lambda(b) : x^* \wedge (a \wedge b) = 0\,; a, b \in L\} \\
&\leq Sup\{\gamma(a) \wedge \gamma(b) : x^* \wedge (a \wedge b) = 0,\ a, b \in L\} \\
&= Sup\{\gamma(a \wedge b) : x^* \wedge (a \wedge b) = 0,\ a, b \in L\} \\
&= \gamma^e(x) = \gamma(x)
\end{aligned}$$

Hence $(\lambda \vee \sigma)^e = sup\{\lambda, \sigma\}$. Thus $(\mathcal{FF}^e(L), \subseteq)$ is a lattice. Since χ_D and χ_L are the smallest and the greatest e-fuzzy filters of $\mathcal{FF}^e(L)$, $(\mathcal{FF}^e(L), \cap, \cup, \chi_D, \chi_L)$ is a bounded lattice. By Corollary 3.8 any subfamily of e-fuzzy filters of $\mathcal{FF}^e(L)$ has infimum in $\mathcal{FF}^e(L)$ and $\mathcal{FF}^e(L)$ has greatest element. Hence $(\mathcal{FF}^e(L), \cap, \cup, \chi_D, \chi_L)$ is a complete bounded lattice. For any λ, σ and $\theta \in \mathcal{FF}^e(L)$, we have $(\lambda \cup \sigma) \cap (\lambda \cup \theta) = (\lambda \vee \sigma)^e \cap (\lambda \vee \theta)^e = ((\lambda \vee \sigma) \cap (\lambda \vee \theta))^e = (\lambda \vee (\sigma \cap \theta))^e = \lambda \cup (\sigma \cap \theta)$. Therefore $(\mathcal{FF}^e(L), \cap, \cup, \chi_D, \chi_L)$ is a bounded and complete distributive lattice. □

In the following, we characterize the e-fuzzy filters

Theorem 3.9. *Let λ be a fuzzy filter of a Stone ADL L. Then, the following are equivalent.*

(1) *λ is an e-fuzzy filter,*

(2) *$\lambda(x) = \lambda(x^{**})$,*

(3) *For $x, y \in L$, $x^* = y^*$ implies $\lambda(x) = \lambda(y)$.*

Proof. (1) $\Rightarrow$ (2). Suppose that λ is an e-fuzzy filter of L. For $x, a \in L$, $\lambda(x) = \lambda^e(x) = sup\{\lambda(a) : x^* \wedge a = 0,\ a \in L\} = sup\{\lambda(a) : x^{***} \wedge a = 0,\ a \in L\} = \lambda^e(x^{**}) = \lambda(x^{**})$.

(2) $\Rightarrow$ (3). Suppose that condition (2) holds. Let $x, y \in L, x^* = y^*$. Then $x^{**} = y^{**}$. Thus $\lambda(x) = \lambda(x^{**}) = \lambda(y^{**}) = \lambda(y)$. Hence $\lambda(x) = \lambda(y)$.

(3) $\Rightarrow$ (1). Suppose that condition (3) holds. $\lambda^e(x) = sup\{\lambda(a) : x^* \wedge a = 0,\ a \in L\} \leq sup\{\lambda(a) : (a \vee x)^* = x^*,\ a \in L\} \leq \lambda(a \vee x) = \lambda(x)$. Since $x^* \wedge a = 0$ implies $x^* = a^* \wedge x^* = (a \vee x)^*$ and by (3) $\lambda(x \vee a) = \lambda(x)$. This implies $\lambda^e \subseteq \lambda$. Clearly $\lambda \subseteq \lambda^e$. Hence λ is an e-filter of L. □

4. Prime e-Fuzzy Filters and Maximal e-fuzzy Filters of a Stone ADL L

In this section, we introduce prime e-fuzzy filters and maximal e-fuzzy filters of a Stone ADL L and we discuss some properties of them.

Definition 4.1. A proper e-fuzzy filter μ in a Stone ADL L is called a prime e-fuzzy filter if for any fuzzy filters λ and ν of L, $\lambda \cap \nu \subseteq \mu \Rightarrow \lambda \subseteq \mu$ or $\nu \subseteq \mu$.

Theorem 4.2. *A proper filter F is a prime e-filter of L and $\alpha \in [0,1)$ if and only if the fuzzy subset given by*

$$F_\alpha^1(x) = \begin{cases} 1 & if\ x \in F \\ \alpha & if\ \ x \notin F \end{cases}$$

is a prime e-fuzzy filter of L.

Proof. Suppose that a proper filter F of L is a prime e-filter of L and $\alpha \in [0,1)$. Clearly F_α^1 is a proper fuzzy filter of L. Since $(F_\alpha^1)_1 = F$ and $(F_\alpha^1)_\alpha = L$ are e-filters of L. This implies by Theorem 3.4, F_α^1 is a proper e-fuzzy filter of L. Now we prove that F_α^1 is a prime e-fuzzy filter. Let ν and θ be any fuzzy filters of L such that $\nu \cap \theta \subseteq F_\alpha^1$. Suppose if possible that $\nu \nsubseteq F_\alpha^1$ and $\theta \nsubseteq F_\alpha^1$. Then there exist $x, y \in L$ such that $\nu(x) > F_\alpha^1(x)$ and $\theta(y) > F_\alpha^1(y)$. This indicates $F_\alpha^1(x) = F_\alpha^1(y) = \alpha$ and so $x \notin F$ and $y \notin F$. Since F is prime, $x \vee y \notin F$ and so $F_\alpha^1(x \vee y) = \alpha$. Now, $(\nu \cap \theta)(x \vee y) = \nu(x \vee y) \wedge \theta(x \vee y) \geq \nu(x) \wedge \theta(y) > \alpha \wedge \alpha = \alpha = F_\alpha^1(x \vee y)$, which is a contradiction to our assumption $\nu \cap \theta \subseteq F_\alpha^1$. Hence F_α^1 is a prime e-fuzzy filter. Conversely, suppose that F_α^1 is a prime e-fuzzy filter. Clearly F_α^1 is an e-fuzzy filter and $(F_\alpha^1)_1 = F$. Hence F is an e-filter of L. Let A and B be any filters of L such that $A \cap B \subseteq F$. Then $(A \cap B)_\alpha^1 = A_\alpha^1 \cap B_\alpha^1 \subseteq F_\alpha^1$. Since F_α^1 is prime, $A_\alpha^1 \subseteq F_\alpha^1$ or $B_\alpha^1 \subseteq F_\alpha^1$. This implies $B \subseteq F$ or $A \subseteq F$. Hence F is a prime e-filter of L. □

Theorem 4.3. *A proper e-fuzzy filter λ of L is a prime e-fuzzy filter if and only if $Img(\lambda) = \{1, \alpha\}$, where $\alpha \in [0,1)$ and the set $\lambda_* = \{x \in L : \lambda(x) = 1\}$ is a prime e-filter of L.*

Proof. The converse part of this theorem follows from Lemma 4.2. Suppose that λ is a prime e-fuzzy filter. Clearly $1 \in Im(\lambda)$. Since λ is proper, there is $x \in L$ such that $\lambda(x) < 1$. We prove that $\lambda(x) = \lambda(y)$ for all $x, y \in L - \lambda_*$. Suppose that $\lambda(x) \neq \lambda(y)$ for some $x, y \in L - \lambda_*$. Without loss of generality we can assume that $\lambda(y) < \lambda(x) < 1$. Define fuzzy subsets θ and ϕ as follows:

$$\theta(z) = \begin{cases} 1 & if\ z \in [x) \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\phi(z) = \begin{cases} 1 & if\ z \in \lambda_* \\ \lambda(x) & \text{otherwise.} \end{cases}$$

for all $z \in L$. Then it can be easily verified that both θ and ϕ are fuzzy filters of L. Let $z \in L$. If $z \in \lambda_*$, then $(\theta \cap \phi)(z) \leq 1 = \mu(z)$. If $z \in [x) - \lambda_*$, then $z = x \vee z$, and we have $(\theta \cap \phi)(z) = \theta(z) \wedge \phi(z) = 1 \wedge \lambda(x) = \lambda(x) \leq \lambda(z)$.
Also if $z \notin [x)$, then $\theta(z) = 0$, so that $(\theta \cap \phi)(z) = 0 \leq \lambda(z)$. Therefore for all $x \in L$, $(\theta \cap \phi)(x) \subseteq \lambda(x)$. But we have $\theta(x) = 1 > \lambda(x)$ and $\phi(y) = \lambda(x) > \lambda(y)$. This implies $\phi \nsubseteq \lambda$ and $\theta \nsubseteq \lambda$, which is a contradiction. Thus $\lambda(x) = \lambda(y)$ for all $x, y \in L - \lambda_*$ and hence $Im(\lambda) = \{1, \alpha\}$ for some $\alpha \in [0, 1)$. Let $P = \{x \in L : \lambda(x) = 1\}$. Since λ is proper, we get that P is a proper e-filter of L such that

$$\lambda(z) = \begin{cases} 1 & if\ z \in P \\ \alpha & if\ z \notin P. \end{cases}$$

for $\alpha \neq 1$. Hence by Lemma 4.2, $P = \lambda_*$. □

THEOREM 4.4. *If λ is a prime e-fuzzy filter of L, then $\lambda(x \vee y) = \lambda(x)$ or $\lambda(x \vee y) = \lambda(y)$ for all $x, y \in L$.*

Proof. Suppose that λ is a prime e-filter of L, then there exists a prime e-filter F of L and $\alpha \in [0, 1)$ such that

$$\lambda(x) = \begin{cases} 1 & if\ x \in F \\ \alpha & if\ \ x \notin F \end{cases}$$

for all $x \in L$. If $x, y \in F$, then $x \vee y \in F$ and so $1 = \lambda(x) = \lambda(y) = \lambda(x \vee y)$. If $x \in F$ and $y \notin F$, then $x \vee y \in F$ and so $1 = \lambda(x) = \lambda(x \vee y)$. If $x \notin F$ and $y \notin F$, then $x \vee y \notin F$ and so $\alpha = \lambda(x) = \lambda(y) = \lambda(x \vee y)$. Hence the Theorem holds. □

DEFINITION 4.5. A proper fuzzy filter λ in a Stone ADL L is called a maximal fuzzy filter if $Img(\lambda) = \{1, \alpha\}$, where $\alpha \in [0, 1)$ and the set λ_* is a maximal filter of L.

DEFINITION 4.6. A proper e-fuzzy filter λ in a Stone ADL L is called a maximal e-fuzzy filter if $Img(\lambda) = \{1, \alpha\}$, where $\alpha \in [0, 1)$ and the set λ_* is a maximal e-filter of L.

COROLLARY 4.7. *Any maximal e-fuzzy filter of L is a prime e-fuzzy filter.*

Proof. Let λ be a maximal e-fuzzy filter of L. Then $Im(\lambda) = \{1, \alpha\}$, and λ_* is a maximal e-filter of L. Since every maximal e-filter of L is a prime e-filter of L. This implies λ_* is a prime e-filter of L. Hence λ is a prime e-fuzzy filter of L. □

THEOREM 4.8. *Every maximal fuzzy filter of a Stone ADL L is an e-fuzzy filter.*

COROLLARY 4.9. *Every maximal fuzzy filter of a Stone ADL L is prime e-fuzzy filter.*

THEOREM 4.10. *If λ is minimal in the class of all prime fuzzy filters L containing a given e-fuzzy filter, then λ is an e-fuzzy filter of L.*

Proof. Suppose that λ is minimal in the class of all prime fuzzy filters containing an e-fuzzy filter θ of L. We prove that λ is an e-fuzzy filter. Since λ is a prime fuzzy filter of L, there exists a prime filter P of L such

$$\lambda(z) = \begin{cases} 1 & if\ z \in P \\ \alpha & \text{otherwise.} \end{cases}$$

for some $\alpha \in [0, 1)$. Suppose that λ is not an e-fuzzy filter of L, then there exist $x, y \in L$, $x^* = y^*$ such that $\lambda(x) \neq \lambda(y)$. Without loss of generality, assume $\lambda(x) = 1$ and $\lambda(y) = \alpha$. Consider a fuzzy ideal ϕ of L defined by

$$\phi(z) = \begin{cases} 1 & if\ z \in (L - P) \vee (x \vee y] \\ \alpha & \text{otherwise.} \end{cases}$$

Then $\theta \cap \phi \leq \alpha$. Otherwise there exists $a \in L$ such that $\phi(a) = 1$ and $\theta(a) > \alpha$. This implies $a \in (L - P) \vee (x \vee y]$.

$$\implies a = r \vee s \text{ for some } r \in L - P \text{ and } s \in (x \vee y]$$
$$\implies a = r \vee s = r \vee ((x \vee y) \wedge s) = (r \vee x \vee y) \wedge (r \vee s) \leq r \vee x \vee y$$

As $x^* = y^*$ implies $(r \vee x \vee y)^* = (r \vee y)^*$. Since θ is an e-fuzzy filter of L, $\alpha < \theta(a) = \theta(r \vee s) \leq \theta(r \vee x \vee y) = \theta(r \vee y) \leq \lambda(r \vee y)$. This implies $1 = \lambda(r \vee y)$.

Hence $\lambda(y) = 1$ or $\lambda(r) = 1$, which is a contradiction. Thus $\theta \cap \phi \leq \alpha$.

This implies there exists a prime fuzzy filter η such that $\eta \cap \phi \leq \alpha$ and $\theta \subseteq \eta$. Clearly $x \vee y \in (L - P) \vee (x \vee y]$. This implies $\phi(x \vee y) = 1$. Since $\phi \cap \eta \leq \alpha$, $\eta(x \vee y) \leq \alpha < \lambda(x \vee y) = 1$. This implies $\lambda \nsubseteq \eta$. This indicates λ is not minimal in the class of all prime fuzzy filters containing a given e-fuzzy filter, which is a contradiction. Therefore, λ is an e-fuzzy filter. □

THEOREM 4.11. *Let λ be a prime fuzzy filter of a Stone ADL L, and $\lambda(0) = 0$. Then a fuzzy subset $\ell(\lambda)$ of L defined as $\ell(\lambda)(x) = \lambda'(x^*)\ \forall x \in L$ is an e-fuzzy filter of L.*

Proof. $\ell(\lambda)(m) = \lambda'(m^*) = 1 - \lambda(m^*) = 1 - \lambda(0) = 1$.

$$\begin{aligned}\ell(\lambda)(x \wedge y) = \lambda'((x \wedge y)^*) &= 1 - \lambda(x^* \vee y^*)\\ &= (1 - \lambda(x^*)) \wedge (1 - \lambda(y^*))\\ &= \lambda'(x^*) \wedge \lambda'(y^*) = \ell(\lambda)(x) \wedge \ell(\lambda)(y)\end{aligned}$$

This implies $\ell(\lambda)$ is a fuzzy filter of L. Next we prove that $\ell(\lambda)$ is an e-fuzzy filter.

$$\begin{aligned}\ell(\lambda)^e(x) &= sup\{\ell(\lambda)(a) : x^* \wedge a = 0,\ a \in L\}\\ &= sup\{\ell(\lambda)(a) : a^* \wedge x^* = x^*,\ a \in L\}\\ &= sup\{1 - \lambda(a^*) : a^* \wedge x^* = x^*,\ a \in L\}\\ &\leq 1 - \lambda(x^*), \text{ since } x^* = a^* \wedge x^* \leq a^* \text{ and } \lambda \text{ is an isotone}\\ &= \ell(\lambda)(x)\end{aligned}$$

Clearly $\ell(\lambda) \subseteq \ell(\lambda)^e$ □

Hence $\ell(\lambda)$ is an e-fuzzy filter of L.

COROLLARY 4.12. *Let λ be a maximal fuzzy filter of Stone ADL L and $\lambda(0) = 0$. Then $\ell(\lambda)$ is an e-fuzzy filter of L.*

DEFINITION 4.13. [6] An ADL L is said to be a disjunctive ADL if for any $x, y \in L$, $Ann\{x\} = Ann\{y\}$ implies $x = y$.

THEOREM 4.14. *Let L be a Stone ADL. If L is a disjunctive ADL, then every fuzzy filter of L is an e-fuzzy filter.*

Proof. Suppose that λ is a fuzzy filter of disjunctive ADL L. Clearly $\lambda \subseteq \lambda^e$

$$\begin{aligned}\text{Conversely, } \lambda^e(x) &= \sup\{\lambda(a) : x^* \wedge a = 0, a \in L\}\\ &\leq \sup\{\lambda(a) : (a \vee x)^* = x^*, a \in L\}\\ &\leq \lambda(a \vee x) = \lambda(x), \text{ since } L \text{ is disjunctive ADL}\end{aligned}$$

and λ is an istone.

This implies $\lambda = \lambda^e$. Hence every fuzzy filter is an e-fuzzy filter. □

THEOREM 4.15. *For any fuzzy filter λ of a Stone ADL L, a fuzzy subset $\lambda^*(x) = sup\{\lambda(b) : x^* \wedge b = 0\ , b \in L\}\ \forall\ x \in L$ is an e-fuzzy filter.*

Proof. For any $x, y \in L$,

$$\lambda^*(1) = sup\{\lambda(b) : 1^* \wedge b = 0,\ b \in L\} \geq \lambda(1) = 1$$

$$\begin{aligned}
\lambda^*(x) \wedge \lambda^*(y) &= sup\{\lambda(a) : x^* \wedge a = 0,\ a \in L\} \wedge sup\{\lambda(b) : y^* \wedge b = 0,\ b \in L\} \\
&= sup\{\lambda(a) \wedge \lambda(b) : x^* \wedge a = 0,\ y^* \wedge b = 0,\ a, b \in L\} \\
&\leq sup\{\lambda(a \wedge b) : (x \wedge y)^* \wedge a \wedge b = 0,\ a, b \in L\} \\
&= \lambda^*(x \wedge y) \\
\lambda^*(x) \vee \lambda^*(y) &= sup\{\lambda(a) : x^* \wedge a = 0,\ a \in L\} \vee sup\{\lambda(b) : y^* \wedge b = 0,\ b \in L\} \\
&= sup\{\lambda(a) \vee \lambda(b) : x^* \wedge a = 0,\ y^* \wedge b = 0,\ a, b \in L\} \\
&\leq sup\{\lambda(a \vee b) : (x \vee y)^* \wedge a \vee b = 0,\ a, b \in L\} \\
&= \lambda^*(x \vee y)
\end{aligned}$$

This implies λ^* is a fuzzy filter of L. Next we prove that λ is an e-fuzzy filter. Now

$\lambda^*(x^{**}) = sup\{\lambda(c) : x^{***} \wedge c = 0,\ c \in L\} = sup\{\lambda(c) : x^* \wedge c = 0,\ c \in L\}$ $= \lambda^*(x)$. Therefore λ^* is an e-fuzzy filter of L. □

References

[1] B. A. Alaba, G. M. Addis, *Fuzzy congruence relations on almost distributive lattices*, Ann. Fuzzy Math. Inform.

[2] B. A. Alaba and T. G. Alemayehu, *e-Fuzzy filters of MS-algebras*, Korean J. Math. **27** (4) (2019), 1159–1180.

[3] H. Hadji-Abadi and M.M. Zahedi, *Some results on fuzzy prime spectrum of a ring* , Fuzzy sets and systems **77** (1996), 235–240.

[4] R. Kumar, *Fuzzy prime spectrum of a ring*, Fuzzy sets and systems **46** (1992), 147–154.

[5] R. Kumar, *Spectrum prime fuzzy ideals*, Fuzzy sets and systems **62** (1994), 101–109.

[6] N. Rafi, Ravi Kumar Bandaru and G.C. Rao, *e-filters in Stone Almost Distributive Lattices*, Chamchuri Journal of Mathematics **7** (2015), 16–28.

[7] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512–517.

[8] U. M. Swamy and D. V. Raju, *Fuzzy ideals and congruences of lattices*, Fuzzy sets and systems **95** (1998), 249–253.

[9] U. M. Swamy and G. C. Rao, *Almost Distributive Lattices*, J. Aust. Math. Soc. (Series A) **31** (1981), 77–91.

[10] U. M. Swamy, G. C. Rao, and G. Nanaji Rao, *Pseudo-complementation on Almost Distributive Lattices*, Southeast Asian Bullettin of Mathematics **24** (2000), 95–104.

[11] U. M. Swamy, G. C. Rao, and G. Nanaji Rao, *Stone Almost Distributive Lattices*, Southeast Asian Bullettin of Mathematics **24** (2000), 513–526.
[12] U. M. Swamy, Ch. Santhi Sundar Raj, and N. Teshale, *Fuzzy ideals of almost distributive lattices*, Ann. Fuzzy Math. Inform., accepted for publication.
[13] U.M. Swamy, Ch. Santhi Sundar Raj, A. Natnael Teshale, *L-Fuzzy Filters of Almost Distributive Lattices*, IJMSC, 8, 1(2018), 35–43.
[14] Bo. Yuan and W. Wu, *Fuzzy ideals on a distributive lattice*, Fuzzy Sets and Systems **35** (1990), 231–240.
[15] L. A. Zadeh, *Fuzzy sets*, Inform. and Control **8** (1965), 338–353.

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