

BESSEL-WRIGHT TRANSFORM IN THE SETTING OF QUANTUM CALCULUS

ILYES KAROUI, LAZHAR DHAOUADI, Wafa BINOUS, AND MENIAR HADDAD

ABSTRACT. This work is devoted to the study of a q -harmonic analysis related to the q -analog of the Bessel-Wright integral transform [6]. We establish some important properties of this transform and we focalise our attention in studying the associated transmutation operator.

1. Introduction

Recently, a new discipline was created and interested a lot of researchers : the q -harmonic analysis [9] who find application in the q -deformed mechanics. This theory was first elaborated by Koornwinder and R.F. Swarttouw [12] and then by Fitouhi et al. [3, 5, 7] who investigated in the q -Bessel function and the q -Fourier transform and related q -harmonic analysis, q -orthogonality relation, q -Plancherel formulae, q -Hardy Theorem [3].

Motivated by these gigantic growth and taking into account the work of the classical Bessel-Wright operator [6, 11], it is quite legitimate to try to extend these results to the quantum calculus case. We define the analog of the Bessel-Wright operator [2, 6, 11] and we try to give a q -integral representation of the q -Bessel-Wright function introduced via the q -hypergeometric functions.

To make this work self containing, we shall try to generalize our results starting by presenting some results associated with the q -Bessel transform, deeply studied on [3–5, 7], which will be useful in the sequel. Next, we present the basic tool in our work: the q -transmutation operator. This q -transmutation operator associated to the q -Bessel-Wright function is deeply studied in section 3 and the most important results are proved as well as the q -Bessel-Wright operator.

Finally, in Section 4, we introduce the q -Bessel-Wright transform. We prove many results that will be a strong step in studying a large class of q -integral transform.

Received July 11, 2020. Revised March 22, 2021. Accepted March 25, 2021.

2010 Mathematics Subject Classification: 33D15, 47A70.

Key words and phrases: q -Harmonic Analysis, q -Bessel-Wright function, q -Bessel-Wright transform, q -Hardy transform.

© The Kangwon-Kyungki Mathematical Society, 2021.

This is an Open Access article distributed under the terms of the Creative Commons Attribution Non-Commercial License (<http://creativecommons.org/licenses/by-nc/3.0/>) which permits unrestricted non-commercial use, distribution and reproduction in any medium, provided the original work is properly cited.

2. Preliminaries on q -harmonic analysis

Throughout this paper, we will assume that $0 < q < 1, \alpha > -1$ and $\beta > 0$. We refer to [9] for the definitions, notations and properties of the q -shifted factorials, the Jackson's q -derivative and the Jackson's q -integrals.

The q -shifted factorial are defined by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k),$$

and

$$\mathbb{R}_q^+ = \{q^n : n \in \mathbb{Z}\}.$$

The q -derivative of a function f is given by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x} \quad \text{if } x \neq 0.$$

The q -Jackson integrals from 0 to a and from 0 to ∞ are defined by [9]

$$\int_0^a f(x) d_q x = (1 - q)a \sum_{n=0}^{\infty} f(aq^n) q^n,$$

$$\int_0^{+\infty} f(x) d_q x = (1 - q) \sum_{n=-\infty}^{\infty} f(q^n) q^n,$$

provided the sums converge absolutely.

The space $\mathcal{L}_{q,p,\alpha}$, $1 \leq p < \infty$ denote the set of functions on \mathbb{R}_q^+ for which

$$\|f\|_{q,p,\alpha} = \left[\int_{-\infty}^{+\infty} |f(x)|^p |x|^{2\alpha+1} d_q x \right]^{1/p} < \infty,$$

and $\tilde{\mathcal{L}}_{q,p,(\alpha,\beta)}(\mathbb{R}_q^+)$ the set of functions on \mathbb{R}_q^+ for which

$$\left[\int_0^{+\infty} |x^{2\beta} f(x)|^p |x|^{2\alpha+1} d_q x \right]^{1/p} < \infty.$$

We define the q -inner product $\langle \cdot, \cdot \rangle_\alpha$ in the Hilbert space $\mathcal{L}_{q,2,\alpha}$ as follow

$$f, g \in \mathcal{L}_{q,2,\alpha} \Rightarrow \langle f, g \rangle_\alpha = \int_{-\infty}^{+\infty} f(x)g(x) |x|^{2\alpha+1} d_q x.$$

Similarly $\mathcal{C}_{q,0}$ is the space of continuous functions at 0 for which

$$\|f\|_{q,\infty} = \sup_{x \in \mathbb{R}_q^+} |f(x)| < \infty$$

and $\mathcal{C}_{q,b}$ the space of continuous functions at 0 and bounded on \mathbb{R}_q^+ .

The space $\mathcal{S}(\mathbb{R}_q^+)$ denote the set of the q -analogue of the Schwartz space f defined on \mathbb{R}_q^+ such

that for all $n \in \mathbb{N}$ $D_q^n f$ is continuous at 0, and verifying

$$\lim_{x \rightarrow \infty} x^k |D_q^n f(x)| = 0, \quad \forall n, k \in \mathbb{N},$$

and $\mathcal{D}_{q,a}(\mathbb{R}_q^+)$ denote the subspace of $\mathcal{S}(\mathbb{R}_q^+)$ with compact support in $[-a, a]$.

The q -Paley-Wiener space is the sets of functions on \mathbb{R}_q^+ defined as follow

$$PW_{q,a}^\alpha = \left\{ f(x) = \int_0^a u(t) j_\alpha(xt; q^2) d_q t, \quad u \in \mathcal{D}_{q,a} \right\}.$$

The normalized q -Bessel function is defined by

$$j_\alpha(x, q^2) = \sum_{n=0}^\infty (-1)^n \frac{q^{n(n+1)}}{(q^{2\alpha+2}, q^2)_n (q^2, q^2)_n} x^{2n}.$$

The q -Bessel function has the following q -Mehler integral representation [7]

$$j_\alpha(x, q^2) = c_{q,\alpha+1} \left(-\frac{1}{2} \right) \int_0^1 W_{\alpha+1}(t; q^2) \cos(xt; q^2) d_q t$$

where

$$(1) \quad c_{q,\alpha}(\xi) = \frac{(q^{2\xi+2}; q^2)_\infty (q^{2\alpha}; q^2)_\infty}{(q^2; q^2)_\infty (q^{2\alpha+2\xi+2}; q^2)_\infty}$$

and W_α the function defined by formula

$$(2) \quad W_\alpha(t; q^2) = \frac{1}{(1-q)} \frac{(q^2 t^2; q^2)_\infty}{(q^{2\alpha} t^2; q^2)_\infty}.$$

And these two formulas (1) and (2) give the following formula

$$(3) \quad c_{q,\alpha}(\xi) \frac{1}{(q^{2\xi+2}; q^2)_n} \int_0^1 W_\alpha(t; q^2) t^{2n} t^{2\xi+1} d_q t = \frac{1}{(q^{2\alpha+2+2\xi}; q^2)_n}.$$

The q -Bessel operator as follow

$$\Delta_{q,\alpha} f(x) = \frac{1}{x^2} [f(q^{-1}x) - (1 + q^{2\alpha})f(x) + q^{2\alpha}f(qx)].$$

The function $x \mapsto j_\alpha(\lambda x, q^2)$ is an eigenfunction of $\Delta_{q,\alpha}$ for the eigenvalue $-\lambda^2$.

The q -Bessel Fourier transform $\mathcal{F}_{q,\alpha}$ is defined by

$$\mathcal{F}_{q,\alpha}(f)(x) = c_{q,\alpha} \int_0^{+\infty} f(t) j_\alpha(xt; q^2) t^{2\alpha+1} d_q t.$$

THEOREM 1. *We have*

$$\mathcal{F}_{q,\alpha} : \mathcal{L}_{q,1,\alpha}(\mathbb{R}_q^+) \rightarrow \mathcal{C}_{q,0}(\mathbb{R}_q^+)$$

and

$$\mathcal{F}_{q,\alpha} : \mathcal{S}(\mathbb{R}_q^+) \rightarrow \mathcal{S}(\mathbb{R}_q^+)$$

and

$$\mathcal{F}_{q,\alpha} : \mathcal{D}_{q,a}(\mathbb{R}_q^+) \rightarrow PW_{q,a}^\alpha.$$

Proof. Using [5, Proposition 3.1], we get

$$\mathcal{F}_{q,\alpha} : \mathcal{L}_{q,1,\alpha}(\mathbb{R}_q^+) \rightarrow \mathcal{C}_{q,0}(\mathbb{R}_q^+).$$

Via [5, Corollary 3.3], we get

$$\mathcal{F}_{q,\alpha} : \mathcal{S}(\mathbb{R}_q^+) \rightarrow \mathcal{S}(\mathbb{R}_q^+)$$

and by [4, Theorem 1], we have

$$\mathcal{F}_{q,\alpha} : \mathcal{D}_{q,a}(\mathbb{R}_q^+) \rightarrow PW_{q,a}^\alpha.$$

□

3. On the q -Bessel-Wright function

We start this section by defining the q -Bessel-Wright function

$$\begin{aligned}
 (4) \quad j_{(\alpha,\beta)}(x, q^2) &= {}_2\Phi_2(q^2, -; q^{2\alpha+2}, q^{2\beta+2}; q^2; q^2 x^2) \\
 &= \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^{2\alpha+2}, q^2)_n (q^{2\beta+2}, q^2)_n} x^{2n}.
 \end{aligned}$$

PROPOSITION 1. *The q -Bessel-Wright function $x \mapsto j_{(\alpha,\beta)}(\lambda x, q^2)$ is an eigenfunction of the q -Bessel-Wright difference operator $\Delta_{(\alpha,\beta)}$ associated to the eigenvalue $-\lambda^2$*

$$\Delta_{(\alpha,\beta)} f(x) = \frac{1}{x^2} [f(q^{-1}x) - (q^{2\beta} + q^{2\alpha}) f(x) + q^{2(\alpha+\beta)} f(qx) - (1 - q^{2\alpha})(1 - q^{2\beta}) f(0)].$$

Proof. We have

$$\frac{(1 - q^{2\alpha+2i})(1 - q^{2\beta+2i})}{q^{2i}} = q^{-2i} - (q^{2\alpha} + q^{2\beta}) + q^{2(\alpha+\beta)+2i}.$$

So we get

$$\begin{aligned}
 \Delta_{(\alpha,\beta)} j_{(\alpha,\beta)}(x, q^2) &= \sum_{n=1}^{\infty} (-1)^n \prod_{i=1}^n \frac{q^{-2n} - (q^{2\alpha} + q^{2\beta}) + q^{2(\alpha+\beta)+2n}}{q^{-2i} - (q^{2\alpha} + q^{2\beta}) + q^{2(\alpha+\beta)+2i}} x^{2n-2} \\
 &= -\Delta_{(\alpha,\beta)} j_{(\alpha,\beta)}(x, q^2),
 \end{aligned}$$

which leads to the result. □

PROPOSITION 2. *The q -Bessel-Wright difference operator could be factorized as follow*

$$\Delta_{(\alpha,\beta)} = \partial_{q,\alpha} \partial_{q,\beta}^*$$

where

$$\begin{aligned}
 \partial_{q,\alpha} f(x) &= \frac{f(x) - q^{2\alpha+1} f(qx)}{x} \\
 \partial_{q,\beta}^* f(x) &= \frac{[f(q^{-1}x) - f(0)] - q^{2\beta} [f(x) - f(0)]}{x}.
 \end{aligned}$$

Proof. A simple calculation leads to the result

$$\begin{aligned}
 \partial_{q,\alpha} \partial_{q,\beta}^* f(x) &= \frac{\frac{[f(q^{-1}x) - f(0)] - q^{2\beta} [f(x) - f(0)]}{x} - q^{2\alpha+1} \frac{[f(x) - f(0)] - q^{2\beta} [f(qx) - f(0)]}{qx}}{x} \\
 &= \frac{f(q^{-1}x) - (q^{2\beta} + q^{2\alpha}) f(x) - q^{2\beta+2\alpha} f(qx) - (1 - q^{2\alpha})(1 - q^{2\beta}) f(0)}{x^2} \\
 &= \Delta_{(\alpha,\beta)} f(x).
 \end{aligned}$$

□

PROPOSITION 3. We have

$$\partial_{q,\alpha} j_{(\alpha+\frac{1}{2},\beta)}(x; q^2) = -q^2 (1 - q^{2\beta+4}) x j_{(\alpha+\frac{1}{2},\beta+1)}(qx; q^2)$$

and

$$\partial_{q,\beta}^* j_{(\alpha,\beta)}(qx; q^2) = -qx (1 - q^{2\alpha+4}) j_{(\alpha+1,\beta)}(qx; q^2) - \frac{1 - q^{2\beta}}{qx}.$$

Proof. The following calculation

$$\begin{aligned} & \partial_{q,\alpha} j_{(\alpha+\frac{1}{2},\beta)}(x; q^2) \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^{2\alpha+3}; q^2)_n (q^{2\beta+2}; q^2)_n} x^{2n-1} (1 - q^{2\alpha+1+2n}) \\ &= - (1 - q^{2\beta+4}) q^2 x \sum_{n=1}^{\infty} (-1)^{n-1} \frac{q^{n(n-1)}}{(q^{2\alpha+3}; q^2)_{n-1} (q^{2\beta+4}; q^2)_{n-1}} (qx)^{2(n-1)} \\ &= -q^2 (1 - q^{2\beta+4}) x j_{(\alpha+\frac{1}{2},\beta+1)}(qx; q^2), \end{aligned}$$

and

$$\begin{aligned} & \partial_{q,\beta}^* j_{(\alpha,\beta)}(qx; q^2) \\ &= \frac{[j_{(\alpha,\beta)}(x; q^2) - 1] - q^{2\beta} [j_{(\alpha,\beta)}(qx; q^2) - 1]}{qx} \\ &= \frac{[j_{(\alpha,\beta)}(x; q^2) - 1] - q^{2\beta} [j_{(\alpha,\beta)}(qx; q^2) - 1]}{qx} - \frac{1 - q^{2\beta}}{qx} \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^{2\alpha+2}; q^2)_n (q^{2\beta+2}; q^2)_n} x^{2n-1} \left(\frac{1 - q^{2\beta+2n}}{q} \right) - \frac{1 - q^{2\beta}}{qx} \\ &= -qx (1 - q^{2\alpha+4}) \sum_{n=1}^{\infty} (-1)^n \frac{q^{n(n-1)}}{(q^{2(\alpha+1)+2}; q^2)_{n-1} (q^{2\beta+2}; q^2)_{n-1}} (qx)^{2(n-1)} - \frac{1 - q^{2\beta}}{qx} \\ &= -qx (1 - q^{2\alpha+4}) j_{(\alpha+1,\beta)}(qx; q^2) - \frac{1 - q^{2\beta}}{qx} \end{aligned}$$

leads to the result. □

In this step of our work, we introduce a q -Transmutation operator. This will allow us to stake out the q -Bessel-Wright function properties from the usual q -Bessel function. The bellow proposition will be a strong step in generalizing the q -Bessel properties (integrability, bounds, etc) to be applied to the q -Bessel-Wright function.

DEFINITION 1. The q -Bessel intertwining operator $\mathcal{R}_{q,\beta}$ is defined by

$$\mathcal{R}_{q,\beta}(f)(x) = c_{q,\beta}(0) \int_0^1 W_\beta(t) t f(xt) d_q t$$

where $c_{q,\beta}$ is defined by the formula (1), and W_β is the function defined by the formula (2).

THEOREM 2. If

$$\alpha + 1 < p$$

then we have

$$\mathcal{R}_{q,\beta} : \mathcal{L}_{q,p,\alpha}(\mathbb{R}_q^+) \rightarrow \mathcal{L}_{q,p,\alpha}(\mathbb{R}_q^+).$$

Proof. Let $f \in \mathcal{L}_{q,p,\alpha}(\mathbb{R}_q^+)$. Using the generalized Minkowski integral inequality [10], we obtain

$$\begin{aligned} \|\mathcal{R}_{q,\beta}(f)\|_{q,p,\alpha} &= \left[\int_0^\infty \left| \int_0^1 W_\beta(t) t f(xt) d_q t \right|^p x^{2\alpha+1} d_q x \right]^{\frac{1}{p}} \\ &\leq \int_0^1 W_\beta(t) t \left[\int_0^{+\infty} |f(xt)|^p x^{2\alpha+1} dx \right]^{\frac{1}{p}} d_q t \\ &\leq \int_0^1 W_\beta(t) t \left[\int_0^{+\infty} |f(x)|^p x^{2\alpha+1} dx \right]^{\frac{1}{p}} t^{1-\frac{2(\alpha+1)}{p}} d_q t \\ &\leq \int_0^1 W_\beta(t) t^{1-\frac{2(\alpha+1)}{p}} d_q t \times \|f\|_{q,p,\alpha} < \infty. \end{aligned}$$

□

PROPOSITION 4. The operator $\mathcal{R}_{q,\beta}$ has the following inverse operator

$$\mathcal{R}_{q,\beta}^{-1}(f)(x) = \frac{(q^{-2\beta}; q^2)_\infty}{(q^2; q^2)_\infty} \int_0^1 W_{-\beta}(t) f(xt) t d_q t.$$

Proof. We introduce the following operator

$$H_{q,\beta}(f)(x) = c_{q,\beta}(0) (1 - q^2)^\beta x^{2\beta} \int_0^1 W_\beta(t) f(xt) t d_q t, \quad \forall x \in \mathbb{R}_q^+.$$

Using [1, lemma 2] we obtain

$$H_{q,\beta}^{-1}(f) = H_{q,-\beta}(f).$$

Note that

$$H_{q,\beta}(f)(x) = (1 - q^2)^\beta x^{2\beta} \mathcal{R}_{q,\beta}(f)(x)$$

which implies

$$H_{q,\beta}^{-1}(f)(x) = (1 - q^2)^{-\beta} x^{-2\beta} \mathcal{R}_{q,\beta}^{-1}(f)(x), \quad \forall x \in \mathbb{R}_q^+.$$

So we get

$$\mathcal{R}_{q,\beta}^{-1}(f)(x) = \frac{(q^{-2\beta}; q^2)_\infty}{(q^2; q^2)_\infty} \int_0^1 W_{-\beta}(t) f(xt) t d_q t$$

□

PROPOSITION 5. The q -Bessel-Wright function is related to the q -Bessel function via the following formula

$$j_{(\alpha,\beta)}(x; q^2) = \mathcal{R}_{q,\beta}(j_\alpha(x; q^2))$$

and we have

$$(5) \quad |j_{(\alpha,\beta)}(x, q^2)| \leq 1, \quad \forall x \in \mathbb{R}_q^+.$$

Proof. Using formula (4), we get

$$\begin{aligned} \mathcal{R}_{q,\beta} (j_\alpha (x; q^2)) &= \sum_{n=0}^{+\infty} (-1)^n \frac{q^{n(n+1)}}{(q^{2\alpha+2}; q^2)_n} x^{2n} \left[c_{q,\alpha} (0) \frac{1}{(q^2; q^2)_n} \int_0^1 W_\beta (t; q^2) t^{2n+1} d_q t \right] \\ &= \sum_{n=0}^{+\infty} (-1)^n \frac{q^{n(n+1)}}{(q^{2\alpha+2}; q^2)_n (q^{2\beta+2}; q^2)_n} x^{2n} \\ &= j_{(\alpha,\beta)} (x; q^2). \end{aligned}$$

□

We define the q -Transmutation operator $\mathcal{W}_{q,\alpha,\beta}$ as follow

$$\mathcal{W}_{q,\alpha,\beta} (f) (x) = c_{q,\beta} (0) x^{-2\alpha} \int_x^\infty \frac{\left(q^2 \frac{x^2}{u^2}; q^2 \right)_\infty}{(1-q) \left(q^{2\beta} \frac{x^2}{u^2}; q^2 \right)_\infty} f (u) u^{2\alpha-1} d_q u,$$

and which can be also written it in the following form

$$\mathcal{W}_{q,\alpha,\beta} (f) (x) = \frac{c_{q,\beta} (0)}{(1-q)} \int_1^\infty \frac{\left(\frac{q^2}{t^2}; q^2 \right)_\infty}{\left(\frac{q^{2\beta}}{t^2}; q^2 \right)_\infty} f (xt) t^{2\alpha-1} d_q t.$$

In the following, we prove that the operator $\mathcal{W}_{q,\alpha,\beta}$ leaves invariant the spaces $\mathcal{S}_q (\mathbb{R}_q^+)$ and $\mathcal{D}_q (\mathbb{R}_q^+)$ and that under some conditions, it send the space $\mathcal{L}_{q,p,\alpha} (\mathbb{R}_q^+)$ to the space $\mathcal{L}_{q,p,\alpha} (\mathbb{R}_q^+)$.

THEOREM 3. *The q -Bessel-Wright Transmutation operator satisfy the following properties*

1. We have

$$(6) \quad \mathcal{W}_{q,\alpha,\beta} : \mathcal{L}_{q,1,\alpha} (\mathbb{R}_q^+) \rightarrow \mathcal{L}_{q,1,\alpha} (\mathbb{R}_q^+)$$

2. If $\alpha + 1 < p$ then we have

$$\mathcal{W}_{q,\alpha,\beta} : \mathcal{L}_{q,p,\alpha} (\mathbb{R}_q^+) \rightarrow \mathcal{L}_{q,p,\alpha} (\mathbb{R}_q^+)$$

3. We have

$$(7) \quad \mathcal{W}_{q,\alpha,\beta} : \mathcal{S}_q (\mathbb{R}_q^+) \rightarrow \mathcal{S}_q (\mathbb{R}_q^+)$$

and

$$\mathcal{W}_{q,\alpha,\beta} : \mathcal{D}_q (\mathbb{R}_q^+) \rightarrow \mathcal{D}_q (\mathbb{R}_q^+).$$

Proof. We have

$$f \in \mathcal{L}_{q,1,\alpha} (\mathbb{R}_q^+)$$

Using the generalized Minkowski integral inequality [10], we obtain

$$\begin{aligned}
 & \| \mathcal{W}_{q,\alpha,\beta} (f) (x) \|_{q,\alpha,p} \\
 = & \left[\int_0^\infty | \mathcal{W}_{q,\alpha,\beta} (f) (x) |^p x^{2\alpha+1} d_q x \right]^{\frac{1}{p}} \\
 = & \frac{c_{q,\beta} (0)}{(1-q)} \left[\int_0^\infty \left| \int_1^\infty \frac{\left(\frac{q^2}{t^2}; q^2 \right)_\infty}{\left(\frac{q^{2\beta}}{t^2}; q^2 \right)_\infty} f (xt) t^{2\alpha-1} d_q t \right|^p x^{2\alpha+1} d_q x \right]^{\frac{1}{p}} \\
 \leq & \frac{c_{q,\beta} (0)}{(1-q)} \int_1^\infty \frac{\left(\frac{q^2}{t^2}; q^2 \right)_\infty}{\left(\frac{q^{2\beta}}{t^2}; q^2 \right)_\infty} \left[\int_0^{+\infty} |f (xt)|^p x^{2\alpha+1} dx \right]^{\frac{1}{p}} t^{2\alpha-1} d_q t \\
 \leq & \frac{c_{q,\beta} (0)}{(1-q)} \int_1^\infty \frac{\left(\frac{q^2}{t^2}; q^2 \right)_\infty}{\left(\frac{q^{2\beta}}{t^2}; q^2 \right)_\infty} \left[\int_0^{+\infty} |f (x)|^p x^{2\alpha+1} d_q x \right]^{\frac{1}{p}} t^{2\alpha-1-\frac{2(\alpha+1)}{p}} d_q t \\
 \leq & \frac{c_{q,\beta} (0)}{(1-q)} \int_1^\infty \frac{\left(\frac{q^2}{t^2}; q^2 \right)_\infty}{\left(\frac{q^{2\beta}}{t^2}; q^2 \right)_\infty} t^{2\alpha-1-\frac{2(\alpha+1)}{p}} d_q t \|f\|_{q,p,\alpha} \\
 \leq & \frac{c_{q,\beta} (0)}{(1-q)} \frac{\left(q^2; q^2 \right)_\infty \left(q^{\frac{2(\alpha+1)}{p}-2\alpha}; q^2 \right)_\infty}{\left(q^{\frac{2(\alpha+1)}{p}-2\alpha}; q^2 \right)_\infty \left(q^{2\beta}; q^2 \right)_\infty} \|f\|_{q,p,\alpha} < \infty, \text{ if } 1 > \alpha (p-1).
 \end{aligned}$$

Hence, for $p = 1$, we have

$$\mathcal{W}_{q,\alpha,\beta} : \mathcal{L}_{q,1,\alpha} (\mathbb{R}_q^+) \rightarrow \mathcal{L}_{q,1,\alpha} (\mathbb{R}_q^+) (\mathbb{R}_q^+).$$

To prove the second result, we have

$$\begin{aligned}
 \mathcal{W}_{q,\alpha,\beta} (f) (x) &= \frac{c_{q,\beta} (0)}{(1-q)} \int_1^\infty \frac{\left(\frac{q^2}{t^2}; q^2 \right)_\infty}{\left(\frac{q^{2\beta}}{t^2}; q^2 \right)_\infty} f (xt) t^{2\alpha-1} d_q t \\
 &= \frac{c_{q,\beta} (0)}{(1-q)} \Phi (x).
 \end{aligned}$$

Let $(p, n) \in \mathbb{N}^2$,

$$D_q^n \Phi (x) = \int_1^\infty \frac{\left(\frac{q^2}{t^2}; q^2 \right)_\infty}{\left(\frac{q^{2\beta}}{t^2}; q^2 \right)_\infty} t^n D_q^n f (xt) t^{2(\alpha)-1} d_q t.$$

Using the Dominated Convergence Theorem, we get

$$\begin{aligned} & \lim_{x \rightarrow +\infty} x^p |D_q^n \Phi(x)| \\ & \leq \lim_{x \rightarrow +\infty} x^p \int_1^\infty \frac{\left(\frac{q^2}{t^2}; q^2\right)_\infty}{\left(\frac{q^{2\beta}}{t^2}; q^2\right)_\infty} t^n |D_q^n f(xt)| t^{2\alpha-1} d_q t \\ & \leq \int_1^\infty \frac{\left(\frac{q^2}{t^2}; q^2\right)_\infty}{\left(\frac{q^{2\beta}}{t^2}; q^2\right)_\infty} t^n \lim_{x \rightarrow +\infty} [(tx)^p |D_q^n f(xt)|] t^{2\alpha-1} d_q t = 0. \end{aligned}$$

So

$$\mathcal{W}_{q,\alpha,\beta} : \mathcal{S}_q(\mathbb{R}_q) \rightarrow \mathcal{S}_q(\mathbb{R}_q).$$

Let $f \in \mathcal{D}_{q,a}(\mathbb{R}_q)$, we have

$$\begin{aligned} & \mathcal{W}_{q,\alpha,\beta}(f)(x) \\ & = c_{q,\beta}(0) x^{-2\alpha} \int_x^\infty \frac{\left(q^2 \frac{x^2}{u^2}; q^2\right)_\infty}{(1-q) \left(q^{2\beta} \frac{x^2}{u^2}; q^2\right)_\infty} f(u) u^{2\alpha-1} d_q u \\ & = \begin{cases} 0, & \text{if } x > a \\ c_{q,\beta}(0) x^{-2\alpha} \int_x^a \frac{\left(q^2 \frac{x^2}{u^2}; q^2\right)_\infty}{(1-q) \left(q^{2\beta} \frac{x^2}{u^2}; q^2\right)_\infty} f(u) u^{2\alpha-1} d_q u, & \text{if } x < a \end{cases} \end{aligned}$$

We conclude that $\mathcal{W}_{q,\alpha,\beta}(f) \in \mathcal{D}_{q,a}(\mathbb{R}_q^+)$. □

PROPOSITION 6. Let $f, g \in \mathcal{L}_{q,p,\alpha}(\mathbb{R}_q^+) \cap \mathcal{L}_{q,\bar{p},\alpha}(\mathbb{R}_q^+)$, such that $\frac{1}{p} + \frac{1}{\bar{p}} = 1$. We get

$$(8) \quad \langle f, \mathcal{R}_{q,\beta}(g) \rangle_\alpha = \langle \mathcal{W}_{q,\alpha,\beta}(f), g \rangle_\alpha.$$

Proof. Indeed,

$$\begin{aligned} \frac{(1-q)}{c_{q,\beta}(0)} \langle g, \mathcal{R}_{q,\beta}(f) \rangle_\alpha & = \int_0^\infty g(x) \left[\int_0^1 \frac{\left(q^2 t^2; q^2\right)_\infty}{\left(q^{2\beta} t^2; q^2\right)_\infty} t f(xt) d_q t \right] x^{2\alpha+1} d_q x \\ & = \int_0^\infty g(x) \left[\int_0^x \frac{\left(q^2 \frac{u^2}{x^2}; q^2\right)_\infty}{\left(q^{2\beta} \frac{u^2}{x^2}; q^2\right)_\infty} u f(u) d_q u \right] x^{2\alpha-1} d_q x \\ & = \int_0^\infty f(u) \left[u^{-2\alpha} \int_u^\infty \frac{\left(q^2 \frac{u^2}{x^2}; q^2\right)_\infty}{\left(q^{2\beta} \frac{u^2}{x^2}; q^2\right)_\infty} g(x) x^{2\alpha-1} d_q x \right] u^{2\alpha+1} d_q u \\ & = \frac{(1-q)}{c_{q,\beta}(0)} \langle \mathcal{W}_{q,\alpha,\beta}(g), f \rangle_\alpha. \end{aligned}$$

The computation are justified by the Fubini's Theorem and Theorem 6 & 7:

$$\begin{aligned} & \int_0^{+\infty} \int_0^1 \left| g(x) \frac{\left(q^2 t^2; q^2\right)_\infty}{\left(q^{2\beta} t^2; q^2\right)_\infty} t f(xt) \right| x^{2\alpha+1} d_q t d_q x \\ & \leq \int_0^{+\infty} |g(x)| x^{2\alpha+1} d_q x \times \|\mathcal{R}_{q,\beta}(f)\|_{q,\bar{p},\alpha} \\ & \leq \|g\|_{q,p,\alpha} \times \|\mathcal{R}_{q,\beta}(f)\|_{q,\bar{p},\alpha} < \infty. \end{aligned}$$

□

4. The q -Bessel-Wright transform

In this section, we define the q -Bessel-Wright transform, and we investigate in the related q -harmonic analysis.

DEFINITION 2. Let $f \in \mathcal{L}_{q,1,\alpha}(\mathbb{R}_q)$, we define the q -Bessel-Wright transform as follow

$$\mathcal{F}_{(\alpha,\beta)}(f)(x; q^2) = c_{q,(\alpha,\beta)} \int_0^{+\infty} f(t) j_{(\alpha,\beta)}(xt; q^2) t^{2\alpha+1} d_q t$$

where

$$c_{q,(\alpha,\beta)} = \left(\frac{1}{1-q} \right)^2 \frac{(q^{2\alpha+2}; q^2)_\infty (q^{2\beta+2}; q^2)_\infty}{(q^2; q^2)_\infty (q^2; q^2)_\infty}.$$

PROPOSITION 7. Let $f, g \in \mathcal{L}_{q,1,\alpha}(\mathbb{R}_q)$ we have

$$\int_0^{+\infty} \mathcal{F}_{(\alpha,\beta)}(f)(x) g(x) x^{2\alpha+1} d_q x = \int_0^{+\infty} f(x) \mathcal{F}_{(\alpha,\beta)}(g)(x) x^{2\alpha+1} d_q x.$$

Proof. Let $f, g \in \mathcal{L}_{q,1,\alpha}(\mathbb{R}_q)$. Then we have

$$\begin{aligned} & \int_0^{+\infty} \mathcal{F}_{(\alpha,\beta)}(f)(x) g(x) x^{2\alpha+1} d_q x \\ &= \int_0^{+\infty} \left[c_{q,(\alpha,\beta)} \int_0^{+\infty} f(t) j_{(\alpha,\beta)}(xt; q^2) t^{2\alpha+1} d_q t \right] g(x) x^{2\alpha+1} d_q x \\ &= \int_0^{+\infty} \left[c_{q,(\alpha,\beta)} \int_0^{+\infty} j_{(\alpha,\beta)}(xt; q^2) g(x) x^{2\alpha+1} d_q x \right] f(t) t^{2\alpha+1} d_q t \\ &= \int_0^{+\infty} f(t) \mathcal{F}_{(\alpha,\beta)}(g)(t) t^{2\alpha+1} d_q t \end{aligned}$$

The computation are justified by the Fubini's Theorem. In fact

$$\begin{aligned} & \int_0^{+\infty} \left| \int_0^{+\infty} f(t) j_{(\alpha,\beta)}(xt; q^2) g(x) \right| t^{2\alpha+1} x^{2\alpha+1} d_q t d_q x \\ & \leq \int_0^{+\infty} \left| \int_0^{+\infty} f(t) g(x) \right| t^{2\alpha+1} x^{2\alpha+1} d_q t d_q x \\ & \leq \int_0^{+\infty} |f(t)| t^{2\alpha+1} d_q t \int_0^{+\infty} |g(x)| x^{2\alpha+1} d_q x \\ & \leq \|f\|_{q,1,\alpha} \times \|g\|_{q,1,\alpha}. \end{aligned}$$

So we get the result. □

COROLLARY 1. Let $f \in \mathcal{L}_{q,p,\alpha}(\mathbb{R}_q^+)$ we have

$$\mathcal{F}_{(\alpha,\beta)}(f) = \mathcal{F}_{q,\alpha} \circ \mathcal{W}_{q,\alpha,\beta}(f)$$

and

$$\mathcal{F}_{(\alpha,\beta)}^{-1}(f)(x; q^2) = \mathcal{W}_{q,\alpha,\beta}^{-1} \circ \mathcal{F}_{q,\alpha}(f)$$

where $\mathcal{F}_{q,\alpha}$ is the q -Bessel transform [4].

Proof. Using (8), we have

$$\begin{aligned} & \frac{1}{c_{q,v}} \mathcal{F}_{(\alpha,\beta)}(f)(x; q^2) \\ &= \int_0^{+\infty} f(t) j_{(\alpha,\beta)}(xt; q^2) t^{2\alpha+1} d_q t \\ &= \langle f, \mathcal{R}_{\alpha,q}(j_\alpha(x.; q^2)) \rangle_\alpha \\ &= \langle \mathcal{W}_{q,\alpha,\beta}(f), j_\alpha(x.; q^2) \rangle_\alpha \\ &= \mathcal{F}_{q,\alpha} \circ \mathcal{W}_{q,\alpha,\beta}(x). \end{aligned}$$

Using $\mathcal{F}_{q,\alpha}^{-1} = \mathcal{F}_{q,\alpha}$, we get

$$\mathcal{F}_{(\alpha,\beta)}^{-1}(f)(x; q^2) = \mathcal{W}_{q,\alpha,\beta}^{-1}(f) \circ \mathcal{F}_{q,\alpha}$$

In fact $\mathcal{W}_{q,\alpha,\beta}^{-1}(f)$ exists because $\mathcal{R}_{q,\beta}^{-1}$ exists and we have

$$\langle f, \mathcal{R}_{q,\beta}(g) \rangle_\alpha = \langle \mathcal{W}_{q,\alpha,\beta}(f), g \rangle_\alpha.$$

□

THEOREM 4. Let $f \in \mathcal{L}_{q,1,\alpha}(\mathbb{R}_q)$ then we have

$$\| \mathcal{F}_{(\alpha,\beta)} f \|_{q,\infty} \leq c_{q,(\alpha,\beta)} \| f \|_{q,1,\alpha}.$$

Proof. Using formula (5)

$$\begin{aligned} \| \mathcal{F}_{(\alpha,\beta)} f \|_{q,\infty} &= \sup_{x \in \mathbb{R}_q^+} | \mathcal{F}_{(\alpha,\beta)} f(x) | \\ &\leq c_{q,(\alpha,\beta)} \int_0^{+\infty} | f(t) | \sup_{x \in \mathbb{R}_q^+} | j_{(\alpha,\beta)}(xt; q^2) | t^{2\alpha+1} d_q t \\ &\leq c_{q,(\alpha,\beta)} \| f \|_{q,1,\alpha}, \end{aligned}$$

so we get the result.

□

THEOREM 5. The q -Bessel-Wright transform verifies the following properties

:

$$\mathcal{F}_{(\alpha,\beta)} : \mathcal{L}_{q,1,\alpha}(\mathbb{R}_q^+) \rightarrow \mathcal{C}_{q,0}(\mathbb{R}_q^+).$$

: If $\alpha + 1 < p$ then we have

$$\mathcal{F}_{(\alpha,\beta)} : \mathcal{L}_{q,p,\alpha}(\mathbb{R}_q^+) \rightarrow \mathcal{L}_{q,p,\alpha}(\mathbb{R}_q^+)$$

:

$$\mathcal{F}_{(\alpha,\beta)} : \mathcal{S}(\mathbb{R}_q^+) \rightarrow \mathcal{S}(\mathbb{R}_q^+)$$

:

$$\mathcal{F}_{(\alpha,\beta)} : \mathcal{D}_a(\mathbb{R}_q^+) \rightarrow PW_{q,a}^\alpha$$

Proof. Using the fact that

$$\mathcal{F}_{q,\alpha} \circ \mathcal{W}_{q,\alpha,\beta},$$

and the fact that the operator $\mathcal{W}_{q,\alpha,\beta}$ send the space $\mathcal{L}_{q,1,\alpha}(\mathbb{R}_q^+)$ to $\mathcal{L}_{q,1,\alpha}(\mathbb{R}_q^+)$. We combine this result with Theorem (1), we get

$$\mathcal{F}_{(\alpha,\beta)} : \mathcal{L}_{q,1,\alpha}(\mathbb{R}_q^+) \rightarrow \mathcal{C}_{q,0}(\mathbb{R}_q^+)$$

Since the operator $\mathcal{W}_{q,\alpha,\beta}$ leaves $\mathcal{S}(\mathbb{R}_q^+)$ and $\mathcal{D}_a(\mathbb{R}_q^+)$ invariant, using the Theorem (1) we get

$$\mathcal{F}_{(\alpha,\beta)} : \mathcal{S}(\mathbb{R}_q^+) \rightarrow \mathcal{S}(\mathbb{R}_q^+)$$

and

$$\mathcal{F}_{(\alpha,\beta)} : \mathcal{D}_a(\mathbb{R}_q^+) \rightarrow PW_{q,a}^\alpha$$

□

5. Example and Application

To highlight the contribution of our work, we propose to investigate in the $(\alpha, \frac{1}{2})$ q -Bessel-Wright transform.

PROPOSITION 8. Let $\beta = \frac{1}{2}$, the function $j_{(\alpha,\frac{1}{2})}$

$$\begin{aligned} j_{(\alpha,\frac{1}{2})}(x; q^2) &= {}_2\Phi_2(q^2, -; q^{2\alpha+2}, q^3; q^2; q^2 x^2) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^{2\alpha+2}, q^2)_n (q^3, q^2)_n} x^{2n} \end{aligned}$$

is a q -analog of the Struve function [8].

PROPOSITION 9. The q -Struve function $x \mapsto j_{(\alpha,\frac{1}{2})}(\lambda x, q^2)$ is an eigenfunction of the q -Struve difference operator $\Delta_{(\alpha,\frac{1}{2})}$ associated to the eigenvalue $-\lambda^2$ where

$$\Delta_{(\alpha,\frac{1}{2})} f(x) = \frac{1}{x^2} [f(q^{-1}x) - (q + q^{2\alpha}) f(x) + q^{2\alpha+1} f(qx) - (1 - q^{2\alpha})(1 - q) f(0)]$$

DEFINITION 3. Let $f \in \mathcal{L}_{q,1,\alpha+\frac{1}{2}}(\mathbb{R}_q)$, we define the q -Struve transform [8] as follow

$$\mathcal{F}_{(\alpha,\frac{1}{2})}(f)(x; q^2) = c_{q,(\alpha,\frac{1}{2})} \int_0^{+\infty} f(t) j_{(\alpha,\frac{1}{2})}(xt; q^2) t^{2\alpha+1} d_q t$$

where

$$c_{q,(\alpha,\frac{1}{2})} = \left(\frac{1}{1 - q} \right)^2 \frac{(q^{2\alpha+2}; q^2)_\infty (q^3; q^2)_\infty}{(q^2; q^2)_\infty (q^2; q^2)_\infty}.$$

LEMMA 1. The q -Struve transform verifies the following properties

1.

$$\mathcal{F}_{(\alpha,\frac{1}{2})} : \mathcal{L}_{q,1,\alpha}(\mathbb{R}_q^+) \rightarrow \mathcal{C}_{q,0}(\mathbb{R}_q^+)$$

2.

$$\mathcal{F}_{(\alpha,\frac{1}{2})} : \mathcal{S}(\mathbb{R}_q^+) \rightarrow \mathcal{S}(\mathbb{R}_q^+)$$

3.

$$\mathcal{F}_{(\alpha,\frac{1}{2})} : \mathcal{D}_a(\mathbb{R}_q^+) \rightarrow PW_{q,a}^\alpha$$

4.

$$\left\| \mathcal{F}_{(\alpha,\frac{1}{2})} f \right\|_{q,\infty} \leq c_{q,(\alpha,\frac{1}{2})} \|f\|_{q,1,\alpha+\frac{1}{2}}.$$

Proof. Since we fixed $\beta = \frac{1}{2}$, We satisfy conditions of Theorem 6 and 7, and so we get the result. □

REMARK 1. In the same way, we can define the following q -analog transform

- The Hankel transform when $\alpha = 0$.
- The \mathcal{Y} transform when $\alpha = \frac{1}{2}$.
- The Hartley transform when $\alpha = -\frac{1}{4}$.
- The Hardy transform which was first introduced by G.H. Hardy [13].

Acknowledgements

The authors would like to thank the anonymous reviewers for their helpful and constructive comments that greatly contributed to improving the final version of the paper.

References

- [1] Bouzeffour F., Inversion formulas for q -Riemann-Liouville and q -Weyl transforms, *J. Math. Anal. Appl.* **336** (2007), 833–848.
- [2] Berkak I., Loualid E.M. and Daher R., An extension of the Bessel–Wright transform in the class of Boehmians. *Arab. J. Math.* **9** (2020), 271–280. <https://doi.org/10.1007/s40065-019-0250-z>.
- [3] Dhaouadi L., On the q -Bessel Fourier Transform, *Bulletin of Mathematical Analysis and Applications* **5** (2013), 42–60.
- [4] Dhaouadia L., Binous W., Fitouhi A., Paley-Wiener theorem for the q -Bessel transform and associated q -sampling formula, *Expo. Math.* **27** (2009), 55–72.
- [5] Dhaouadi L., Fitouhi A. and El Kamel J., Inequalities in q -Fourier Analysis, *Journal of Inequalities in Pure and Applied Mathematics*, **7** (2006), 171.
- [6] Fitouhi A., Dhaouadi L. and Karoui I., On the Bessel–Wright Transform. *Anal. Math.* **45** (2019), 291–309. <https://doi.org/10.1007/s10476-018-0659-1>
- [7] Fitouhi A., Hamza M. and Bouzeffour F., The q - j Bessel function, *J. Appr. Theory*, **115** (2002), 144–166.
- [8] Gasmi A. and Sifi M., The Bessel-Struve intertwining operator on \mathbb{C} and mean periodic functions, *IJMMS* **59** (2004), 3171–3185.
- [9] Gasper G., Rahman M., *Basic Hypergeometric Series Encyclopedia of Mathematics and its Application*, second ed., vol. **35**, Cambridge University Press, Cambridge, UK, (2004).
- [10] Hardy G. H., Littlewood J. E., and Polya G., *Inequalities*, Cambridge University Press, New York (1934).
- [11] Karoui I., Binous W., Fitouhi A., On the Bessel-Wright Operator and Transmutation with Applications. In: Kravchenko V., Sitnik S. (eds) *Transmutation Operators and Applications. Trends in Mathematics*. Birkhäuser, Cham (2020). https://doi.org/10.1007/978-3-030-35914-0_18.
- [12] Koornwinder T. H., Swarttouw R. F., On q -analogues of the Fourier and Hankel transforms, *Trans. Amer. Math. Soc.* **333** (1992), 445–461.
- [13] Zayed A. I., *Handbook of Function and Generalized Function Transformations*, Boca Raton, Fla. CRC Press (1996).

Ilyes Karoui

Faculté des Sciences de Tunis, 1060 Tunis, Tunisia.

E-mail: ilyes.karoui@gmail.com

Lazhar Dhaouadi

I.P.E.Bizerte, 7021 Zarzouna, Tunisia

E-mail: Lazhar.Dhaouadi@ipeib.rnu.tn

Wafa Binous

Institut préparatoire aux études d'ingénieurs de Tunis, 1060 Tunis, Tunisia.

E-mail: wafabinous@yahoo.fr

Meniar Haddad

Faculté des Sciences de Tunis, 1060 Tunis, Tunisia.

E-mail: Meniar.Haddad@fst.rnu.tn