Korean J. Math. **28** (2020), No. 3, pp. 587–601
http://dx.doi.org/10.11568/kjm.2020.28.3.587

ENHANCING EIGENVALUE APPROXIMATION WITH BANK–WEISER ERROR ESTIMATORS

Kwang-Yeon Kim

Abstract. In this paper we propose a way of enhancing eigenvalue approximations with the Bank–Weiser error estimators for the $P1$ and $P2$ conforming finite element methods of the Laplace eigenvalue problem. It is shown that we can achieve two extra orders of convergence than those of the original eigenvalue approximations when the corresponding eigenfunctions are smooth and the underlying triangulations are strongly regular. Some numerical results are presented to demonstrate the accuracy of the enhanced eigenvalue approximations.

1. Introduction

We consider the Laplace eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

which describes a transverse vibration of a membrane over a polygonal domain Ω in $\mathbb{R}^2$ with its boundary $\partial\Omega$. The variational formulation of this problem is to find $(u, \lambda) \in H_0^1(\Omega) \times \mathbb{R}$ such that $b(u, u) = 1$ and

$$a(u, v) = \lambda b(u, v) \qquad \forall v \in H_0^1(\Omega), \tag{2}$$

Received July 27, 2020. Revised September 13, 2020. Accepted September 14, 2020.
2010 Mathematics Subject Classification: 65N15, 65N25, 65N30.
Key words and phrases: eigenvalue problem, finite element method, error estimator, superconvergence.

where

$$a(u,v) = (\nabla u, \nabla v)_\Omega, \qquad b(u,v) = (u,v)_\Omega,$$

and $(u,v)_S$ denotes the standard L^2 inner product over S. It is well known that there exist positive eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ with $\lim_{j\to\infty} \lambda_j = \infty$ and corresponding eigenfunctions $u_1, u_2, u_3, \cdots$ such that $b(u_i, u_j) = 0$ for $i \neq j$. The eigenfunctions of (2) corresponding to a particular eigenvalue λ form a finite-dimensional vector space which will be denoted by

$$M(\lambda) = \{u \in H_0^1(\Omega) : a(u,v) = \lambda b(u,v) \ \ \forall v \in H_0^1(\Omega)\}.$$

As discussed in [2, 3, 6, 14], a priori error analysis for finite element methods of the problem (2) is much more complicated than that for the source problem $-\Delta u = f$. It is rather straightforward to extend the error estimators for the source problem to the eigenvalue problem by replacing f with the discrete function $\lambda_h u_h$, which results in the extra higher order term $\|\lambda u - \lambda_h u_h\|_{0,\Omega}$ in the reliability of the error estimator. We refer to [7, 14] for the residual-based error estimator and [18, 20] for the error estimator by gradient recovery.

There have been various ways to improve the accuracy of eigenfunction and/or eigenvalue approximations of (2). In [17,18] the error estimator by gradient recovery was utilized to increase the order of convergence in eigenvalue approximations for the $P1$ conforming FEM. The two-space method proposed in [19] involves a global postprocessing using a higher order finite element. In [9] the authors considered combining the method of [18] or [19] with the two-grid method [22]. Another well-known techniques are the Richardson extrapolation based on asymptotic error expansion of eigenvalue approximations [1] and combination of lower and upper bounds for the eigenvalues [10].

In this paper we aim to enhance the eigenvalue approximations of (2) computed by the $P1$ and $P2$ conforming FEMs. We basically follow the approach of [17] which recovers the eigenfunction value and then employs the Rayleigh quotient. To this end, the Bank–Weiser error estimator proposed in [4, 13] for the source problem is extended to the eigenvalue problem (2). This error estimator is computed by solving local Neumann problems using higher-order correction spaces and can yield asymptotically exact estimates of the eigenfunction error under proper conditions on the regularity of the eigenfunctions and the structure of the underlying triangulations. This enables us to recover higher order H_0^1-conforming approximations of the eigenfunctions which is then used to enhance the eigenvalue approximation via the Rayleigh quotient. Our approach is similar in spirit to the two-space method of [19] in that both uses higher order approximation spaces. But the two-space method solves a global problem and works even for unstructured triangulations, while our method is local in nature and requires the triangulations to be mildly structured in order to achieve a higher order of convergence as in [17].

The rest of the paper is organized as follows. In Section 2 we introduce the finite element methods for the eigenvalue problem and derive some superconvergence result. In Section 3 the Bank–Weiser error estimator is presented for the $P1$ and $P2$ conforming FEMs and its asymptotic exactness is established under some conditions. This error estimator is then used to enhance the eigenvalue approximation in Section 4. Finally, in Section 5, some numerical results are provided to demonstrate the accuracy of the enhanced eigenvalue approximations.

2. Finite Element Method

Let $\mathcal{T}_h$ be a regular triangulation of Ω and let $\mathcal{E}_h^\Omega$ be the collection of all interior edges of $\mathcal{T}_h$. We denote the diameter of a triangle $T \in \mathcal{T}_h$ by h_T, the unit outward normal vector to ∂T by $\boldsymbol{n}_T$, and the length of an edge $e \in \mathcal{E}_h^\Omega$ by h_e. The mesh size of $\mathcal{T}_h$ is defined as $h = \max_{T \in \mathcal{T}_h} h_T$.

For an interior edge $e = \partial T \cap \partial T'$, we define the jump of a function v across e as

$$[\![v]\!] = v|_T - v|_{T'}$$

(the sign of $[\![v]\!]$ does not matter) and the jump and mean value of the normal derivative of v across e as

$$\left[\!\!\left[\frac{\partial v}{\partial \boldsymbol{n}}\right]\!\!\right] = \nabla v|_T \cdot \boldsymbol{n}_T + \nabla v|_{T'} \cdot \boldsymbol{n}_{T'}, \qquad \left\langle \frac{\partial v}{\partial \boldsymbol{n}_T} \right\rangle = \frac{1}{2}(\nabla v|_T + \nabla v|_{T'}) \cdot \boldsymbol{n}_T.$$

Let $\mathbb{P}_k(T)$ be the space of all polynomials of degree up to k on T and let $\mathcal{N}_k(T)$ be the set of standard Lagrange nodes of $\mathbb{P}_k(T)$. The subspace of $\mathbb{P}_{k+1}(T)$ defined by

$$\mathbb{P}_{k+1}^0(T) = \{v \in \mathbb{P}_{k+1}(T) : v(\boldsymbol{x}) = 0 \ \ \forall \boldsymbol{x} \in \mathcal{N}_k(T)\}$$

acts as a $(k+1)$th-order *correction space* for $\mathbb{P}_k(T)$, i.e., we have $\mathbb{P}_{k+1}(T) = \mathbb{P}_k(T) \oplus \mathbb{P}^0_{k+1}(T)$. The usual scaling argument gives the estimates

$$\|v\|_{0,T} + h_T^{1/2}\|v\|_{0,\partial T} \le Ch_T\|\nabla v\|_{0,T} \qquad \forall v \in \mathbb{P}^0_{k+1}(T) \tag{3}$$

with some constant $C > 0$ independent of the mesh size h.

Let $V_h^k \subset H_0^1(\Omega)$ be the standard conforming finite element space of degree $k \ge 1$ over $\mathcal{T}_h$. The finite element discretization of (2) seeks $(u_h, \lambda_h) \in V_h^k \times \mathbb{R}$ such that $b(u_h, u_h) = 1$ and

$$a(u_h, v_h) = \lambda_h b(u_h, v_h) \qquad \forall v \in V_h^k. \tag{4}$$

The following a priori error estimates can be found, for example, in [2, 3, 6]: if $M(\lambda) \subset H^{t+1}(\Omega)$ and h is sufficiently small, then for each discrete eigenpair (u_h, λ_h) of (4), there exists an eigenpair (u, λ) of (2) such that

$$|\lambda-\lambda_h| \le Ch^{2s}, \quad \|\nabla(u-u_h)\|_{0,\Omega} \le Ch^s, \quad \|u-u_h\|_{0,\Omega} \le C\|u-P_hu\|_{0,\Omega}, \tag{5}$$

where $s = \min(t, k)$ and $P_hu \in V_h^k$ is the elliptic projection of u defined by

$$a(P_hu, v_h) = a(u, v_h) \qquad \forall v_h \in V_h^k.$$

If Ω is $(1+r)$-regular for some $0 < r \le 1$, then one can apply the duality argument to obtain

$$\|u - P_hu\|_{0,\Omega} \le Ch^r\|\nabla(u - P_hu)\|_{0,\Omega} \le Ch^{s+r},$$

which gives by (5)

$$\|u - u_h\|_{0,\Omega} \le Ch^{s+r}. \tag{6}$$

The constants $C > 0$ in (5)–(6) depend on λ. Moreover, the eigenfunction $u = u(h)$ satisfying (5) may depend on h but its Sobolev norms are bounded uniformly in h due to the normalization $b(u_h, u_h) = 1$ and finite dimensionality of the eigenspace $M(\lambda)$.

Throughout the rest of the paper we will use the notation $a \lesssim b$ (resp. $a \gtrsim b$) to mean the inequality $a \le Cb$ (resp. $a \ge Cb$) with some constant $C > 0$ possibly depending on u and λ but independent of the mesh size h.

Before closing this section, we extend the superconvergence results of [5,11,21] for the source problem to the eigenvalue problem. For this sake it is assumed that $M(\lambda) \subset H^{k+2}(\Omega) \cap W^{k+1,\infty}(\Omega)$ and the triangulation $\mathcal{T}_h$ is mildly structured in the following sense: (cf. [5, 21])

Condition (α,σ): There exists a subset $\mathcal{T}_{1,h}\subset\mathcal{T}_h$ and some positive constants α,σ such that

- every pair of adjacent triangles $T,T'\in\mathcal{T}_{1,h}$ form an $O(h_T^{1+\alpha})$ approximate parallelogram, which means that the lengths of two opposite edges of $T\cup T'$ differ only by $O(h_T^{1+\alpha})$.
- $\sum_{T\in\mathcal{T}_h\backslash\mathcal{T}_{1,h}}|T|\lesssim h^\sigma$, where $|T|$ denotes the area of T.

The well-known three-line triangulations whose edges are parallel to three fixed directions (like the one shown in Figure 1 of Section 5) satisfy the condition (α,σ) with the best values $\alpha=\sigma=\infty$. It is also known that a sequence of triangulations generated from an arbitrary initial triangulation by dividing every triangle into four congruent subtriangles satisfy the condition (α,σ) with $\alpha=2$ and $\sigma=1$.

Let (u,λ) and (u_h,λ_h) be solutions of (2) and (4), respectively, which satisfy the estimates (5), and let $u_I\in V_h^k$ denote the standard Lagrange interpolation of u. The superconvergence result of the source problem proved in [5, 11, 21] for $k=1,2$ asserts that

$$\|\nabla(u_I-P_hu)\|_{0,\Omega}\lesssim h^{k+\rho}(\|u\|_{k+2,\Omega}+|u|_{k+1,\infty,\Omega}) \tag{7}$$

with $\rho=\min(\alpha,\sigma/2,1)$. From the equality

$$a(P_hu-u_h,v_h)=a(u-u_h,v_h)=b(\lambda u-\lambda_hu_h,v_h)\qquad\forall v_h\in V_h^k,$$

it follows that

$$\|\nabla(P_hu-u_h)\|_{0,\Omega}\lesssim\|\lambda u-\lambda_hu_h\|_{0,\Omega}\le\lambda\|u-u_h\|_{0,\Omega}+|\lambda-\lambda_h|. \tag{8}$$

The first term could be bounded by (6) (which is valid without the condition (α,σ)) but this yields the convergence rate $O(h^{k+r})$ which may be worse than $O(h^{k+\rho})$ due to the geometry of Ω. Instead we follow the proof of [20, Theorem 3.1] to obtain by (5)

$$\begin{aligned}\lambda\|u-u_h\|_{0,\Omega}+|\lambda-\lambda_h|&\lesssim\|u-P_hu\|_{0,\Omega}+|\lambda-\lambda_h|\\&\le\|u-u_I\|_{0,\Omega}+\|u_I-P_hu\|_{0,\Omega}+|\lambda-\lambda_h|\\&\lesssim h^{k+1}|u|_{k+1,\Omega}+\|\nabla(u_I-P_hu)\|_{0,\Omega}.\end{aligned} \tag{9}$$

Combining (7)–(9) leads to the following superconvergence result for the eigenvalue problem

$$\|\nabla(u_I-u_h)\|_{0,\Omega}\lesssim h^{k+\rho}(\|u\|_{k+2,\Omega}+|u|_{k+1,\infty,\Omega}). \tag{10}$$

3. Bank–Weiser Error Estimator

From now on the degree of the finite element space V_h^k is fixed at $k = 1, 2$ as in [4, 13]. The following a posteriori error estimator of the Bank–Weiser type is a straightforward extension of the ones proposed in [4, 13] to the eigenvalue problem (2).

Definition 1. For every $T \in \mathcal{T}_h$, find $\varepsilon_h|_T \in \mathbb{P}^0_{k+1}(T)$ such that $\varepsilon_h|_{\partial T \cap \partial\Omega} = 0$ and

$$(\nabla \varepsilon_h, \nabla v)_T = (\Delta u_h + \lambda_h u_h, v)_T - \frac{1}{2}\int_{\partial T} \left[\!\left[\frac{\partial u_h}{\partial \boldsymbol{n}_T} \right]\!\right] v \, ds \tag{11}$$

for all $v \in \mathbb{P}^0_{k+1}(T)$ with $v|_{\partial T \cap \partial\Omega} = 0$. The Bank–Weiser error estimator is then defined as

$$\eta = \|\nabla_h \varepsilon_h\|_{0,\Omega}, \tag{12}$$

where $(\nabla_h w)|_T = \nabla(w|_T)$ for $T \in \mathcal{T}_h$.

Remark 1. The imposition of the Dirichlet boundary condition $\varepsilon_h|_{\partial T \cap \partial\Omega} = 0$ will lead to the improved estimates with $\rho = \min(\alpha, \sigma/2, 1)$ in subsequent results. Otherwise we would have $\rho = \min(\alpha, \sigma/2, 1/2)$ as in [13].

Remark 2. By taking $v = \varepsilon_T$ in (11) and applying (3), it is easy to see that

$$\|\nabla \varepsilon_h\|_{0,T} \lesssim h_T \|\Delta u_h + \lambda_h u_h\|_{0,T} + h_T^{1/2} \left\| \left[\!\left[\frac{\partial u_h}{\partial \boldsymbol{n}_T} \right]\!\right] \right\|_{0,\partial T \setminus \partial\Omega}.$$

Then the local lower bound for standard residuals (cf. [7]) yields

$$\|\nabla \varepsilon_h\|_{0,T} \lesssim \|\nabla(u - u_h)\|_{0,\omega_T} + h_T \|\lambda u - \lambda_h u_h\|_{0,\omega_T},$$

where ω_T is the union of triangles that share an edge with T. It is difficult to prove a global upper bound of η and we may need the saturation assumption as was done in [4] for the $P1$ FEM of the source problem.

Next we turn to the proof of asymptotic exactness of the Bank–Weiser error estimator which will be used in the next section to enhance a given eigenpair approximation (u_h, λ_h) of (4). The argument is almost the same as those of [8, 13] for the source problem and exploits an auxiliary function $q_w|_T \in \mathbb{P}^0_{k+1}(T)$ with $q_w|_{\partial T \cap \partial\Omega} = 0$ which is the solution of

$$(\nabla q_w, \nabla v)_T = -(\Delta w, v)_T + \int_{\partial T} \left\langle \frac{\partial w_I}{\partial \boldsymbol{n}_T} \right\rangle v \, ds - (\nabla w_I, \nabla v)_T \tag{13}$$

for all $v \in \mathbb{P}^0_{k+1}(T)$ with $v|_{\partial T\cap\partial\Omega} = 0$, where w is a given function in $H^2(\Omega)$.

The following estimate for $\|\nabla(u - u_I - q_u)\|_{0,T}$ was derived in [13, Lemma 4.1] for $T \in \mathcal{T}_{1,h}$ having no boundary edges and $k = 2$. Thanks to the Dirichlet boundary condition $q_u|_{\partial T\cap\partial\Omega} = 0$, the same result is valid even when T has boundary edges.

Lemma 1. *Let $T \in \mathcal{T}_h$ be such that $\omega_T \subset \bigcup_{T\in\mathcal{T}_{1,h}} \overline{T}$. If $u \in H^{k+2}(\omega_T)$ and $u = 0$ on $\partial T \cap \partial\Omega$, then we have*

$$\|\nabla(u - u_I - q_u)\|_{0,T} \lesssim h_T^{k+\min(\alpha,1)} \|u\|_{k+2,\omega_T}.$$

Proof. The proof is essentially the same as that of [13, Lemma 4.1], so we briefly describe some modifications due to consideration of the Dirichlet boundary condition $q_u|_{\partial T\cap\partial\Omega} = 0$.

As was done for the inequality (11) of [13], it follows from (13) that for all $w \in H^{k+1}(\omega_T)$,

$$\|\nabla(w-w_I-q_w)\|_{0,T} \le \|\nabla(w-w_I)\|_{0,T} + \|\nabla q_w\|_{0,T} \lesssim h_T^k |w|_{k+1,\omega_T}. \tag{14}$$

Next we verify that if $\phi \in \mathbb{P}_{k+1}(\omega_T)$ and $\phi = 0$ on $\partial T \cap \partial\Omega$, then

$$\|\nabla(\phi - \phi_I - q_\phi)\|_{0,T} \lesssim h_T^{k+\alpha} |\phi|_{k+1,\omega_T}. \tag{15}$$

Write the equation (13) as

$$(\nabla(\phi - \phi_I - q_\phi), \nabla v)_T = \sum_{e\subset\partial T\setminus\partial\Omega} \int_e \left(\frac{\partial\phi}{\partial \boldsymbol{n}_T} - \left\langle \frac{\partial\phi_I}{\partial \boldsymbol{n}_T} \right\rangle\right) v\, ds.$$

For $k = 2$ it was shown in the proof of [13, Lemma 4.1] that

$$\left| \int_e \left(\frac{\partial\phi}{\partial \boldsymbol{n}_T} - \left\langle \frac{\partial\phi_I}{\partial \boldsymbol{n}_T} \right\rangle\right) v\, ds \right| \lesssim h_T^{k+\alpha} |\phi|_{k+1,\omega_T} \|\nabla v\|_{0,T}$$

for $v \in \mathbb{P}^0_{k+1}(T)$ with $v|_{\partial T\cap\partial\Omega} = 0$. For $k = 1$ we can use [16, Lemma 7.1] to get the same result. Then the estimate (15) is obtained by taking $v = \phi - \phi_I - q_\phi$ (which is possible because $\phi - \phi_I \in \mathbb{P}^0_{k+1}(T)$ and $\phi = \phi_I = 0$ on $\partial T \cap \partial\Omega$).

Now choose $\phi \in \mathbb{P}_{k+1}(T)$ to be the $(k+1)$th-order standard Lagrange interpolation of u on T which is extended in a natural way to ω_T. Note that $\phi = 0$ on $\partial T \cap \partial\Omega$ (as u does) and $|u - \phi|_{k+1,\omega_T} \lesssim h_T |u|_{k+2,\omega_T}$. The rest of the proof is based on (14) and (15); see the proof of [13, Lemma 4.1]. □

Now we are ready to prove the main results of this section. The following two theorems correspond to [13, Theorems 4.2–4.3] and thus can be proved in a similar way.

THEOREM 1. *Assume that $\mathcal{T}_h$ satisfies the condition (α,σ) and $M(\lambda) \subset H^{k+2}(\Omega)\cap W^{k+1,\infty}(\Omega)$. Let $(u,\lambda) \in H_0^1(\Omega)\times\mathbb{R}$ and $(u_h,\lambda_h) \in V_h^k \times \mathbb{R}$ be solutions of (2) and (4) for $k = 1,2$, respectively, which satisfy the estimates (5). Then we have with $\rho = \min(\alpha, \sigma/2, 1)$*

$$\|\nabla_h(u-u_h-\varepsilon_h)\|_{0,\Omega} \lesssim h^{k+\rho}(\|u\|_{k+2,\Omega} + |u|_{k+1,\infty,\Omega}).$$

Proof. We begin by splitting $u-u_h-\varepsilon_h$ into three terms

$$u-u_h-\varepsilon_h = (u-u_I-q_u)+(q_u-\varepsilon_h)+(u_I-u_h).$$

By using Lemma 1, it can be shown that (see the proof of [13, Theorem 4.2])

$$\|\nabla_h(u-u_I-q_u)\|_{0,\Omega} \lesssim h^{k+\rho}(\|u\|_{k+2,\Omega} + |u|_{k+1,\infty,\Omega}). \tag{16}$$

By (11) and (13) it holds that

$$(\nabla(q_u-\varepsilon_h),\nabla v)_T = (\Delta(u_I-u_h),v)_T+(\lambda u-\lambda_h u_h, v)_T - \frac{1}{2}\int_{\partial T}\left[\!\left[\frac{\partial(u_I-u_h)}{\partial \boldsymbol{n}_T}\right]\!\right] v\, ds$$

for all $v \in \mathbb{P}_{k+1}^0(T)$ with $v|_{\partial T\cap\partial\Omega} = 0$. Applying the Cauchy–Schwarz inequality, the estimates (3) and the inverse inequalities successviely and then taking $v = q_u - \varepsilon_h$, we obtain

$$\|\nabla(q_u-\varepsilon_h)\|_{0,T} \lesssim \|\nabla(u_I-u_h)\|_{0,\omega_T} + h_T\|\lambda u - \lambda_h u_h\|_{0,T}. \tag{17}$$

Hence it follows that

$$\begin{aligned}\|\nabla_h(u-u_h-\varepsilon_h)\|_{0,\Omega} \lesssim\ & h^{k+\rho}(\|u\|_{k+2,\Omega}+|u|_{k+1,\infty,\Omega}) + \|\nabla(u_I-u_h)\|_{0,\Omega} \\ & + h(\lambda\|u-u_h\|_{0,\Omega}+|\lambda-\lambda_h|),\end{aligned}$$

which proves the desired result by applying the estimates (10) and (5). □

THEOREM 2. *Assume the conditions of Theorem 1 and let η be defined by (12). If $\mathcal{T}_h$ is quasi-uniform, then we have*

$$\left|\frac{\eta}{\|\nabla(u-u_h)\|_{0,\Omega}} - 1\right| \lesssim h^\rho.$$

Proof. By [15, Corollary 3.3] it is known that $\|\nabla(u-u_h)\|_{0,\Omega} \gtrsim h^k$. The rest of the proof goes in a standard way using Theorem 1 (see, e.g., [13, Theorem 4.3] or [21, Theorem 5.1]). □

4. Enhancing Eigenvalue Approximation

The piecewise quadratic or cubic function ε_h computed by (11) is discontinuous across the edges of $\mathcal{T}_h$. The following lemma gives an estimate on the jumps of ε_h across the edges.

LEMMA 2. *Under the conditions of Theorem 1, the following estimate holds*

$$\left(\sum_{e\in\mathcal{E}_h^\Omega} h_e^{-1}\|[\![\varepsilon_h]\!]\|_{0,e}^2\right)^{1/2} \lesssim h^{k+\rho}(\|u\|_{k+2,\Omega} + |u|_{k+1,\infty,\Omega}).$$

Proof. For the quadratic element ($k = 2$), we use the fact that $\int_e v\,ds = 0$ for $v \in \mathbb{P}_3^0(T)$ to obtain

$$\begin{aligned}&\left(\sum_{e\in\mathcal{E}_h^\Omega} h_e^{-1}\|[\![\varepsilon_h]\!]\|_{0,e}^2\right)^{1/2}\\ &= \left(\sum_{e\in\mathcal{E}_h^\Omega} h_e^{-1}\left\|[\![u-u_h-\varepsilon_h]\!] - \frac{1}{h_e}\int_e [\![u-u_h-\varepsilon_h]\!]\,ds\right\|_{0,e}^2\right)^{1/2}\\ &\lesssim \|\nabla_h(u-u_h-\varepsilon_h)\|_{0,\Omega}.\end{aligned}$$

Then the desired result follows directly from Theorem 1.

Now consider the linear element ($k = 1$). Fix $e = \partial T_1 \cap \partial T_2$ and let $\omega_e = T_1 \cup T_2$. The triangle inequality gives

$$h_e^{-1}\|[\![\varepsilon_h]\!]\|_{0,e}^2 \lesssim h_e^{-1}\|[\![\varepsilon_h - q_u]\!]\|_{0,e}^2 + h_e^{-1}\|[\![q_u]\!]\|_{0,e}^2.$$

By using (3) and (17) the first term is bounded as follows:

$$h_e^{-1}\|[\![\varepsilon_h-q_u]\!]\|_{0,e}^2 \lesssim \|\nabla_h(\varepsilon_h-q_u)\|_{0,\omega_e}^2 \lesssim \|\nabla(u_I-u_h)\|_{0,\omega_{T_1}\cup\omega_{T_2}}^2 + h_e^2\|\lambda u-\lambda_h u_h\|_{0,\omega_e}^2$$

For the second term we use the trace inequality to get

$$h_e^{-1}\|[\![q_u]\!]\|_{0,e}^2 = h_e^{-1}\|[\![u-u_I-q_u]\!]\|_{0,e}^2 \lesssim h_e^{-2}\|u-u_I-q_u\|_{0,\omega_e}^2 + \|\nabla_h(u-u_I-q_u)\|_{0,\omega_e}^2.$$

Let $\phi|_{T_i} \in \mathbb{P}_2(T_i)$ be the $P2$ Lagrange interpolation of $u|_{T_i}$ for $i = 1, 2$ which satisfies

$$\|u-\phi\|_{0,\omega_e} + h_e\|\nabla(u-\phi)\|_{0,\omega_e} \lesssim h_e^3|u|_{3,\omega_e}.$$

Then we have $\phi - u_I|_{T_i} \in \mathbb{P}_2^0(T_i)$ for $i = 1, 2$ and it follows by (3) that

$$\begin{aligned} h_e^{-2}\|u - u_I - q_u\|_{0,\omega_e}^2 &\lesssim h_e^{-2}\|u - \phi\|_{0,\omega_e}^2 + h_e^{-2}\|\phi - u_I - q_u\|_{0,\omega_e}^2 \\ &\lesssim h_e^4|u|_{3,\omega_e}^2 + \|\nabla_h(\phi - u_I - q_u)\|_{0,\omega_e}^2 \\ &\lesssim h_e^4|u|_{3,\omega_e}^2 + \|\nabla_h(u - u_I - q_u)\|_{0,\omega_e}^2. \end{aligned}$$

Collecting the above result, we obtain

$$\begin{aligned} \left(\sum_{e\in\mathcal{E}_h^\Omega} h_e^{-1}\|[\![\varepsilon_h]\!]\|_{0,e}^2\right)^{1/2} \lesssim &\|\nabla(u_I - u_h)\|_{0,\Omega} + h(\|u - u_h\|_{0,\Omega} + |\lambda - \lambda_h|) \\ &+ h^2|u|_{3,\Omega} + \|\nabla_h(u - u_I - q_u)\|_{0,\Omega}. \end{aligned}$$

The proof is completed by applying the estimates (10), (5) and (16). $\square$

The most straightforward way to get a continuous function from a piecewise polynomial function is to take the average of nodal values at every Lagrange node of $\mathcal{T}_h$, which is often called the Oswald interpolation in literature. Let $\widehat{\varepsilon}_h$ denote the continuous piecewise quadratic ($k = 1$) or cubic ($k = 2$) function over $\mathcal{T}_h$ obtained from ε_h in this way. Notice that we have $\widehat{\varepsilon}_h \in H_0^1(\Omega)$ because $\varepsilon_h = 0$ on $\partial\Omega$.

THEOREM 3. *Under the conditions of Theorem 1, the following estimate holds*

$$\|\nabla(u - u_h - \widehat{\varepsilon}_h)\|_{0,\Omega} \lesssim h^{k+\rho}(\|u\|_{k+2,\Omega} + |u|_{k+1,\infty,\Omega}).$$

Proof. The triangle inequality gives

$$\|\nabla(u - u_h - \widehat{\varepsilon}_h)\|_{0,\Omega} \le \|\nabla_h(u - u_h - \varepsilon_h)\|_{0,\Omega} + \|\nabla_h(\varepsilon_h - \widehat{\varepsilon}_h)\|_{0,\Omega}.$$

From [12, Theorem 2.2] we see that the second term is bounded by

$$\|\nabla_h(\varepsilon_h - \widehat{\varepsilon}_h)\|_{0,\Omega} \lesssim \left(\sum_{e\in\mathcal{E}_h^\Omega} h_e^{-1}\|[\![\varepsilon_h]\!]\|_{0,e}^2\right)^{1/2}.$$

The proof is completed by applying Theorem 1 and Lemma 2. $\square$

Theorem 3 implies that $\widehat{u}_h = u_h + \widehat{\varepsilon}_h \in H_0^1(\Omega)$ is a superconvergent approximation to an eigenfunction $u \in M(\lambda)$ and thus can be used to enhance the eigenvalue approximation via the Rayleigh quotient (see, e.g., (5.9) of [17])

$$\widehat{\lambda}_h = \frac{a(\widehat{u}_h, \widehat{u}_h)}{b(\widehat{u}_h, \widehat{u}_h)}. \tag{18}$$

To estimate the enhanced eigenvalue error $\widehat{\lambda}_h - \lambda$, we invoke the following identity from [3, Lemma 9.1]

$$\frac{a(\phi,\phi)}{b(\phi,\phi)} - \lambda = \frac{a(u-\phi,u-\phi)}{b(\phi,\phi)} - \lambda\frac{b(u-\phi,u-\phi)}{b(\phi,\phi)} \tag{19}$$

which is valid for any solution (u,λ) of (2) and any nonzero function $\phi \in H_0^1(\Omega)$. Theorem 3 enables us to get an extra order of convergence $O(h^{2\rho})$ for the enhanced eigenvalue approximation $\widehat{\lambda}_h$ in comparison with the original approximation λ_h (see (5)).

THEOREM 4. *Assume the conditions of Theorem 1 and let $\widehat{\lambda}_h$ be defined by (18). For sufficiently small h, we have*

$$|\lambda - \widehat{\lambda}_h| \lesssim h^{2(k+\rho)}.$$

Proof. Taking $\phi = \widehat{u}_h = u_h + \widehat{\varepsilon}_h$ in (19) and applying Theorem 3, we obtain

$$|\lambda - \widehat{\lambda}_h| \lesssim \frac{\|\nabla(u-u_h-\widehat{\varepsilon}_h)\|_{0,\Omega}^2 + \|u-u_h-\widehat{\varepsilon}_h\|_{0,\Omega}^2}{\|u_h+\widehat{\varepsilon}_h\|_{0,\Omega}^2} \lesssim \frac{h^{2(k+\rho)}}{\|u_h+\widehat{\varepsilon}_h\|_{0,\Omega}^2}.$$

Since $\widehat{\varepsilon}_h|_T \in \mathbb{P}_{k+1}^0(T)$ for $T \in \mathcal{T}_h$, it follows by (3), (5) and Theorem 3 that

$$\|\widehat{\varepsilon}_h\|_{0,\Omega} \lesssim h\|\nabla\widehat{\varepsilon}_h\|_{0,\Omega} \le h(\|\nabla(u-u_h-\widehat{\varepsilon}_h)\|_{0,\Omega} + \|\nabla(u-u_h)\|_{0,\Omega}) \lesssim h^{k+1},$$

and hence

$$\|u_h + \widehat{\varepsilon}_h\|_{0,\Omega} \ge \|u_h\|_{0,\Omega} - \|\widehat{\varepsilon}_h\|_{0,\Omega} \gtrsim 1 - h^{k+1} \gtrsim 1$$

if h is sufficiently small. This completes the proof. □

5. Numerical Results

In this section we numerically investigate the effectiveness of using the Bank–Weiser error estimators to enhance the eigenvalue approximations. All computations are performed with MATLAB R2018b and the discrete algebraic system (4) is solved by the command `eigs` with the normalization $\|u_h\|_{0,\Omega} = 1$.

We consider the Laplace eigenvalue problem (1) on the L-shaped domain $\Omega = (-1,1)^2 \backslash [0,1] \times [-1,0]$ which has a re-entrant corner at the origin. The first and second eigenvalues are approximately 9.639723844021941 and 15.197251926454343 when rounded to 15 decimal places (cf. [1]),

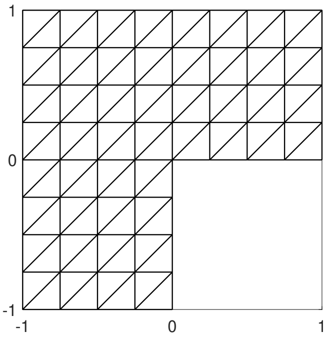

FIGURE 1. Initial triangulation on $\Omega = (-1,1)^2 \setminus [0,1] \times [-1,0]$ with the mesh size $h = \frac{1}{4}$

while the third eigenvalue is exactly $2\pi^2 = 19.739208802178716\cdots$. It is known that the first and second eigenfunctions have the leading term $O(r^{2/3})$ and $O(r^{4/3})$ in a series expansion using the polar coordinates (r,θ) near the origin. This implies that the first and second eigenfunctions belong to $H^{t+1}(\Omega)$ with $t = \frac{2}{3} - \epsilon$ and $t = \frac{4}{3} - \epsilon$, respectively, for any $\epsilon > 0$. On the other hand, the third eigenfunction has the explicit form $u(x,y) = \sin(\pi x)\sin(\pi y)$ which is also the first eigenfunction of the Laplace operator on the unit square $(0,1)^2$.

The $P1$ and $P2$ finite element solutions (u_h, λ_h) of (4) are computed on a sequence of uniform regular triangulations generated by successive refinement of every triangle into four congruent subtriangles starting with the initial triangulation of equal right isosceles triangles as shown in Figure 1. These triangulations satisfy the condition (α,σ) with $\alpha = \sigma = \infty$, and thus we have $\rho = 1$. This implies that the actual orders of convergence for the eigenvalue approximations λ_h and $\widehat{\lambda}_h$ are dictated by the regularity of the corresponding eigenfunctions.

We report the values of the eigenvalue errors in Tables 1–3 for the smallest three eigenvalues mentioned above. The numerical order of convergence next to the error $|\lambda - \lambda_h|$ is evaluated by

$$\text{Order} = \log_2 \frac{|\lambda - \lambda_{2h}|}{|\lambda - \lambda_h|},$$

and similarly for $|\lambda - \widehat{\lambda}_h|$. From Table 1 we observe that the order of convergence for $|\lambda - \lambda_h|$ is about $\frac{4}{3}$ for both $P1$ and $P2$ elements as predicted by the theoretical estimate (5) with $t = \frac{2}{3} - \epsilon$. It is clearly seen that the order of convergence is not improved by the eigenvalue enhancement, although the error becomes slightly smaller, because the eigenfunction is not sufficiently smooth. In the case of the second eigenvalue we have $t = \frac{4}{3} - \epsilon$, so the eigenvalue approximation λ_h is optimal for the $P1$ element but not for the $P2$ element as observed in Table 2. Besides it appears from Table 2 that the enhanced eigenvalue approximation $\widehat{\lambda}_h$ gains an extra order of convergence $O(h^{\frac{2}{3}})$ for the $P1$ element but nothing for the $P2$ element. On the other hand, Table 3 shows that the eigenvalue enhancement achieves the full extra order of convergence $O(h^2)$ for the third eigenvalue. This is in accordance with Theorem 4 as the third eigenfunction is smooth.

TABLE 1. Eigenvalue errors for the 1st eigenvalue $\lambda \approx 9.639723844021941$

	$P1$				$P2$			
$1/h$	$\lvert\lambda - \lambda_h\rvert$	Order	$\lvert\lambda - \widehat{\lambda}_h\rvert$	Order	$\lvert\lambda - \lambda_h\rvert$	Order	$\lvert\lambda - \widehat{\lambda}_h\rvert$	Order
4	9.342e–01	—	1.151e–01	—	6.327e–02	—	3.330e–02	—
8	2.768e–01	1.755	4.509e–02	1.352	2.388e–02	1.406	1.336e–02	1.318
16	8.865e–02	1.643	1.813e–02	1.315	9.427e–03	1.341	5.319e–03	1.329
32	3.009e–02	1.559	7.234e–03	1.325	3.741e–03	1.333	2.112e–03	1.332
64	1.069e–02	1.493	2.876e–03	1.331	1.485e–03	1.333	8.385e–04	1.333
128	3.933e–03	1.443	1.142e–03	1.333	5.893e–04	1.333	3.328e–04	1.333

TABLE 2. Eigenvalue errors for the 2nd eigenvalue $\lambda \approx 15.197251926454343$

	$P1$				$P2$			
$1/h$	$\lvert\lambda - \lambda_h\rvert$	Order	$\lvert\lambda - \widehat{\lambda}_h\rvert$	Order	$\lvert\lambda - \lambda_h\rvert$	Order	$\lvert\lambda - \widehat{\lambda}_h\rvert$	Order
4	1.750e+00	—	7.946e–02	—	3.714e–02	—	2.012e–03	—
8	4.360e–01	2.005	6.134e–03	3.695	2.972e–03	3.643	2.220e–04	3.180
16	1.093e–01	1.996	6.014e–04	3.350	2.765e–04	3.426	3.338e–05	2.733
32	2.742e–02	1.995	7.426e–05	3.018	3.144e–05	3.137	5.235e–06	2.673
64	6.873e–03	1.996	1.049e–05	2.823	4.195e–06	2.906	8.241e–07	2.667
128	1.721e–03	1.998	1.580e–06	2.731	6.136e–07	2.773	1.298e–07	2.667

TABLE 3. Eigenvalue errors for the 3rd eigenvalue $\lambda = 2\pi^2 \approx 19.739208802178716$

	$P1$				$P2$			
$1/h$	$\lvert\lambda-\lambda_h\rvert$	Order	$\lvert\lambda-\widehat{\lambda}_h\rvert$	Order	$\lvert\lambda-\lambda_h\rvert$	Order	$\lvert\lambda-\widehat{\lambda}_h\rvert$	Order
4	3.080e+00	—	1.892e–01	—	6.564e–02	—	2.447e–03	—
8	7.631e–01	2.013	1.107e–02	4.095	4.435e–03	3.888	4.353e–05	5.813
16	1.904e–01	2.003	6.653e–04	4.056	2.831e–04	3.969	7.114e–07	5.935
32	4.757e–02	2.001	4.114e–05	4.015	1.779e–05	3.992	1.131e–08	5.975
64	1.189e–02	2.000	2.565e–06	4.004	1.114e–06	3.998	1.783e–10	5.988
128	2.973e–03	2.000	1.602e–07	4.001	6.964e–08	3.999	2.206e–12	6.336

References

[1] P. Amore, J. P. Boyd, F. M. Fernández, and B. Rösler, *High order eigenvalues for the Helmholtz equation in complicated non-tensor domains through Richardson extrapolation of second order finite differences*, J. Comput. Phys. **312** (2016), 252–271.

[2] I. Babuška and J. E. Osborn, *Finite element-Galerkin approximation of the eigenvalues and eigenvectors of selfadjoint problems*, Math. Comp. **52** (1989), 275–297.

[3] I. Babuška and J. E. Osborn, *Eigenvalue Problems, in Handbook of Numerical Analysis II, Finite Element Methods (Part 1), edited by P.G. Lions and P.G. Ciarlet*, North-Holland, Amsterdam, 1991, 641–787.

[4] R. E. Bank and A. Weiser, *Some a posteriori error estimators for elliptic partial differential equations*, Math. Comp. **44** (1985), 283–301.

[5] R. E. Bank and J. Xu, *Asymptotically exact a posteriori error estimators, Part I: Grids with superconvergence*, SIAM J. Numer. Anal. **41** (2003), 2294–2312.

[6] D. Boffi, *Finite element approximation of eigenvalue problems*, Acta Numer. **19** (2010), 1–120.

[7] R. G. Durán, C. Padra, and R. Rodríguez, *A posteriori error estimates for the finite element approximation of eigenvalue problems*, Math. Models Methods Appl. Sci. **13** (2003), 1219–1229.

[8] R. G. Durán and R. Rodríguez, *On the asymptotic exactness of Bank–Weiser's estimator*, Numer. Math. **62** (1992), 297–303.

[9] H. Guo, Z. Zhang, and R. Zhao, *Superconvergent two-grid methods for elliptic eigenvalue problems*, J. Sci. Comput. **70** (2017), 125–148.

[10] J. Hu, Y. Huang, and Q. Shen, *A high accuracy post-processing algorithm for the eigenvalues of elliptic operators*, Numer. Math. **52** (2012), 426–445.

[11] Y. Huang and J. Xu, *Superconvergence of quadratic finite elements on mildly structured grids*, Math. Comp. **77** (2008), 1253–1268.

[12] O. Karakashian and F. Pascal, *A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems*, SIAM J. Numer. Anal. **41** (2003), 2374–2399.
[13] K.-Y. Kim and J.-S. Park, *Asymptotic exactness of some Bank–Weiser error estimator for quadratic triangular finite element*, Bull. Korean Math. Soc. **57** (2020), 393–406.
[14] M. G. Larson, *A posteriori and a priori error analysis for finite element approximations of self-adjoint elliptic eigenvalue problems*, SIAM J. Numer. Anal. **38** (2000), 608–625.
[15] Q. Lin, H. Xie, and J. Xu, *Lower bounds of the discretization error for piecewise polynomials*, Math. Comp. **83** (2014), 1–13.
[16] A. Maxim, *Asymptotic exactness of an a posteriori error estimator based on the equilibrated residual method*, Numer. Math. **106** (2007), 225–253.
[17] A. Naga and Z. Zhang, *Function value recovery and its application in eigenvalue problems*, SIAM J. Numer. Anal. **50** (2012), 272–286.
[18] A. Naga, Z. Zhang, and A. Zhou, *Enhancing eigenvalue approximation by gradient recovery*, SIAM J. Sci. Comput. **28** (2006), 1289–1300.
[19] M. R. Racheva and A. B. Andreev, *Superconvergence postprocessing for eigenvalues*, Comp. Methods Appl. Math. **2** (2002), 171–185.
[20] H. Wu and Z. Zhang, *Enhancing eigenvalue approximation by gradient recovery on adaptive meshes*, IMA J. Numer. Anal. **29** (2009), 1008–1022.
[21] J. Xu and Z. Zhang, *Analysis of recovery type a posteriori error estimators for mildly structured grids*, Math. Comp. **73** (2004), 1139–1152.
[22] J. Xu and A. Zhou, *A two-grid discretization scheme for eigenvalue problems*, Math. Comp. **70** (2001), 17–25.

Kwang-Yeon Kim
Department of Mathematics, Kangwon National University
Chuncheon 24341, Korea
E-mail: eulerkim@kangwon.ac.kr