# ASYMPTOTIC STABILITY OF COMPETING SPECIES

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Abstract. Large-time asymptotic behavior of the solutions of interacting population reaction-diffusion systems are considered. Polynomial stability was proved.

## 1. Introduction

In this paper we consider a system of two competing species with Dirichlet boundary conditions. The system of equations are:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \sigma_1 \Delta u_1 = u_1[a + \tilde{f}_1(t, u_1, u_2)], \\ \dfrac{\partial u_2}{\partial t} - \sigma_2 \Delta u_2 = u_2[a + \tilde{f}_2(t, u_1, u_2)] \end{cases} \tag{1.1}$$

for $x \in \Omega,\ t > 0$. Here $u_i(x,t),\ i = 1, 2$, represents the concentration of two species at position $x$ and time $t$. The parameters $a, b, \sigma_1, \sigma_1$ are positive constants, with $a$ and $b$ representing growth rates when no interaction occurs, $\sigma_1$ and $\sigma_2$ representing diffusion rates. The functions $\tilde{f}_i : \mathbb{R}^2 \to \mathbb{R},\ i = 1, 2$ have Hölder continuous partial derivatives up to second order in compact sets. Further, we assume that

$$\tilde{f}_1(t, 0, 0) = \tilde{f}_2(t, 0, 0) = 0. \tag{1.2}$$

For $(u_1, u_2)$ in the first open quadrant, the first partial derivatives of $\tilde{f}_1,\ \tilde{f}_2$ satisfy:

$$\frac{\partial \tilde{f}_i}{\partial u_j} < 0 \quad \text{for each } i, j = 1, 2. \tag{1.3}$$

We assume here that there are two functions $f_1,\ f_2$ such that

$$|\tilde{f}_i(t, u_1, u_2) - f_i(u_1, u_2)| \le K(1 + t)^{-\gamma}, \quad i = 1, 2 \tag{1.4}$$

for some positive constants $K$ and $\gamma$ and

$$1 < \min_{x,t} \left| \frac{\overline{u}_i(x)}{\overline{u}_j(x)} \cdot \frac{(\partial \tilde{f}_i / \partial u_i)(t, \overline{u}_1(x), \overline{u}_2(x))}{(\partial \tilde{f}_i / \partial u_j)(t, \overline{u}_1(x), \overline{u}_2(x))} \right| \tag{1.5}$$

for each $x \in \overline{\Omega}, t > 0,\ i \ne j,\ 1 \le i, j \le 2$. Here $\overline{u}_i,\ i = 1, 2$, are the equilibrium solution of (1.1). Concerning the stability of the equilibrium solution of (1.1) when the reaction functions $\tilde{f}_i(t, u_1, u_2),\ i = 1, 2$, are independent of time we have the following result.

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THEOREM 1.1([L]). *Let $(\overline{u}_1(x), \overline{u}_2(x))$ be an equilibrium solution to (1.1) when $\tilde{f}_i(t, u_1, u_2) \equiv f_i(u_1, u_2),\ i = 1, 2$. Suppose that the conditions (1.2) and (1.3) hold and that*

$$|\frac{\overline{u}_i(x)}{\overline{u}_j(x)} \frac{(\partial f_j/\partial u_i)(\overline{u}_1(x), \overline{u}_2(x))}{(\partial f_j/\partial u_j)(\overline{u}_1(x), \overline{u}_2(x))}| < \min |\frac{\overline{u}_i(x)}{\overline{u}_j(x)} \frac{(\partial f_i/\partial u_i)(\overline{u}_1(x), \overline{u}_2(x))}{(\partial f_i/\partial u_j)(\overline{u}_1(x), \overline{u}_2(x))}| \tag{1.6}$$

*for each $x \in \overline{\Omega},\ 1 \le i, j \le 2,\ i \ne j$, then $(\overline{u}_1(x), \overline{u}_2(x))$ is asymptotically stable.*

Hence, it is our purpose to study the stability of the equilibrium solution of (1.1) under the conditions stated above.

## 2. Main Result

We use the standard upper and lower solution methods( [L], [P]) to prove the following stability result.

THEOREM 2.1. *Let $(\overline{u}_1(x), \overline{u}_2(x))$ be an equilibrium solution to (1.1) under the following boundary conditions;*

$$\overline{u}_i(x, t) \equiv 0,\ i = 1, 2 \text{ for all } t > 0, x \in \partial\Omega.$$

*Suppose that $\tilde{f}_1, \tilde{f}_2$ satisfy the conditions (1.2)–(1.5). Then $(\overline{u}_1, \overline{u}_2)$ is asymptotically stable.*

*Proof.* Let

$$w_2 = (1 + p(t))\overline{u}_2(x),$$
$$v_1 = (1 - p(t))\overline{u}_1(x)$$

Then

$$\begin{aligned}
&\frac{\partial w_2}{\partial t} - \sigma_2\Delta w_2 - w_2[b + \tilde{f}_2(t, v_1, w_2)] \\
&\quad = p'\overline{u}_2 \\
&\qquad - (1 + p(t))\sigma_2\Delta\overline{u}_2 - (1 + p(t))[b + \tilde{f}_2(t, v_1, w_2)] \\
&\quad = p'\overline{u}_2 + \\
&\qquad (1 + p(t))\big(-\sigma_2\Delta\overline{u}_2 - \overline{u}_2\tilde{f}_2(t, v_1, w_2)\big) \\
&\quad = p'\overline{u}_2 + (1 + p(t))\big(-\sigma_2\Delta\overline{u}_2 - \overline{u}_2 f_2(\overline{u}_1, \overline{u}_2)\big) - \\
&\qquad (1 + p(t))\overline{u}_2[\tilde{f}_2(t, v_1, w_2) - f_2(\overline{u}_1, \overline{u}_2)] \\
&\quad = p'\overline{u}_2 + \\
&\qquad (1 + p(t))\overline{u}_2[\tilde{f}_2(t, v_1, w_2) - f_2(\overline{u}_1, \overline{u}_2)] \\
&\quad = p'\overline{u}_2 + \\
&\qquad (1 + p(t))\overline{u}_2[\tilde{f}_2(t, v_1, w_2) - \tilde{f}_2(t, \overline{u}_1, \overline{u}_2)] \\
&\qquad - (1 + p(t))\overline{u}_2[\tilde{f}_2(t, \overline{u}_1, \overline{u}_2) - f_2(\overline{u}_1, \overline{u}_2)] \\
&\geq \overline{u}_2 \times \big(p' - (1 + p(t))\, p(t)\, \overline{u}_1|\frac{\partial \tilde{f}_2}{\partial u_1}(t, \eta_1, \eta_2)| + \\
&\qquad (1 + p(t))\, p(t)\, \overline{u}_2|\frac{\partial \tilde{f}_2}{\partial u_2}(t, \eta_1, \eta_2)| \\
&\qquad - (1 + p(t))[\tilde{f}_2(t, \overline{u}_1, \overline{u}_2) - f_2(\overline{u}_1, \overline{u}_2)]\big)
\end{aligned}$$

At this point we choose $p(t)$ to be

$$p(t) := (1 + (1 + t)^{-\gamma}) \qquad \gamma < c.$$

Then, since

$$|\tilde{f}_2(t, \overline{u}_1, \overline{u}_2) - f_2(\overline{u}_1, \overline{u}_2)| \leq | \leq K(1 + t)^{-c},$$

We have

$$\begin{aligned}
&\frac{\partial w_2}{\partial t} - \sigma_2\Delta w_2 - w_2[b + \tilde{f}_2(t, v_1, w_2)] \geq \\
&\qquad (1 + t)^{-\gamma}\overline{u}_2 \times \big(-p(1 + t)^{-1} - [1 + (1 + t)^{-\gamma}]\overline{u}_1|\frac{\partial \tilde{f}_2}{\partial u_1}(t, \eta_1, \eta_2)| + \\
&\qquad + [1 + (1 + t)^{-\gamma}]\overline{u}_2|\frac{\partial \tilde{f}_2}{\partial u_2}(t, \eta_1, \eta_2)| - c[1 + (1 + t)^{-\gamma}](1 + t)^{-c+\gamma}\big).
\end{aligned}$$

Therefore

$$\frac{\partial w_2}{\partial t} - \sigma_2\Delta w_2 - w_2[b + \tilde{f}_2(t, v_1, w_2)] \geq 0$$

if

$$\begin{aligned}
\overline{u}_2(x)|\frac{\partial \tilde{f}_2}{\partial u_2}(t, \eta_1, \eta_2)| \geq \frac{p}{(1 + t)[1 + (1 + t)^{-\gamma}]} + \frac{K}{(1 + t)^{c-\gamma}} + \\
\overline{u}_1(x)|\frac{\partial \tilde{f}_2}{\partial u_1}(t, \eta_1, \eta_2)|.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\frac{\partial v_1}{\partial t} - \sigma_1 \Delta v_1 - v_1[a + \tilde{f}_1(t, v_1, w_2)] \\
&\quad = -\gamma(1+t)^{-\gamma-1}\,\overline{u}_1 \\
&\qquad - \sigma_1[1-(1+t)^{-\gamma}]\Delta\overline{u}_1 \\
&\qquad\quad - [1-(1+t)^{-\gamma}]\overline{u}_1[a + \tilde{f}_1(t, v_1, w_2)] \\
&\quad = (1+t)^{-\gamma}\overline{u}_1(x) \times \qquad \big(-\gamma(1+t)^{-1} \\
&\qquad\quad - [1-(1+t)^{-\gamma}]\overline{u}_1(x)|\frac{\partial \tilde{f}_1}{\partial u_2}(t, \eta_1, \eta_2)| \\
&\qquad\quad + [1-(1+t)^{-\gamma}]\overline{u}_2(x)|\frac{\partial \tilde{f}_1}{\partial u_1}(t, \eta_1, \eta_2)| \\
&\qquad\quad - K[1+(1+t)^{-\gamma}](1+t)^{-c+\gamma}\big).
\end{aligned}$$

Therefore

$$\frac{\partial v_1}{\partial t} - \sigma_1 \Delta v_1 - v_1[a + \tilde{f}_1(t, v_1, w_2)] \geq 0$$

if

$$\begin{aligned}
\overline{u}_1(x)|\frac{\partial \tilde{f}_1}{\partial u_1}(t, \eta_1, \eta_2)| \geq \frac{\gamma}{(1+t)[1-(1+t)^{-\gamma}]} + \frac{K}{(1+t)^{c-\gamma}} \\
+ \qquad\qquad \overline{u}_2(x)|\frac{\partial f_1}{\partial u_2}|.
\end{aligned}$$

Hence it suffices to find constant $\gamma$ satisfying the following two inequalities;

$$\begin{cases}
\overline{u}_2(x)|\dfrac{\partial \tilde{f}_2}{\partial u_2}(t, \eta_1, \eta_2)| \geq \gamma + c + \overline{u}_1(x)|\dfrac{\partial \tilde{f}_2}{\partial u_1}(t, \eta_1, \eta_2)|, \\
\overline{u}_1(x)|\dfrac{\partial \tilde{f}_1}{\partial u_1}(t, \eta_1, \eta_2)| \geq \gamma + c + \overline{u}_2(x)|\dfrac{\partial \tilde{f}_1}{\partial u_2}(t, \eta_1, \eta_2|,
\end{cases}$$

which is possible if we can choose positive constants $\alpha$ and $\beta$ such that

$$\begin{cases}
\gamma + c \leq \overline{u}_2(x)|\dfrac{\partial \tilde{f}_2}{\partial u_2}(t, \overline{u}_1, \overline{u}_2)| - \dfrac{\beta}{\alpha}\overline{u}_1(x)|\dfrac{\partial \tilde{f}_2}{\partial u_1}(t, \overline{u}_1, \overline{u}_2)|, \\
\gamma + c \leq \overline{u}_1(x)|\dfrac{\partial \tilde{f}_1}{\partial u_1}(t, \overline{u}_1, \overline{u}_2)| - \dfrac{\alpha}{\beta}\overline{u}_2(x)|\dfrac{\partial \tilde{f}_1}{\partial u_2}(t, \overline{u}_1, \overline{u}_2)|
\end{cases}$$

Hence we can choose a small positive number $\gamma$ provided we have the inequalities (2.4). This completes the proof. □ □

## References


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